Questions from the Lévy processes reading group 3

In all the following questions, you may assume that $X = (X_t)_{t \geq 0}$ is a Lévy process in $\mathbb{R}^d$ with characteristic exponent $\Psi$.

1. Make sure you understand the proof of the fact that: $\Psi$ bounded $\Rightarrow \Pi(\mathbb{R}^d) < \infty$.

2. (Markov property) Show that

$$
\mathbb{E}(f(X_{t+s}) | F_t) = \mathbb{E}(f(X_{t+s}) | \sigma(X_t))
$$

for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$.

3. Fix $t \geq 0$. Check that $X' = (X'_s)_{s \geq 0}$, where $X'_s := X_{t+s} - X_t$, is independent of $F_t$ and has the same distribution as $X$.

4. Let $T$ be a stopping time. Check that

$$
\mathcal{F}_T := \{ A \in \mathcal{F} : A \cap \{ T \leq t \} \in \mathcal{F}_t \text{ for all } t \geq 0 \}
$$

defines a $\sigma$-algebra.

5. Let $T$ be a stopping time such that $\mathbb{P}(T < \infty) > 0$. On $\{ T < \infty \}$, define $X' = (X'_s)_{s \geq 0}$ by setting $X'_s := X_{T+s} - X_T$. Check that, for $0 \leq u \leq v \leq s \leq t$, $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$
\mathbb{E}\left( e^{i(\lambda_1 X'_s - X'_u) + i(\lambda_2 X'_v - X'_s)} \mathbb{1}_{A \cap \{ T < \infty \}} \right) = e^{i(t-s)\Psi(\lambda_1) + i(v-u)\Psi(\lambda_2)} \mathbb{P}(A \cap \{ T < \infty \}).
$$

6. In the setting of the previous question, explain why the following statements hold (interpret the final three as holding conditionally on $\{ T < \infty \}$):

   (a) $X'$ is independent of $\mathcal{F}_T$,
   (b) $X'_t - X'_s$ is independent of $X'_u - X'_v$,
   (c) $X'_t - X'_s \sim X'_{t-s}$,
   (d) the characteristic exponent of $X'$ is $\Psi$.

7. Use the previous results to establish the Strong Markov property: If $T$ is a stopping time such that $\mathbb{P}(T < \infty) > 0$, then conditionally on $\{ T < \infty \}$, $X' = (X'_s)_{s \geq 0}$, where $X'_s := X_{T+s} - X_T$, is independent of $\mathcal{F}_T$ and has the same distribution as $X$. 

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