Exercise Sheet 1

July 21, 2010

1 Steve Evans: Trickle-down growth models, Doob-Martin boundaries, and random matrices

Lecture 1

1. Suppose that $(Z_1, Z_2, \ldots, Z_n)$ has Dirichlet distribution with parameter $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Compute the joint moment $\mathbb{P}\left[\prod_{k=1}^n Z_k^{m_k}\right]$.

2. Construct a "bare-hands" (i.e. without using the Blackwell-MacQueen theorem) proof that Pólya sequences are spreadable.

3. Let ξ be a stationary sequence. Show that the invariant events form a σ-field $\mathcal{I}_\xi$ and that this σ-field is generated by the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\xi_k, \ldots, \xi_{k+m-1})$$

for $m \in \mathbb{N}$ and bounded Borel functions $f : E^m \to \mathbb{R}$. (Why do these limits exist?)

4. Show that the extreme points of the convex set of exchangeable probability measures on E^∞, where E is a Polish space, are the product measures with identical factors.

5. Suppose that E is the Hilbert space $\ell^2 := \{(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sum_n x_n^2 < \infty\}$. Put $K := \{(x_1, x_2, \ldots) \in E : \sum_n 2^n x_n^2 \leq 1\}$. Show that K is compact and convex. Show that $\mathrm{ex}K = \{(x_1, x_2, \ldots) \in E : \sum_n 2^n x_n^2 = 1\}$ and that the closure of $\mathrm{ex}K$ is K, so Krein-Milman doesn't say much in this case.

6. Give an example of a compact, convex subset $K \subseteq \mathbb{R}^d$ for which $\mathrm{ex}K$ is not closed.

7. Suppose that P is a $n \times n$ matrix that is doubly stochastic; that is, P has non-negative entries and each row and column add to 1.

   (a) Use Krein-Milman to show that there are permutation matrices Π_k and $p_k \geq 0$ with $\sum_k p_k = 1$ such that $P = \sum_k p_k \Pi_k$ (a permutation matrix is a matrix that has a single 1 in each row an column and zeros elsewhere).
   Hint: Note that if P is not a permutation matrix, then for some N there are pairs of indices $(i_1, j_1), (i_2, j_2), \ldots, (i_{2N}, j_{2N})$ such that $P(i_k, j_k) > 0$, $i_k = i_{k+1}$ if k is odd, and $j_k = j_{k+1}$ if k is even (with the convention $2N+1 = 1$).

(b) Recall that $[n] := \{1, 2, \ldots, n\}$. Set

$$S := \{(k_1, \ldots, k_n) \in [n]^n : k_i \neq k_j, i \neq j\}.$$

Show that there is an S-valued Markov chain $\{X_m\}_{m=0}^\infty = \{(X_m(1), \ldots, X_m(n)\}_{m=0}^\infty$ such that for each i the process $\{X_m(i)\}_{m=0}^\infty$ is a Markov chain with transition matrix P.

8. Let $\{X_n\}_{n=0}^\infty$ be a Pólya sequence with parameter μ. Put $Y_n = \sum_{k=1}^n \delta_{X_k}$ and $Y_0 := 0$. Show that $\{Y_n\}_{n=0}^\infty$ is a Markov chain.

9. Suppose that $E = \{1, 2, \ldots, r\}$, so that the state space of $Y_n = \sum_{k=1}^n \delta_{X_k}$ may be thought of as $(\mathbb{N}_0)^r$.

   (a) Show that

$$\begin{aligned} h_\nu(y) &:= \frac{d\mathbb{F}^{\mu+y}}{d\mathbb{F}^\mu}(\nu) \\ &= \frac{(\sum_k(\mu_k + y_k) - 1)_{\sum_k y_k}}{\prod_k(\mu_k + y_k - 1)_{y_k}} \prod_k \nu_k^{y_k}, \end{aligned}$$

   where $(a)_\ell = a(a-1)\ldots(a-\ell+1)$ is the usual Pochhammer symbol.

   (b) Show by direct calculation that the function h_ν is harmonic for Y.

   (c) Let $P(y, z)$ be the transition matrix for Y. Show that

$$\frac{1}{h_\nu(y)} P(y, z) h_\nu(z)$$

   is also a transition matrix. What is the corresponding Markov chain?

Lecture 2

1. Suppose that E is a locally convex topological vector space and K is a non-empty, metrizable, compact, convex subset of E. Show that each probability measure μ on K has a unique barycenter.

2. Prove *Proposition 5:* Suppose that $x \in K$. Then x is an extreme point of K if and only if the point mass δ_x is the only probability measure on K with barycenter x.

3. Let $K := \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |y| \leq 1, z = 1\}$.

   (a) Show that there are points of K that are not unique convex combinations of the extreme points.

   (b) Show that $\tilde{K}$ is not a lattice.

   (c) Show that there exist $a', a'' \geq 0$ and $b', b'' \in \mathbb{R}^3$ such that $(a'K + b') \cap (a''K + b'')$ is non-empty but not of the form $aK + b$ for some $a \geq 0$ and $b \in \mathbb{R}^3$.

4. Let κ be the Martin kernel (with reference state b).

   (a) Show that

$$\kappa(i, j) = \frac{\mathbb{P}^i\{X \text{ hits } j\}}{\mathbb{P}^b\{X \text{ hits } j\}}.$$

(b) Show that
$$\kappa(i,j) \le \kappa(j,j) < \infty, \ \forall i,j.$$

(c) Show using the Strong Markov property that
$$\kappa(i,j) \le \kappa(i,i) < \infty, \ \forall i,j.$$

5. Show that if f is excessive (resp. regular) then $\{f(X_n)\}_{n=0}^{\infty}$ is a super-martingale (resp. martingale).

2 Martin Hairer: Convergence of Markov processes

Lecture 1

1. Use the Foster-Lyapunov criteria to show that a random walk on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d \ge 3$.

2. Think of suitable notions of transience, recurrence and positive recurrence in uncountable state-spaces. For example, generalise to $\mathbb{R}^n$ under the assumption that $\mathcal{P}(x, dy) = p(x, y)dy$ where p is continuous in both x and y.

3. Construct "counter-examples" (e.g. to the notion that recurrence and transience are "class properties") in the situation where this assumption does not hold. What should be the appropriate idea of irreducibility?

4. Find an example of a closed, bounded convex subset of a Banach space which has no extremal points.

Lecture 2

1. Prove that $|||\varphi|||_\beta = \inf_{C\in\mathbb{R}} \|\varphi + C\|_\beta$. *Hint:* $C = \inf_x\{1 + \beta V(x) - \varphi(x)\}$.

2. Prove Harris' theorem by working with probability measures instead of functions. (This is, in some sense, the "dual" proof.)

3. Prove Harris' theorem using a coupling argument. (The relevant distance here will not be be weighted total variation but the total variation distance itself.)

4. For the particle in a potential model, with $V(q) \sim |q|^{2k}$ for some $k \in [1/2, 1)$, write down the details to show that the control condition in Harris' theorem is satisfied with an appropriate Lyapunov function.