# Exercise Sheet 2

July 22, 2010

## 1 Steve Evans: Trickle-down growth models, Doob-Martin boundaries, and random matrices

### Lecture 3

1. Prove that if $\nu_1$ and $\nu_2$ are regular then $\nu_1 \wedge \nu_2$ is excessive. Use this to write down the details of the proof of Lemma 12 in the notes.

2. Suppose that $I = \mathbb{N}\times\mathbb{N}$, $\Pi((i,1),(i+1,1)) = 1/2$, $\Pi((i,1),(i,2)) = 1/2$, and $\Pi((i,j),(i,j+1)) = 1$ for $j \geq 2$.

   (a) Compute $\Gamma$ and $\kappa$ for this chain (there is only one possible choice of reference point $b$).

   (b) Describe the regular functions explicitly.

   (c) Compute explicitly the greatest lower bound of two regular functions in the intrinsic order.

### Lecture 4

1. Claim: every $f \in S$ may be written as

$$f = \int_F \kappa(\cdot,\xi)\nu(d\xi)$$

   for some probability measure $\nu$ on $\mathcal{B}(F)$. In the proof of this claim, show that in order to find a measure $\beta$ such that $0 < \Gamma\beta(i) < \infty$ for all $i$, it is sufficient to choose $\beta$ such that $\beta(j) > 0$ for all $j$ and $\sum \Gamma(j,j)\beta(j) < \infty$.

2. Recall the "comb chain" from Lecture 3 Question 2.

   (a) Find the set $F$.

   (b) Verify the "by hand" identification of the regular functions in this case.

   (c) Let $1 = \int_{F_e} \kappa(\cdot,\xi)\nu_1(d\xi)$ be the Martin representation of the constant function 1. What is $\nu_1$?

3. Consider the space-time coin-tossing example.

(a) Verify, using Stirling's formula, that the possible limits of $\kappa((m,n);(r,s))$ obtained by "sending $(r,s)$ to infinity" are given by letting $s \to \infty$ and $r/s \to t \in [0,1]$, and that this gives

$$\kappa((m,n);(s,t)) \to h_t(m,n) = 2^n t^m (1-t)^{n-m}.$$

(b) Check "bare-hands" that $h_t$ is regular.

4. Consider the Pólya sequence $\{X_n\}_{n=1}^{\infty}$ with values in a finite set $E$ for some parameter $\mu$. Put $Y_n = \sum_{k=1}^{n} \delta_{X_k}$ and $Y_0 := 0$.

   (a) Show that the points added in the Martin compactification may be identified with the space of probability measures on $E$ and the trace of the Martin topology on these points is homeomorphic to the usual topology of weak convergence of probability measures.

   (b) Show that with this identification, the $h$-transformed process associated with a probability measure $\nu$ is the process constructed in the same manner as $Y$ from an i.i.d. sequence with common distribution $\nu$.

# 2 Martin Hairer: Convergence of Markov processes

## Lecture 3

1. Show that, for the renewal process discussed in lectures, the rate of convergence to the invariant distribution is slower than $t^{-1}$.
   *Hint: The invariant measure is $dx/x^2$. Suppose that $1 < \alpha < 2$. Construct a Lyapunov function which behaves like $x^\alpha$. Applying the generator to it gives something less than a constant $K$. So $\mathbb{E}[X(t)^\alpha] \leq X(0)^\alpha + Kt$. For $\alpha > 1$, $x^\alpha$ isn't integrable with respect to the invariant measure. Use this to give an argument by contradiction.*

2. Apply the method used in lectures to diffusions

$$dx = -\nabla V(x)dt + dW$$

   such that $V$ is sublinear at infinity e.g. $V(x) \sim |x|^\alpha$ for some $\alpha \in (0,1)$.

   (a) Show that the rate of convergence is subexponential.

   (b) Harder: obtain lower bounds with the same power in the stretched exponential.

## Lecture 4

[No exercises.]