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Exotic derivatives under stochastic volatility models with jumps

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Abstract. In equity and foreign exchange markets the risk-neutral dynamics of the underlying asset are commonly represented by stochastic volatility models with jumps. In this paper we consider a dense subclass of such models and develop analytically tractable formulae for the prices of a range of first-generation exotic derivatives. We provide closed form formulae for the Fourier transforms of vanilla and forward starting option prices as well as a formula for the slope of the implied volatility smile for large strikes. A simple explicit approximation formula for the variance swap price is given. The prices of volatility swaps and other volatility derivatives are given as a one-dimensional integral of an explicit function. Analytically tractable formulae for the Laplace transform (in maturity) of the double-no-touch options and the Fourier-Laplace transform (in strike and maturity) of the double knock-out call and put options are obtained. The proof of the latter formulae is based on extended matrix Wiener-Hopf factorisation results. We also provide convergence results.

1.1 Introduction

A key step in the valuation and hedging of exotic derivatives in financial markets is to decompose these in terms of simpler securities, e.g. vanilla options, which trade in larger volumes, are generally very liquid and therefore have a well defined price. Such a decomposition is often achieved in two steps. First a model for the underlying asset under a risk neutral measure is calibrated to the implied volatility surface. In this step the current state of the market, as described by the prices of vanilla derivatives, is expressed in terms of the parameter values of the model. In other words the chosen model is used to impose a structure on the option prices. The second step consists of pricing the exotic derivative of interest in the calibrated model.

It is well known that in equity and foreign exchange markets stochastic volatility models with jumps can be used to accomplish the first step described above (see e.g. [17], [24]). In the present paper we consider

forward starting vanilla options, volatility derivatives and barrier options, which are among the most widely traded exotic derivatives in the equity and foreign exchange markets.

The desired properties of the model in each of the two steps described above place diametrically opposite restrictions on the choice of modelling framework. This is because in the calibration step one requires a flexible stochastic process that can describe well the current state of the vanilla market (i.e. can calibrate accurately to the observed implied volatility surface), while such flexibility can be a source of problems in the second step, where one needs to compute expectations of path-dependent functionals of the process. A more rigid modelling framework with, say, continuous trajectories and some distributional properties (e.g. independence of increments) could yield the structure of the process needed to establish efficient pricing algorithms for exotic derivatives.

We investigate two families of stochastic volatility models with jumps: the time-changed exponential Lévy models and the stochastic volatility models driven by Lévy processes, where the volatility process is independent of the Lévy driver. In these two families of models, the pricing of European derivatives is well understood and efficient calibration methods have been developed (see for example [12] and [24]), i.e. the first step of the two-stage procedure outlined above. However once the model is calibrated, the problem of pricing the first-generation exotic derivatives (e.g. barrier options) is quite involved. The law of the first-exit time from a bounded interval in stochastic volatility models with jumps, for instance, is not usually available in analytically tractable form. Because of the lack of structural properties that can be exploited to find the laws of the path-dependent functionals of interest, one would typically need to resort to Monte Carlo methods for the pricing of such derivatives in this setting. It is well-known that these methods are time-consuming and yield unstable results, especially when used to calculate the sensitivities of derivative securities. The method proposed in this paper to calculate the prices of such contracts consists of two steps: (i) a Markov chain approximation of the volatility process and (ii) an analytically tractable solution of the value function of the contract of interest in the approximating model. We provide proofs for the convergence of option prices under this approximation, and derive explicit expressions of Laplace and/or Fourier transforms of the value functions under the approximating model.

The approximating class of stochastic volatility processes with jumps considered in this paper retains the structural properties required for the semi-analytic pricing (i.e. up to an integral transform) of forward starting options, volatility derivatives and barrier options. In the case of double-barrier option prices we will show that the process considered here admits explicit formulae for the Laplace/Fourier transforms in terms of the solutions of certain quadratic matrix equations. The main mathematical contribution of the present paper, which underpins the derivation of these closed form formulae, is the proof of the existence and uniqueness of the matrix Wiener-Hopf factorisation of a class of complex valued matrices related to the approximating model (see Theorem 4). These results extend those of [21] where the corresponding results for the real-valued case are established. In the context of noisy fluid flow models the matrix Wiener-Hopf factorisation for the case of a regime-switching Brownian motion is studied by [4, 28].

It should be noted that the matrix Wiener-Hopf factorisation results developed in this paper can also be applied to the pricing of American call and put options in model (1.19) as follows. First we apply the main result in [21] to obtain the price of the perpetual American call or put and then, via the randomization algorithm introduced in [11], find the actual price of the option.

Related Markov chain mixture models, which are special cases of the model considered in this paper, have been studied before in the mathematical finance literature. In [19] and [22] explicit formulae were derived for the price of a perpetual American put option under a regime-switching Brownian motion model. The same process was used in [18] to model stochastic dividend rates where the problem of the pricing of barrier options on equity was considered. Finite maturity American put options were considered in [10] under a regime-switching Brownian motion model. More generally in [8], [7] numerical algorithms were developed in the case of regime-switching Lévy processes. Furthermore extensive work has been done on derivative pricing under stochastic volatility models with and without jumps (see the standard references [24], [17], [14] and [9]).

The remainder of the paper is organized as follows. In Section 1.2 we state and prove the properties of continuous-time Markov chains and phase-type distributions that are needed to define the class of stochastic volatility models with jumps studied in this paper. In Section 1.3 we give a precise definition of this class of models and describe an explicit construction of an approximating sequence of models, based on Markov chains, which converges to the given stochastic volatility model with jumps. The models are set against the backdrop of a foreign exchange market which allows us to include naturally the stochastic foreign and domestic discount factors. In Section 1.4 we provide explicit formulae for the Fourier transforms in model (1.19) for vanilla and forward starting options. This section also gives an approximate explicit formula for the pricing of variance swaps, a one-dimensional integral representation of the price of a volatility swap and formulae for the asymptotic behaviour of the implied volatility smile for large strikes. Section 1.5 is devoted to the first-passage times of regime-switching processes. Section 1.6 discusses the pricing of double-no-touch and double-barrier knock-out options. It provides a formula for the single Laplace transform (in maturity) and the Laplace-Fourier transform (in strike and maturity) of the double-no-touch and the double-barrier knock-out options respectively in terms of the quantity that can be obtained from the complex matrix Wiener-Hopf factorisation (see Theorem 6). Section 1.7 describes the fluid embedding of the model in (1.19), which plays a central role in the Wiener-Hopf factorisation. The key mathematical results of the paper, which allow us to price barrier options in the setting of stochastic volatility, are contained in Section 1.8 where matrix Wiener-Hopf factorisation is defined and the theorems asserting its uniqueness and existence are stated. The proofs of these (and other) results are contained in the appendix.

1.2 Markov chains and phase-type distributions

1.2.1 Finite state Markov chains

We start by collecting some useful and well-known properties of finite state Markov chains that will be important in the sequel (see e.g. [16]). For completeness we will also present the proofs. Throughout the paper we will denote by

$$M(i,j) = M_{ij} = e_i' M e_j, \qquad m(j) = m_j = m' e_j, \qquad i, j = 1, \ldots, n,$$

the ijth element of an $n \times n$ matrix M and the jth element of an n-dimensional vector m, where the vectors e_i, $i = 1, \ldots, n$, denote the standard basis of $\mathbb{C}^n$ and where $'$ means transposition. Throughout the paper I will denote an identity matrix of appropriate size, and $\mathbb{R}_+ = [0, \infty)$ the non-negative real line.

Lemma 1 *Let Z be a Markov chain on a state space $E^0 := \{1, \ldots, N_0\}$, where $N_0 \in \mathbb{N}$, and let $B : E^0 \to \mathbb{C}$ be any function. If Q denotes the generator of Z and Λ_B is a diagonal matrix of size N_0 with diagonal elements equal to $B(i)$, $i = 1, \ldots, N_0$, then it holds that*

$$\mathbb{E}_i \left[\exp \left(\int_0^t B(Z_s) ds \right) I_{\{Z_t = j\}} \right] = \exp \left(t(Q + \Lambda_B) \right)(i,j) \qquad \text{for any} \quad i, j \in E^0, \ t \geq 0, \qquad (1.1)$$

where $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | Z_0 = i]$, $\mathbb{P}_i[\cdot] = \mathbb{P}[\cdot | Z_0 = i]$, and $I_{\{\cdot\}}$ is the indicator of the set $\{\cdot\}$.

Proof. Let $(P_t)_{t \geq 0}$ be a family of N_0-dimensional square matrices with entries $P_t(i,j)$, $i, j = 1, \ldots, N_0$, given by the left-hand side of (1.1). It is clear that $P_0 = I$, where I is the N_0-dimensional identity matrix. The Markov property of the chain Z yields the Chapman-Kolmogorov equation $P_{t+s} = P_s P_t = P_t P_s$ for all $s, t \geq 0$. If we show that the family of matrices $(P_t)_{t \geq 0}$ satisfies the system of ODEs with constant coefficients

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = (Q + \Lambda_B) P_t, \qquad P_0 = I, \qquad (1.2)$$

then the lemma will follow, since equation (1.2) is well-known to have a unique solution given by the right-hand side of (1.1). The Chapman-Kolmogorov equation implies that $P_{t+h} - P_t = (P_h - I)P_t$ and it is therefore enough to show $\lim_{h \to 0}(P_h - I)/h = Q + \Lambda_B$. In other words we need to prove

$$\lim_{h \to 0} (P_h(i,j) - I(i,j))/h = \begin{cases} Q(i,j) & \text{if } i \neq j, \\ Q(i,j) + B(j) & \text{if } i = j. \end{cases} \qquad (1.3)$$

The random variables $B(Z_s)$ are bounded uniformly in s and hence the Taylor expansion of the exponential yields

$$P_h(i,j) = \mathbb{E}_i \left[I_{\{Z_h = j\}} \left(1 + \int_0^h B(Z_s) \mathrm{d}s \right) \right] + o(h) \qquad \text{for all} \quad i, j \in \{1, \ldots N_0\}.$$

It is clear that

$$\lim_{h \to 0} I_{\{Z_h = j\}} \frac{1}{h} \int_0^h B(Z_s) \mathrm{d}s = I_{\{Z_0 = j\}} B(Z_0) \quad \mathbb{P}_i\text{-a.s.} \qquad \text{for all} \quad i, j \in \{1, \dots N_0\},$$

since the paths of Z are $\mathbb{P}_i$-a.s constant for exponentially distributed amount of time. The dominated convergence theorem and the well-known fact $\mathbb{E}_i \left[I_{\{Z_h = j\}} \right] = h(I(i,j) + Q(i,j)) + o(h)$ therefore imply (1.3). This concludes the proof. $\qquad\square$

We now apply Lemma 1 to esatblish a simple but important property of the spectrum of a discounted generator.

Lemma 2 *Let Q be a generator of a Markov chain with $N_0 \in \mathbb{N}$ states and let D be a complex diagonal matrix of dimension N_0. Then every eigenvalue $\lambda \in \mathbb{C}$ of the matrix $Q - D$ (i.e. a solution of the equation $(Q - D)x = \lambda x$ for some non-zero element x in $\mathbb{C}^{N_0}$) satisfies the inequality*

$$\Re(\lambda) \leq - \min \left\{ \Re(d_i) \ : \ i = 1, \dots, N_0 \right\},$$

where $d_i = D(i,i)$, $i = 1, \dots, N_0$, are diagonal elements of D. In particular if $\min \left\{ \Re(d_i) \ : \ i = 1, \dots, N_0 \right\} > 0$, then the matrix $Q - D$ is invertible. Furthermore, the real part of every eigenvalue of Q is non-positive.

Proof. Let λ be an eigenvalue of the matrix $Q - D$ that corresponds to the eigenvector $x \in \mathbb{C}^{N_0}$. Then x is also an eigenvector with eigenvalue $\exp(\lambda)$ of the matrix $\exp(Q - D)$. Lemma 1 implies that if Z is the chain generated by Q then the following identity holds

$$e_i' \exp(Q - D)x \ = \ \sum_{j=1}^{N_0} x_j \mathbb{E}_i \left[\exp \left(- \int_0^1 d_{Z_t} dt \right) I_{\{Z_1 = j\}} \right], \quad i = 1, \dots, N_0, \tag{1.4}$$

where e_i (resp. d_i) denotes the i-th basis vector in $\mathbb{C}^{N_0}$ (resp. diagonal element of the matrix D).

Assume now without loss of generality that the norm $\|x\|_\infty := \max\{|x_i| \ : \ i = 1, \dots, N_0\}$ of the vector x is one. Then identity (1.4) implies the estimate

$$\exp(\Re(\lambda)) = |\exp(\lambda)| = \| \exp(Q - D)x \|_\infty \leq \exp(- \min \left\{ \Re(d_i) \ : \ i = 1, \dots, N_0 \right\})$$

which proves the lemma. $\qquad\square$

Lemma 3 *Let $q \in \mathbb{C}$ be such that $\Re(q) > 0$ and M a matrix whose eigenvalues all have non-positive real part. Then the matrix $qI - M$ is invertible and the following formula holds*

$$\int_0^\infty \mathrm{e}^{-qt} \exp(tM) \mathrm{d}t = (qI - M)^{-1}. \tag{1.5}$$

Proof. The following identity holds for any $T \in (0, \infty)$ by the fundamental theorem of calculus

$$\int_0^T \exp((M - qI)t) \mathrm{d}t = (M - qI)^{-1} \left(\exp((M - qI)T) - I \right)$$

and, since the real part of the spectrum of the matrix $M - qI$ is strictly negative by Lemma (2), in the limit as $T \to \infty$ we obtain

$$\int_0^\infty \exp((M - qI)t) \mathrm{d}t = (qI - M)^{-1}.$$

$\qquad\square$

1.2.2 (Double) phase-type distributions

In this section we review basic properties of phase-type distributions, as these will play an important role in the sequel. We refer to Neuts [27] and Asmussen [2] for further background on phase-type distributions.

A distribution function $F : \mathbb{R}_+ \to [0, 1]$ is called *phase-type* if it is a distribution of the absorption time of a continuous-time Markov chain on $(m + 1)$ states, for some $m \in \mathbb{N}$, with one state absorbing and the remaining states transient. The distribution F is uniquely determined by the matrix $A \in \mathbb{R}^{m \times m}$, which is the generator of the chain restricted to the transient states, and the initial distribution of the chain on the transient states $\alpha \in \mathbb{R}^m$ (i.e. the coordinates of α are non-negative and the inequalities $0 \leq \alpha'\mathbf{1} \leq 1$ hold, where $\mathbf{1}$ is the m-dimensional vector with each coordinate equal to one and $'$ denotes transposition). The notation $X \sim PH(\alpha, A)$ is commonly used for a random variable X with cumulative distribution function F. Note also that the law of the original chain on the entire state space is given by

$$\text{the initial distribution} \quad \begin{pmatrix} \alpha \\ 1 - \alpha'\mathbf{1} \end{pmatrix} \quad \text{and the generator matrix} \quad \begin{pmatrix} A & (-A)\,\mathbf{1} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $\mathbf{0}$ denotes a row of m zeros. It is clear from this representation that the cumulative distribution function F and its density f are of the form

$$F(t) = 1 - \alpha' e^{tA} \mathbf{1} \quad \text{and} \quad f(t) = -\alpha' e^{tA} A\mathbf{1} \quad \text{for any} \quad t \in \mathbb{R}_+. \tag{1.6}$$

Note also that 0 is an atom of the distribution if and only if $\alpha'\mathbf{1} < 1$ in which case the function f is a densitiy of a sub-probability measure on $(0, \infty)$. The n-th moment of the random variable $X \sim PH(\alpha, A)$ is given by

$$\mathbb{E}[X^n] = n!\, \alpha'\, (-A)^{-n}\, \mathbf{1}.$$

It follows from the definition the phase-type distribution that the matrix A can be viewed as a generator of a killed continuous-time Markov chain on m states. Therefore we can express the matrix A as $A = Q - D$, where Q is the generator of a chain on m states and D is a diagonal matrix with non-negative diagonal elements that are equal to the coordinates of the vector $-A\mathbf{1}$. Lemma 2 therefore implies that the real part of each eigenvalue of A is non-positive. The next proposition gives a characterisation of the existence of exponential moments of a phase-type distribution in terms of the eigenvalues of the matrix A.

Proposition 1 *Let $X \sim PH(\alpha, A)$ be a phase-type random varaible as defined above and let λ_0 be the eigenvalue of the matrix A with the largest real part, i.e. $\Re(\lambda_0) = \max\{\Re(\lambda) : \lambda \text{ eigenvalue of } A\}$. Then, for any $u \in \mathbb{C}$, the exponential moment $\mathbb{E}[\exp(uX)]$ exists and is finite if and only if $\Re(u) < -\Re(\lambda_0)$ in which case the following formula holds*

$$\mathbb{E}[\exp(uX)] = \alpha'(A + uI)^{-1}A\mathbf{1} + (1 - \alpha'\mathbf{1}),$$

where I denotes an m-dimensional dentity matrix.

Proof. It is clear that the identity

$$\mathbb{E}[\exp(uX)] = \mathbb{P}(X = 0) + \int_0^\infty \exp(tu)f(t)dt \tag{1.7}$$

must hold for all $u \in \mathbb{R}$ where f is the density of X on the interval $(0, \infty)$. Hence the question of existence of $\mathbb{E}[\exp(uX)]$ is equivalent to the question of convergence of the integral. Using Formula (1.6) for the density f, the fact $\exp(t(A + uI)) = \exp(tA)\exp(tu)$ for all $u \in \mathbb{C}$ and the Jordan canonical decomposition of the matrix A we can conclude that

$$\mathbb{E}[|\exp(uX)|] < \infty \quad \Longleftrightarrow \quad -\alpha'\left(\int_0^\infty \exp((A + \Re(u)I)t)dt\right)A\mathbf{1} < \infty \quad \Longleftrightarrow \quad \Re(\lambda_0 + u) < 0,$$

where λ_0 is as defined above. This proves the equivalence in the proposition.

Note that the condition $\Re(u) < -\Re(\lambda_0)$ implies, by Lemma (2), that the matrix $A + uI$ is invertible. For any $T \in \mathbb{R}_+$ the fundamental theorem of calculus therefore yields the matrix identity

$$\int_0^T \exp((A + uI)t)dt = (A + uI)^{-1}[\exp((A + uI)T) - I]. \tag{1.8}$$

Since all the eigenvalues of $A + uI$ have a strictly negative real part, it follows from Jordan canonical decomposition of $A + uI$ that $\lim_{T\to\infty} \exp((A + uI)T) = 0$. Therefore identities (1.7) and (1.8) conclude the proof of the proposition. $\qquad\square$

More generally, a *double phase-type* jump distribution $DPH(p, \beta^+, B^+, \beta^-, B^-)$ is defined to have density

$$f(x) := pf^+(x)I_{(0,\infty)}(x) + (1 - p)f^-(-x)I_{(-\infty,0)}(x) \quad \text{such that} \tag{1.9}$$
$$p \in [0, 1], \quad f^\pm \sim PH(\beta^\pm, B^\pm), \quad f^\pm(x) = -(\beta^\pm)'e^{xB^\pm}B^\pm\mathbf{1} \quad \text{and} \quad \mathbf{1}'\beta^\pm = 1,$$

where the phase-type distributions $PH(\beta_i^\pm, B_i^\pm)$ are as described above, $\mathbf{1}$ is a vector of the appropriate size with all coordinates equal to 1 and as usual I_A denotes the indicator of the set A. The condition $\mathbf{1}'\beta_i^\pm = 1$ ensures that the distribution of jump sizes has no atom at zero.

The class of double phase-type distributions is vast. Not only does it contain double exponential distributions

$$f_i(x) := p_i\alpha_i^+e^{-x\alpha_i^+}I_{(0,\infty)}(x) + (1 - p_i)\alpha_i^-e^{x\alpha_i^-}I_{(-\infty,0)}(x) \quad \text{where} \quad \alpha_i^\pm > 0 \text{ and } p_i \in [0, 1], \tag{1.10}$$

mixtures of double exponential distributions and Erlang distributions but this class is in fact dense in the sense of weak convergence in the space of all probability distributions on $\mathbb{R}$.

Proposition 2 *Let F be a probability distribution function on $\mathbb{R}$. Then there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of double-phase-type distributions F_n such that $F_n \Rightarrow F$ as $n \to \infty$.*[3]

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[3] We write $F_n \Rightarrow F$ for a sequence of distribution functions F_n and a distribution function F if F_n converges in distribution to F, that is, $\lim_{n\to\infty} F_n(x) = F(x)$ for all x where F is continuous.

This result directly follows from the three observations that (a) any probability distribution on the real line can be approximated in distribution arbitrarily closely by a random variable taking only finitely many values and (b) any constant random variable is the limit in distribution of Erlang or the negative of Erlang random variables, and (c) a mixture of Erlang distributions is a phase-type distribution.

An important property of exponential distributions is the lack-of-memory property, which can be generalised to stopping times as follows:

Lemma 4 *Let $(\mathcal{F}_t)_{t\geq 0}$ a filtration and let ρ be any stopping time[4] with respect to this filtration. Let $\mathbf{e}_q$ be an exponentially distributed random variable with parameter $q > 0$ which is independent of the σ-algebra generated by $\cup_{t\geq 0}\mathcal{F}_t$. Then the equality*

$$\mathbb{E}\left[I_{\{\rho < \mathbf{e}_q\}}\exp(-\lambda(\mathbf{e}_q - \rho))\Big|\mathcal{F}_\rho\right] = \frac{q}{\lambda + q}\mathrm{e}^{-q\rho}\quad \textit{holds for all}\quad \lambda \geq 0$$

and hence the positive random variable $\mathbf{e}_q - \rho$ defined on the event $\{\rho < \mathbf{e}_q\}$ is, conditional on $\mathcal{F}_\rho$, exponentially distributed with parameter q.

Remarks. (i) This lemma can be viewed as a generalisation of the lack of memory property,

$$\mathbb{P}(\mathbf{e}_q > t + s|\mathbf{e}_q > s) = \mathbb{P}(\mathbf{e}_q > t),$$

of the exponential random variable $\mathbf{e}_q$ when the constant time s is substituted by a stopping time ρ. Note also that it follows from the lemma that the conditional probability of the event $\{\rho < \mathbf{e}_q\}$ equals

$$\mathbb{P}(\mathbf{e}_q > \rho|\mathcal{F}_\rho) = \exp(-q\rho).$$

(ii) Phase-type distributions enjoy a similar property that can be seen as a generalisation of the lack-of-memory of the exponential distribution. More specifically, let T follow a $\mathrm{PH}(\alpha, B)$ distribution independent of the σ-algebra generated by $\cup_{t\geq 0}\mathcal{F}_t$. Than for any stopping time ρ with respect to $(\mathcal{F}_t)_{t\geq 0}$, the random variable $T - \rho$ defined on the event $\{\rho < T\}$, conditional on $\mathcal{F}_\rho$, is $\mathrm{PH}(\alpha_\rho, B)$ distributed where

$$\alpha_\rho = (\alpha'\mathrm{e}^{\rho B}\mathbf{1})^{-1}\alpha' \exp\{\rho B\},$$

since the identity

$$\mathbb{E}\left[I_{\{\rho < T\}}\exp(-\lambda(T - \rho))\Big|\mathcal{F}_\rho\right] = \alpha' \exp{(\rho B)}{(B - \lambda I)}^{-1}B\mathbf{1}$$

holds for all $\lambda \geq 0$. This follows by the same argument as in the proof of Lemma 4. Furthermore we have the following expression for the conditional probability of the event $\{\rho < T\}$:

$$\mathbb{P}(T > \rho|\mathcal{F}_\rho) = \alpha' \exp{(\rho B)}\mathbf{1}.$$

Proof. The following direct calculation based on Fubini's theorem, which is applicable since all the functions are non-negative,

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[4] By definition the stopping time ρ takes values in $[0, \infty]$ and satisfies the condition $\{\rho \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)$. The σ-algebra $\mathcal{F}_\rho$ consists of all events A such that $A \cap \{\rho \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)$.

$$\mathbb{E}\left[I_A I_{\{\rho < \mathbf{e}_q\}} \exp(-\lambda(\mathbf{e}_q - \rho))\right] = \mathbb{E}\left[I_A I_{\{\rho < \infty\}} e^{\lambda\rho} q \int_\rho^\infty e^{-(\lambda+q)t} dt\right] = \mathbb{E}\left[I_A \frac{q}{\lambda+q} e^{-q\rho}\right], \quad \text{where} \quad A \in \mathcal{F}_\rho,$$

proves the identity in the lemma for all non-negative λ. Since the Laplace transform uniquely determines the distribution of a random variable the lemma follows. $\square$

1.3 Stochastic volatility models with jumps

We next describe in detail the two classes of stochastic volatility models with jumps that we will consider.

Let $v = \{v_t\}_{t \geq 0}$ be a Markov process that takes positive values, modelling the underlying stochastic variance, and let X be a Lévy process[5] which drives the noise in the log-price process. The processes are taken to be mutually independent and are both defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The law of X is determined by its characteristic exponent $\psi : \mathbb{R} \to \mathbb{R}$ which is according to the Lévy-Khintchine formula given by

$$\mathbb{E}[e^{iuX_t}] = e^{t\psi(u)} \tag{1.11}$$

with

$$\psi(u) = icu - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^\infty [e^{iux} - 1 - iuxI_{\{|x| \leq 1\}}]\nu(dx), \tag{1.12}$$

where $\sigma^2 \geq 0$ and c are constants and ν is the Lévy measure that satisfies the integrability condition $\int_\mathbb{R}(1 \wedge x^2)\nu(dx) < \infty$. The triplet (c, σ^2, ν) is also called the characteristic triplet of X.

To guarantee that the option prices be finite we impose the usual restriction that X admits (positive) exponential moments; more precisely, we assume that for some $p > 1$

$$\int_1^\infty e^{px}\nu(dx) < \infty, \tag{1.13}$$

which implies that $\mathbb{E}[e^{pX_t}] < \infty$ for all $t \geq 0$. In this case the identity (1.11) remains valid for all u in the strip $\{u \in \mathbb{C} : \Im(u) \in (-p, 0]\}$ in the complex plan, where the function ψ is analytically extended to this strip.

In this setting a candidate stochastic volatility process with jumps $S = \{S_t\}_{t \in [0,T]}$, where $T > 0$ denotes a maturity or time-horizon, is given by

$$S_t := S_0 \exp\left((r-d)t + \int_0^t \sqrt{v_u}dX_u - \int_0^t \psi(-i\sqrt{v_s})ds\right), \qquad S_0 = s > 0, \tag{1.14}$$

where r and d are the instantaneous interest rate and dividend yield respectively. Here we assume that the variance process v satisfies the following integrability condition:

$$\int_0^T |\psi(-i\sqrt{v_s})|ds < \infty \quad a.s. \tag{1.15}$$

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[5] A Lévy process $X = \{X_t\}_{t \geq 0}$ is a stochastic process that has independent and stationary increments, and has right-continuous paths with left-limits with $X_0 = 0$.

It is easy to see by conditioning on the filtration generated by the variance process v that the integrability condition in (1.15) implies the martingale property of the discounted process $\{\mathrm{e}^{-(r-d)t}S_t\}_{t\in[0,T]}$.

Note that if we take for example X to be a Brownian motion with drift and v an independent square-root process, the process S reduces to a Heston model with zero correlation between the driving Brownian factors (see e.g. [17]). The class of models described by (1.14) is quite flexible, and contains for example the stochastic volatility models with jumps described in Lipton [24], as long as there is no correlation between the driving Brownian motions.

A related class of models that has been proposed in the literature is the one where the effect of stochasticity of volatility is achieved by randomly changing the time-scale (see e.g. Carr et al. [12]); in the setting above the price process $\{S_t\}_{t\in[0,T]}$ is defined by

$$S_t := S_0 \exp\left((r - d)t + X_{V_t} - \psi(-\mathtt{i})V_t\right), \qquad S_0 = s > 0, \qquad \text{where} \quad V_t := \int_0^t v_u \mathrm{d}u, \qquad (1.16)$$

and we assume that v satisfies the integrability condition

$$V_T < \infty \quad a.s. \tag{1.17}$$

Also in this case the discounted process $\{\mathrm{e}^{-(r-d)t}S_t\}_{t\in[0,T]}$ is a martingale.

It is clear from the definitions that in the case where X is a Brownian motion with drift, the classes of models in (1.14) and (1.16) coincide, due to the scaling property of Brownian motion. Whereas the effect of the variance process v on the Brownian motion with drift is the same in both classes of models, the effect of the process v on the behaviour of jumps is different. In (1.16) the Markov process v modulates only the intensity of the jumps of X while in model (1.14) the volatility scales the distribution of size of the jumps but does not affect the intensity.

In the next section we will describe a modelling framework in which any model in the classes given by (1.14) and (1.16) can be approximated. The approximation in Section 1.3.2 retains the structural properties required for the semi-analytic pricing (i.e. up to an integral transform) of barrier options, forward starting options and volatility derivatives.

1.3.1 A class of regime-switching models

Let the set $E^0 := \{1, \ldots, N_0\}$ be the state-space of a continuous-time Markov chain $Z = (Z_t)_{t\geq0}$ and let the the process $W = (W_t)_{t\geq0}$ denote a standard Brownian motion which is independent of the chain Z. For each $i \in E^0$ let the process $J^i := (J^i_t)_{t\geq0}$ be a compound Poisson process with intensity $\lambda_i \geq 0$ and jump-sizes distributed according to a double-phase-type distribution $DPH(p_i, \beta_i^+, B_i^+, \beta_i^-, B_i^-)$. In particular the jump-size distributions have no atom at zero, i.e. $(\beta_i^+)'\mathbf{1} = 1$ for all $i \in E^0$ such that $\lambda_i p_i > 0$ and analogously for β_i^-. Assume further that the processes J^i are mutually independent as well as independent from the Brownian motion W and the chain Z.

In this setting consider the following model for the underlying price process $S = (S_t)_{t\geq0}$, the (domestic) money market account $B^D = (B^D_t)_{t\geq0}$ and the cumulative dividend yield $B^F = (B^F_t)_{t\geq0}$:

$$B_t^D := \exp\left(\int_0^t R_D(Z_s)ds\right), \quad B_t^F := \exp\left(\int_0^t R_F(Z_s)ds\right), \quad S_t := \exp(X_t), \tag{1.18}$$

where

$$X_t := x + \int_0^t \mu(Z_s)ds + \int_0^t \sigma(Z_s)dW_s + \sum_{i \in E^0} \int_0^t I_{\{Z_s=i\}}dJ_s^i. \tag{1.19}$$

In the case of the Foreign Exchange market the process B^F can be interpreted as a foreign money market account. The point $x \in \mathbb{R}$ is the starting value of the process X and $R_D, R_F, \mu, \sigma : E^0 \to \mathbb{R}$ are given real-valued functions on E^0 such that R_D, R_F are non-negative and σ is strictly postitive. To price derivatives in our model, we need to understand the law of the Markov process (X, Z), which is determined by the characteristic matrix exponent K, defined as follows.

Definition. The *characteristic matrix exponent* $K : \mathbb{R} \to \mathbb{C}^{N_0 \times N_0}$ of (X, Z) is given by

$$K(u) = Q + \Lambda(u),$$

where Q denotes the generator of the chain Z and, for $u \in \mathbb{R}$, $\Lambda(u)$ is a diagonal matrix of size $N_0 \times N_0$, where the i-th diagonal element equals the characteristic exponent of the process X in regime i, given by

$$\psi_i(u) := \mathtt{i}u\mu_i - \sigma_i^2 u^2/2 + \lambda_i \left[p_i(\beta_i^+)'(B_i^+ + \mathtt{i}uI)^{-1}B_i^+\mathbf{1} + (1-p_i)(\beta_i^-)'(B_i^- - \mathtt{i}uI)^{-1}B_i^-\mathbf{1} - 1 \right]. \tag{1.20}$$

where I and $\mathbf{1}$ are an identity matrix and a vector with all coordinates equal to one of the appropriate dimensions.

Remarks. (i) Note that the functions ψ_i defined in (1.20) can be analytically extended to the strip in the complex plane $\Im(u) \in (-\alpha_i^+, \alpha_i^-)$ where

$$\alpha_i^\pm := \min\{-\Re(\lambda) : \lambda \text{ eigenvalue of } B_i^\pm\} \quad \text{for any state} \quad i \in E^0. \tag{1.21}$$

(ii) In the special case that the jumps follow a double exponential distribution the diagonal elements of the matrix $\Lambda(u)$ take the simpler form

$$\psi_i(u) = \mathtt{i}u\mu_i - \sigma_i^2 u^2/2 + \lambda_i p_i \left(\frac{\beta_i^+}{\beta_i^+ - \mathtt{i}u} - 1 \right) + \lambda_i(1-p_i) \left(\frac{\beta_i^-}{\beta_i^- + \mathtt{i}u} - 1 \right),$$

where $\beta_i^\pm$ and p_i are the parameters of the double exponential distribution.

(iii) Throughout the paper we will use $\mathbb{E}_{x,i}[\cdot]$ to denote the conditional expectation $\mathbb{E}[\cdot|X_0 = x, Z_0 = i]$ and on occasion $\mathbb{E}_i[\cdot]$ to represent $\mathbb{E}_{0,i}[\cdot]$.

We now define two matrices the will play an important role in the sequel.

Definition. The *discount rate matrix* Λ_D is the diagonal matrix with elements $\Lambda_D(i, i) := R_D(i)$ where $i \in E^0$. The *dividend yield matrix* Λ_F is the diagonal matrix given by $\Lambda_F(i, i) := R_F(i)$ for $i \in E^0$.

Theorem 1 *The discounted characteristic function of the Markov process (X, Z) is given by the formula*

$$\mathbb{E}_{x,i}\left[\frac{\exp(\mathrm{i}uX_t)}{B_t^D}I_{\{Z_t=j\}}\right] = \exp(\mathrm{i}ux)\cdot\exp(t(K(u)-\Lambda_D))(i,j) \tag{1.22}$$

for all $u \in \mathbb{R}$.

Remarks. (i) The left-hand side is finite for all $u \in \mathbb{C}$ in the strip $\Im(u) \in (-\alpha_*^+, \alpha_*^-)$ where

$$\alpha_*^+ = \min\{\alpha_k^+ : \lambda_k p_k > 0, k \in E^0\} \quad \text{and} \quad \alpha_*^- = \min\{\alpha_k^+ : \lambda_k(1-p_k) > 0, k \in E^0\}, \tag{1.23}$$

the quantities $\alpha_i^{\pm}$ are defined in (1.21) and the minimum over the empty set is taken to be $+\infty$. It follows, by analytical continuation, that the identity (1.22) remains valid for all u in this strip. Furthermore, the slope of the implied volatility smile in model (1.19) is determined by α_*^+ and α_*^- (see Subsection 1.4.2).

(ii) The Markov property and Theorem 1 imply that the process $\{S_t B_t^F/B_t^D\}_{t\geq 0}$ is a martingale if the following two conditions hold:

$$1 < \alpha_k^+, \quad \text{for all } k \in E^0 \text{ such that } \lambda p_k > 0, \tag{1.24}$$

$$\Lambda(-\mathrm{i}) = \Lambda_D - \Lambda_F. \tag{1.25}$$

Condition (1.24) ensures that $\mathbb{E}_{i,x}[S_T]$ is finite for all $T \geq 0$ and hence by Theorem 1 takes the form $\mathbb{E}_{i,x}[S_T] = \mathrm{e}^x\left[\exp\left(TK(-\mathrm{i})\right)\mathbf{1}\right](i)$. The equality in (1.25) guarantees that S has instantaneous drift given by the rates $\Lambda_D - \Lambda_F$. Any model from the class (1.18)–(1.19) that satisfies conditions (1.24) and (1.25) can be taken as a specification of the price process of the risky-asset under a pricing measure. From now on we assume that model (1.19) is specified under the pricing measure given by condition (1.24)–(1.25).

(iii) For later reference we record that, under a pricing measure, the price at time s of a zero coupon bond maturing at time $t \geq s$ is given by

$$\mathbb{E}_i\left[\frac{1}{B_t^D}\bigg|\mathcal{F}_s^{(X,Z)}\right] = \frac{1}{B_s^D}\cdot\left(\exp((t-s)(Q-\Lambda_D))\mathbf{1}\right)(Z_s), \tag{1.26}$$

where $\mathcal{F}_s^{(X,Z)} = \sigma\{(X_u, Z_u)\}_{u\leq s}$ denotes the standard filtration generated by (X, Z). In particular, at time 0 the price is given by

$$\mathbb{E}_i\left[\left(B_t^D\right)^{-1}\right] = \left(\exp(t(Q-\Lambda_D))\mathbf{1}\right)(i).$$

(iv) The infinitesimal generator $\mathcal{L}$ of the Markov process (X, Z) acts on sufficiently smooth functions[6] $f : \mathbb{R} \times E^0 \to \mathbb{R}$ as

$$\mathcal{L}f(x,i) = \frac{\sigma^2(i)}{2}f''(x,i) + \mu(i)f'(x,i) + \lambda(i)\left[\int_{\mathbb{R}} f(x+z,i)g_i(z)\mathrm{d}z - f(x,i)\right]$$
$$+ \sum_{j\in E^0} q_{ij}[f(x,j) - f(x,i)], \tag{1.27}$$

---

[6] For example, functions f with $f(\cdot, i) \in C_c^2(\mathbb{R})$ for $i \in E^0$, where $C_c^2(\mathbb{R})$ are the twice continuously differentiable functions with compact support.

where g_i is the density of the double phase-type distribution $DPH(p_i, \beta_i^+, B_i^+, \beta_i^-, B_i^-)$, q_{ij} is the ijth element of Q and $'$ denotes differentiation with respect to x.

(v) For a specific regime-switching model (namely the case where the Markov chain Z has two states only) the calibration is studied in [26].

Proof. It is clear from the definition of (X, Z) that it is a Markov process. Let $\mathcal{F}_t^Z := \sigma(Z_s : s \in [0, t])$ be the σ-algebra generated by the chain Z up to time t. Since the compound Poisson processes and Brownian motion in model (1.19) are mutually independent as well as independent of the σ-algebra $\mathcal{F}_t^Z$, it is easy to see that by conditioning on $\mathcal{F}_t^Z$ for any $i \in E^0$ we obtain

$$\mathbb{E}_{x,i}\left[\exp(uX_t)|\mathcal{F}_t^Z\right] = \exp\left(ux + u\int_0^t \mu(Z_s)ds + \frac{u^2}{2}\int_0^t \sigma(Z_s)^2 ds + \int_0^t \nu(Z_s, u)ds\right), \quad (1.28)$$

$$\nu(i, u) := \lambda_i \left[\mathbb{E}[\exp(uJ_i)] - 1\right]$$

$$= \lambda_i \left[p_i(\beta_i^+)'(B_i^+ + uI)^{-1}B_i^+\mathbf{1} + (1 - p_i)(\beta_i^-)'(B_i^- - uI)^{-1}B_i^-\mathbf{1} - 1\right],$$

where the random variable J_i denotes the size of jumps of the compound Poisson process J^i. The last equality in this calculation is a consequence of the choice (1.9) of the distribution of jump sizes and Proposition 1. Therefore the complex number u must be contained in all intervals $(-\alpha_k^-, \alpha_k^+)$, $k \in E^0$, where $\alpha_k^\pm$ are defined in Theorem 1. The identity in (1.28) holds more generally for any jump-distribution that admits a moment generating function. The well-known identity from the theory of Markov chains given in Lemma 1 can now be applied to obtain the expectations of the expressions on both sides of (1.28). This concludes the proof of Theorem 1. $\qquad \square$

From Theorem 1 one may obtain an explicit expression for the marginal distributions of (X, Z) by inverting the Fourier transform (1.22):

Proposition 3 *For any $T > 0$, the joint distribution $q_T^{x,i}(y, j) = \frac{d}{dy}\mathbb{P}_{x,i}[X_T \leq y, Z_T = j]$ of (X_T, Z_T) is given by*

$$q_T^{x,i}(y, j) = \frac{1}{2\pi}\int_\mathbb{R} e^{\mathrm{i}\xi(x-y)}\exp\left(K\left(\xi\right)T\right)(i, j)\,d\xi, \quad y \in \mathbb{R}, i, j \in E^0, \quad (1.29)$$

In particular, X_T is a continuous random variable with probability density function $q_T^{x,i}(y) = \frac{\mathbb{P}_{x,i}[X_T \in dy]}{dy}$ given by

$$q_T^{x,i}(y) = \frac{1}{2\pi}\int_\mathbb{R} e^{\mathrm{i}\xi(x-y)}\left[\exp\left(K\left(\xi\right)T\right)\mathbf{1}\right](i)\,d\xi, \quad y \in \mathbb{R}, i \in E^0. \quad (1.30)$$

Proof. It is well-known that a probability law on the real line $\mathbb{R}$ has a density with respect to the Lebesgue measure if its characteristic function is in $L^1(\mathbb{R})$. The characteristic function of X_T by Theorem 1 equals

$$\mathbb{E}_{x,j}\left[\exp(\mathrm{i}\xi X_T)\right] = e^{\mathrm{i}\xi x}[\exp(K(\xi)T)\mathbf{1}](j). \quad (1.31)$$

We now show that this characteristic function is asymptotically equal to $\exp(-c\xi^2)$, as $|\xi| \to \infty$, for some positive constant c. Note first that the volatility vector σ in model (1.19) has non-zero coordinates by

assumption and the spectra of matrices $B_i^{\pm}$, $i = 1, \ldots, N_0$, do not contain any points of the form $\mathtt{i}\xi$, for $\xi \in \mathbb{R}$, by Lemma 2. Therefore the functions $\xi \mapsto \psi_i(\xi)$, $i = 1, \ldots, N_0$, defined in (1.20) are asymptotically equal to downward facing parabolas. A further application of Lemma 2 implies that the characteristic function has the desired asymptotic behaviour. This further implies that the density $q_T^{x,i}$ exists and is given by the inversion formula (1.30).

Theorem 1 and a similar argument to the one outlined in the previous paragraph imply that the function

$$\xi \mapsto e^{\mathtt{i}\xi x} \exp(TK(\xi))(i,j) = \mathbb{E}_{x,i}\left[\exp(\mathtt{i}\xi X_T) I_{\{Z_T = j\}}\right]$$

is in $L^1(\mathbb{R})$ and that the two equalities hold. Therefore the Fourier inversion formula is valid and the identity in (1.29) follows.   □

1.3.2 Two step approximation procedure

The construction of a regime-switching Lévy process with jump sizes distributed according to a double phase-type distribution that approximates a given stochastic volatility process with jumps from either of the two-classes (1.14) and (1.16) takes place in two steps:

(i) Approximation of the variance process v by a finite-state continuous-time Markov chain and
(ii) Approximation of the Lévy process X by a Lévy process with double-phase-type jumps.

By approximating the variance process by a finite state Markov chain the resulting approximating process is a regime-switching Lévy process. The approximation of the jump part of X by a compound Poisson process with double phase-type jumps will enable us to employ matrix Wiener-Hopf factorisation results, needed to obtain tractable formulae for the prices of barrier-type options. The two steps will be described in detail in the present section.

Markov chain approximation of the variance process

The first step of the approximation procedure that was outlined above is to approximate the variance process v by a finite-state continuous-time Markov chain on some grid contained in the positive real line. We will restrict ourselves to the case that the variance process v is a Feller process on the state space $\mathbb{R}_+ = [0, \infty)$. This assumption implies that v is a Markov process satisfying some regularity properties.

The Feller property is phrased in terms of the semi-group $(P_t)_{t \geq 0}$ of v acting on $C_0(\mathbb{R}_+)$, the set of continuous functions on $\mathbb{R}_+$ that tend to zero at infinity. Recall that, for any Borel function f on $\mathbb{R}_+$ and $t \geq 0$, the map $P_t f : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$P_t f(x) = \mathbb{E}_x[f(v_t)].$$

Assumption 1 *The Markov process $v = \{v_t\}_{t \geq 0}$ is a Feller process; that is, for any $f \in C_0(\mathbb{R}_+)$, the family $(P_t f)_{t \geq 0}$ satisfies the following two properties:*

(i) $P_t f \in C_0(\mathbb{R}_+)$ *for any $t > 0$;*

(ii)$\lim_{t\downarrow 0} P_t f(v) = f(v)$ *for any $v \in \mathbb{R}_+$.*

An approximating Markov chain Z with generator Q on a state-space $E^0 = \{x_1, \ldots, x_{N_0}\}$ can be constructed by choosing E^0 to be some appropriate (non-uniform) grid in $\mathbb{R}_+$, and specifying the generator Q such that an appropriate set of instantaneous (local) moments of the chain Z and the target process v are matched. See [25] for details on this procedure.

Denote by $\mathcal{G} : \mathcal{D} \to C_0(\mathbb{R}_+)$ the infinitesimal generator of v defined on its domain $\mathcal{D}$, and let $Z^{(n)}$ be a sequence of Markov chains with generators $Q^{(n)}$ and state-spaces $E^{0(n)} = \{x_1^{(n)}, \ldots, x_{N^{(n)}}^{(n)}\}$, and denote by $Q^{(n)} f_n$ the vector with coordinates

$$Q^{(n)} f_n(x_i) = \sum_{x_j \in E^{0(n)}} Q^{(n)}(x_i, x_j) f(x_j), \qquad x_i \in E^{0(n)}.$$

The sequence $Z^{(n)}$ weakly approximates the variance process v if the range of the state-spaces $E^{0(n)}$ grows sufficiently fast as n tends to infinity, and if, for all regular functions f, $Q^{(n)} f_n$ converges uniformly to $\mathcal{G} f$, that is $\epsilon_n(f) \to 0$ where

$$\epsilon_n(f) := \max_{x \in (E^{0(n)})^o} \left| Q^{(n)} f_n(x) - \mathcal{G} f(x) \right|$$

and $(E^{0(n)})^o$ is equal to $E^{0(n)}$ without the smallest and the largest elements. The precise statement reads as follows:

Theorem 2 *Assume that the following two conditions are satisfied for any function in a core[7] of $\mathcal{L}$:*

$$\epsilon_n(f) \to 0 \qquad \text{as } n \to \infty, \tag{1.32}$$

$$\text{either (i) } \lim_{y \searrow 0} \mathcal{G} f(y) = 0 \text{ or (ii) } \lim_{n \to \infty} \mathbb{P}_x \left[\tau_{(E^{0(n)})^o}^{(n)} > T \right] = 1, \tag{1.33}$$

where for any set $G \subset \mathbb{R}_+$ we define $\tau_G^{(n)} = \inf\{t \geq 0 : X_t^{(n)} \notin G\}$. Then it holds that, as $n \to \infty$, $v^{(n)} \stackrel{\mathcal{L}}{\Rightarrow} v$.[8]

Remark. (i) Note that the convergence in law implies in particular that

$$\mathbb{E}_x \left[g\left(Z_T^{(n)} \right) \right] \longrightarrow \mathbb{E}_x \left[g(v_T) \right]$$

for any bounded continuous function g.

(ii) A proof of this statement can be found in [25].

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[7] A core $\mathcal{C}$ of the operator $\mathcal{L}$ is a subspace of the domain of $\mathcal{L}$ that is (i) dense in $C_0(\mathbb{R}_+)$ and (ii) there exists $\lambda > 0$ such that the set $\{(\lambda - \mathcal{L})f : f \in \mathcal{C}\}$ is dense in $C_0(\mathbb{R}_+)$.

[8] By $v^{(n)} \stackrel{\mathcal{L}}{\Rightarrow} v$ we denote the convergence in law of $v^{(n)}$ to v in the Skorokhod topology, i.e. convergence of the distributions of $v^{(n)}$ to those of v in the set of probability measures on the Skorokhod space endowed with the Skorokhod topology.

Approximation of a Lévy process

The second stage of the aforementioned approximation procedure amounts to an approximation of a Lévy process by a compound Poisson process with double phase-type jumps.

Proposition 4 *For any Lévy process X there exists a sequence $(X^{(n)})_{n \in \mathbb{N}}$ of Lévy processes with double phase-type jumps such that $X^{(n)} \overset{\mathcal{L}}{\Rightarrow} X$ as $n \to \infty$.*

Remarks. (i) A proof of this result can be found in e.g. Jacod and Shiryaev [20, Section VII.3]. It is based on the fact that a sequence $(X^{(n)})_{n \in \mathbb{N}}$ of Lévy processes weakly converges to a given Lévy process X if and only if $X_1^{(n)}$ converges in distribution to X_1 (see e.g. [20, Corollary VII.3.6] for a proof).

(ii) Furthermore [20, Corollary VII.3.6] implies that a sufficient condition to guarantee that $X_1^{(n)}$ converges in distribution to X_1 is that the characteristic triplets (c_n, σ_n^2, ν_n) of $X^{(n)}$ converges to the triplet (c, σ^2, ν) of X as follows as $n \to \infty$: for some $a > 0$ that is a continuity point of $\nu(\mathrm{d}x)$ and $\nu(-\mathrm{d}x)$ it holds that

$$c_n \to c, \qquad \sigma_n^2 + \int_{(-a,a)} x^2 \nu_n(\mathrm{d}x) \to \sigma^2 + \int_{(-a,a)} x^2 \nu(\mathrm{d}x) \tag{1.34}$$

$$\int_{(0,\infty)} (x^2 \wedge a)|\overline{\nu}_n(x) - \overline{\nu}(x)|\mathrm{d}x \; + \; \int_{(-\infty,0)} (|x|^2 \wedge a)|\underline{\nu}_n(x) - \underline{\nu}(x)|\mathrm{d}x \to 0 \tag{1.35}$$

where, for any measure m on $\mathbb{R}$, $\overline{m}$ and $\underline{m}$ are the left and right tails, $\overline{m}(x) = m([x,\infty))$, $\underline{m}(x) = m((-\infty, x])$.

Suppose now that the Lévy process X is a model for the log of a stock price. Then X has a triplet (c, σ^2, ν) satisfying the exponential moment condition in (1.13). An example of a sequence of Lévy processes with DPH distributed jumps that weakly converges to X is then given as follows. Let $\lambda_n = \nu((-1/n, 1/n)^c)$ and F_n be a double phase-type distribution that approximates in distribution the probability measure $\widetilde{F}_n(\mathrm{d}x) = I_{\{|x| \geq 1/n\}}\nu(\mathrm{d}x)/\lambda_n$, and define the measure ν_n by $\nu_n(\mathrm{d}x) = \lambda_n F_n(\mathrm{d}x)$. Here the F_n and σ_n^2 are to be chosen such that (1.35) and the second requirement in (1.34) hold true. See e.g. [3] for a fitting procedure based on the EM algorithm. Then the sequence of Lévy processes $X^{(n)}$ with triplets $(c, \sigma_n^2, \lambda_n F_n)$ satisfies the conditions (1.35) and thus approximates X in law as n tends to infinity.

1.3.3 Convergence of the approximation procedure

Combining the two steps in the approximation we can now identify candidate sequences of regime-switching processes that converge to either of the stochastic volatility processes with jumps (1.14) and (1.16) and establish the convergence.

Let $(X^{(n)})_{n \in \mathbb{N}}$ be a sequence of Lévy processes with DPH jumps, and let $(Z^{(n)})_{n \in \mathbb{N}}$ be a sequence of Markov chains that is independent of X and $(X^{(n)})_{n \in \mathbb{N}}$. Let ψ_n denote the characteristic exponent of $X^{(n)}$, and (c_n, σ_n^2, ν_n) the characteristic triplet. Consider then the sequences of stochastic processes $(S^{(int-n)})_{n \in \mathbb{N}}$ and $(S^{(tc-n)})_{n \in \mathbb{N}}$ with $S^{(int-n)} = \{S_t^{(int-n)}\}_{t \in [0,T]}$ and $S^{(tc-n)} = \{S_t^{(tc-n)}\}_{t \in [0,T]}$ given by

$$S_t^{(int-n)} := S_0 \exp\left((r-d)t + \int_0^t \sqrt{Z_s^{(n)}} dX_s^{(n)} - \int_0^t \psi_n\left(-\mathtt{i}\sqrt{Z_s^{(n)}}\right) ds \right),$$

$$S_t^{(tc-n)} := S_0 \exp\left((r-d)t + X^{(n)}\left(V_t^{(n)}\right) - \psi_n(-\mathtt{i})V_t^{(n)} \right),$$

$$\text{where} \qquad V_t^{(n)} = \int_0^t Z_s^{(n)} ds.$$

Observe that $S^{(int-n)}$ and $S^{(tc-n)}$ are both in law equal to exponential regime-switching Lévy processes. In the case of $S^{(int-n)}$ this is clear, while in the case of the process $S^{(tc-n)}$ this follows by evaluating the characteristic function and comparing with (1.28).

To establish the convergence in law of $(S^{(int-n)})_{n\in\mathbb{N}}$ and $(S^{(tc-n)})_{n\in\mathbb{N}}$ we will first show convergence of the finite-dimensional distributions.[9]

Proposition 5 *Assume that $X^{(n)} \overset{\mathcal{L}}{\Rightarrow} X$ and $Z^{(n)} \overset{\mathcal{L}}{\Rightarrow} v$ as n tends to infinity. Then the following holds true:*
(a) $(Z^{(n)}, S^{(int-n)})_n \overset{fidis}{\Rightarrow} (v, S)$ where S is the model given in (1.14).
(b) $(V^{(n)}, S^{(tc-n)})_n \overset{fidis}{\Rightarrow} (V, S)$ where S is the time-change model given in (1.16).

Remark. The convergence of European put option prices (and hence, by put-call parity, also of European call option prices) under the approximating models to those under the limiting models is a direct consequence of the convergence in finite dimensional distributions. To establish the convergence of path-dependent option prices such as barrier option prices it is required to prove that the approximating models converge in law.

Proof. (b) To prove the convergence of the finite dimensional distributions it suffices, in view of the Markov property, to show that, for each fixed $t \in [0,T]$, the characteristic functions χ_n of $(V_t^{(n)}, X^{(n)}(V_t^{(n)}))$ converge point-wise to the characteristic function χ of $(V_t, X(V_t))$ as n tends to infinity. By conditioning and using the independence of $V^{(n)}$ from $X^{(n)}$ and of V from X we find that

$$\chi_n(u,v) = \mathbb{E}_{x,i}\left[\exp\left\{ (\mathtt{i}u + \psi_n(v))V_t^{(n)} \right\} \right], \qquad \chi(u,v) = \mathbb{E}_{x,i}[\exp((\mathtt{i}u + \psi(v))V_t)].$$

Since $v^{(n)}$ converges in law to v in the Skorokhod topology, the Skorokhod representation theorem implies that on some probability space $Z^{(n)} \to v$, almost surely, with the convergence with respect to the Skorokhod metric. Since, for any $t \in [0,T]$, the map $i_t : D_{\mathbb{R}}[0,T] \to \mathbb{R}$ given by $i_t : x \mapsto \int_0^t x(s)ds$, is continuous in the Skorokhod topology, we deduce that $V_t^{(n)} \to V_t$ almost surely. In particular, χ_n converges point-wise to χ.

The proof of (a) is similar and omitted. $\qquad\square$

The next result concerns the convergence in law of the sequences $(S^{(int-n)})_{n\in\mathbb{N}}$ and $(S^{(tc-n)})_{n\in\mathbb{N}}$:

Theorem 3 *The following statements hold true:*

---

[9] A sequence of processes $(Y_n)_{n\in\mathbb{N}}$ converges in finite dimensional distribution to the process $Y = \{Y_t\}_{t\in[0,T]}$, if, for any partition $t_1 < \ldots < t_m$ of $[0,T]$, $\mathbb{P}(Y_{t_1}^{(n)} \leq x_1, \ldots, Y_{t_m}^{(n)} \leq x_m) \to \mathbb{P}(Y_{t_1} \leq x_1, \ldots, Y_{t_m} \leq x_m)$. We will denote this convergence by $Y_n \overset{fidis}{\Rightarrow} Y$.

(a) *Assume that (v, S) is a Feller process, where S is the model given in (1.14), that $Z^{(n)}$ satisfies the conditions in Theorem 2, and that the characteristics of $X^{(n)}$ satisfy conditions (1.34) and (1.35). Then, as $n \to \infty$,*

$$S^{(int-n)} \overset{\mathcal{L}}{\Rightarrow} S.$$

(b) *Assume that $X^{(n)} \overset{\mathcal{L}}{\Rightarrow} X$ and $Z^{(n)} \overset{\mathcal{L}}{\Rightarrow} v$ as $n \to \infty$. Then it holds that*

$$S^{(tc-n)} \overset{\mathcal{L}}{\Rightarrow} S,$$

where S is the time-change model given in (1.16).

Remark. The convergence in law stated above carries over to the convergence of barrier option prices under the respective models, if the boundaries are continuity points of the limiting model. For instance, if we denote by $\tau_A = \inf\{t \geq 0 : S_t \notin A\}$ the first time that S leaves the set $A := [\ell, u]$, and $\mathbb{P}(S_T \in \{\ell, u\}) = 0$, then, for any bounded continuous pay-off functions $g, h : \mathbb{R}_+ \to \mathbb{R}$, we have that as $n \to \infty$ the double knock-out option and rebate option prices under the approximating models converge to those under the limiting model:

$$\mathbb{E}_x \left[g\left(S_T^{(n)} \right) I_{\{\tau_A^{(n)} > T\}} \right] \longrightarrow \mathbb{E}_x \left[g(S_T) I_{\{\tau_A > T\}} \right],$$

$$\mathbb{E}_x \left[e^{-r\tau_A^{(n)}} h\left(S_{\tau_A^{(n)}}^{(n)} \right) I_{\{\tau_A^{(n)} \leq T\}} \right] \longrightarrow \mathbb{E}_x \left[e^{-r\tau_A} h(S_{\tau_A}) I_{\{\tau_A \leq T\}} \right],$$

where $S^{(n)}$ denotes $S^{(int-n)}$ or $S^{(tc-n)}$. A proof of this result was given in [25].

Proof. In view of Proposition 5, it suffices[10] to verify that the sequences $(S^{(int-n)})_{n \in \mathbb{N}}$ and $(S^{(tc-n)})_{n \in \mathbb{N}}$ are relatively compact in $D_{\mathbb{R}}[0, T]$.

(a) We will establish relative compactness of the sequence $(X^{(int-n)})_{n \in \mathbb{N}} = (\log S^{(int-n)})_{n \in \mathbb{N}}$. Let $X' = \log S$. It is straightforward to check that the set of functions $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ of the form $f(x, v) = g(x)h(v)$ with h in the core of $\mathcal{G}$, the infinitesimal generator of v, and with $g \in C_c^\infty(\mathbb{R})$[11] is dense in $C_0(\mathbb{R}_+ \times \mathbb{R})$ and is contained in the domain $\mathcal{D}(\mathcal{L}')$ of the infinitesimal generator $\mathcal{L}'$ of (v, X'). Furthermore, $\mathcal{L}'$ acts on such f as

$$\mathcal{L}' f(x, v) = \frac{1}{2} v h(v) \sigma^2 g''(x) + \left[(r - d) + c\sqrt{v} - \psi(-\mathtt{i}\sqrt{v}) \right] h(v) g'(x)$$
$$+ h(v) \int_{\mathbb{R}} \left[g(x + z\sqrt{v}) - g(x) - z\sqrt{v} g'(x) I_{\{|z| \leq 1\}} \right] \nu(\mathrm{d}z) + g(x)\mathcal{G}h(v), \qquad x \in \mathbb{R}, v > 0,$$

since by construction the stochastic integral $\left\{ \int_0^t \sqrt{v_s} \mathrm{d}X_s \right\}_{t \geq 0}$ jumps if and only if the Lévy process X jumps and, if the jump occurs at time t, the quotient of the jump sizes equals $\sqrt{v_t}$. On the other hand, the infinitesimal generator $\mathcal{L}^{(n)}$ of the regime-switching processes $(Z^{(n)}, X^{(int-n)})$ acts on $f(x, v) = g(x)h(v)$ as

$$\mathcal{L}^{(n)} f(x, v) = \frac{1}{2} v \sigma_n^2 h(v) g''(x) + \left[(r - d) + c_n\sqrt{v} - \psi_n(-\mathtt{i}\sqrt{v}) \right] h(v) g'(x)$$
$$+ h(v) \int_{\mathbb{R}} \left[g(x + z\sqrt{v}) - g(x) - z\sqrt{v} g'(x) I_{\{|z| \leq 1\}} \right] \nu_n(\mathrm{d}z) + g(x)Q^{(n)}h(v), \qquad x \in \mathbb{R}, v \in E^{0(n)}.$$

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[10] See e.g. Theorem 3.7.8 in Ethier and Kurtz [15] for a proof of this well known fact.

[11] $C_c^\infty(\mathbb{R}_+)$ denotes the set of infinitely differentiable functions with compact support contained in $\mathbb{R}_+$.

$\mathcal{L}^{(n)}f$ converges to $\mathcal{L}'f$ uniformly as $n \to \infty$:

$$\epsilon'_n(f) := \sup_{x \in \mathbb{R}, v \in E^{0(n)}} |\mathcal{L}'f(x,v) - \mathcal{L}^{(n)}f(x,v)| \to 0. \tag{1.36}$$

To see why this is true note that the triangle inequality and integration by parts imply that

$$\begin{aligned}
\epsilon'_n(f) &\leq \|\mathcal{G}h - Q^{(n)}h\|_n \|g\|_\infty + C_1 |\overline{\nu}(a) - \overline{\nu}_n(a)| + C_2 \max\{|\sigma_n^2 - \sigma^2|, |c_n - c|\} \\
&+ C_3 \left\{ \int_a^\infty |\overline{\nu}(z) - \overline{\nu}_n(z)| \mathrm{d}z + \int_{-\infty}^{-a} |\underline{\nu}(z) - \underline{\nu}_n(z)| \mathrm{d}z \right\} \\
&+ C_4 \left\{ \int_{(0,a)} z^2 |\overline{\nu}(z) - \overline{\nu}_n(z)| \mathrm{d}z + \int_{(-a,0)} z^2 |\underline{\nu}(z) - \underline{\nu}_n(z)| \mathrm{d}z \right\},
\end{aligned} \tag{1.37}$$

where a is a continuity point of the measures $\nu(\mathrm{d}x)$ and $\nu(-\mathrm{d}x)$ and $C_1, \ldots, C_4$, are certain finite constants independent of n, and we denoted $\|f\|_n = \sup_{x \in E^{0(n)}} |f(x)|$ and $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. In view of the conditions (1.34) and (1.35) $\epsilon'_n(f)$ tends to zero as n tends to infinity. Corollary 4.8.6 in Ethier and Kurtz [15] implies then that $(Z^{(n)}, X^{(int-n)})_{n \in \mathbb{N}}$ and hence $(Z^{(n)}, S^{(int-n)})_{n \in \mathbb{N}}$ is relatively compact in $D_{\mathbb{R}^2}[0,T]$.

(b) Denote by $\widetilde{X}^{(n)} = \{\widetilde{X}_t^{(n)}\}_{t \geq 0}$ and $\widetilde{X} = \{\widetilde{X}_t\}_{t \geq 0}$ the Lévy processes given by

$$\widetilde{X}_t^{(n)} = X_t^{(n)} - \psi^{(n)}(-\mathtt{i})t \qquad \text{and} \qquad \widetilde{X}_t = X_t - \psi(-\mathtt{i})t.$$

We will verify[12] the relative compactness of the sequence $(Y^{(n)})_{n \in \mathbb{N}}$ with $Y_t^{(n)} = \widetilde{X}^{(n)}(V_t^{(n)})$. In view of the Skorokhod embedding theorem and the convergence in law of $(Z^{(n)}, \widetilde{X}^{(n)})$ to $(v, \widetilde{X})$, we may and shall assume that $(Z^{(n)}, \widetilde{X}^{(n)})_{n \in \mathbb{N}}$ and $(v, \widetilde{X})$ are defined on the same probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and that, $\mathbb{P}'$-almost surely, $(Z^{(n)}, X^{(n)}) \to (v, X)$ with respect to the Skorokhod metric. Fix an $\omega \in \Omega'$ for which this convergence holds true.

Observe that, for any $U > 0$, $\overline{z} = \sup_n Z_U^{(n)}(\omega)$ is finite, as $v_U(\omega)$ is finite and $|Z_U^{(n)}(\omega) - v_U(\omega| \to 0$ as n tends to infinity. For $x \in D_{\mathbb{R}}[0,U]$, $\delta > 0$, $U > 0$, denote by $w'(x, \delta, U)$ the modulus of continuity

$$w'(x, \delta, U) = \inf_{\{t_i\}} \max_i \sup_{s,t \in [t_{i-1}, t_i)} |x(s) - x(t)|,$$

where $\{t_i\}$ ranges over all partitions $0 = t_0 < t_1 < \ldots < t_{n-1} < U \leq t_n$ with $\min_{1 \leq i \leq n} |t_i - t_{i-1}| > \delta$. Note that $w'(x, \delta, U)$ is non-decreasing in δ and U. Therefore it is straightforward to check that

$$w'_n(U) := w'(Y^{(n)}(\omega), \delta, U) \leq w'(\widetilde{X}^{(n)}(\omega), \delta\overline{z}, U\overline{z}). \tag{1.38}$$

Furthermore, observe that

$$\mathcal{Y}_n(\omega) := \{Y_s^{(n)}(\omega) : s \leq U\} \subset \{\widetilde{X}_s^{(n)}(\omega) : s \leq U\overline{z}\}. \tag{1.39}$$

---

[12] The proof draws on and combines a number of results from the theory of weak convergence of probability measures that can be found in Ethier and Kurtz [15, Chapters 3, 6]

Since $\{\widetilde{X}^{(n)}(\omega)\}_{n\in\mathbb{N}}$ is convergent in $D_{\mathbb{R}}[0,\infty)$, it follows that (I) for every rational $t \in [0,U]$ there exists a compact set $C_t \subset \mathbb{R}$ such that $Y_t^{(n)}(\omega) \in C_t$ for all n and (II) for every $U > 0$, $\lim_{\delta\to 0}\sup_n w'_n(U) = 0$ and as a consequence[13] $\mathcal{Y}_n$ is relatively compact in $D_{\mathbb{R}}[0,\infty)$. $\qquad\qquad\square$

1.4 European and volatility derivatives

1.4.1 Call and put options

We first turn to the valuation of a call option price. In the model under consideration a closed form expression is available, in terms of the original parameters, for the Fourier transform c_T^* in log-strike $k = \log K$ of the call prices $C_T(K)$ with maturity T,

$$c_T^*(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi k} C_T(\mathrm{e}^k)\mathrm{d}k \quad \text{where} \quad \Im(\xi) < 0.$$

Proposition 6 *Define for any $\xi \in \mathbb{C}\backslash\{0,\mathtt{i}\}$, $x \in \mathbb{R}$ and $j \in E^0$ the value $D(\xi,x,j)$ is defined by the formula*

$$D(\xi,x,j) := \frac{\mathrm{e}^{(1+\mathrm{i}\xi)x}}{\mathrm{i}\xi - \xi^2} \cdot \left[\exp\left\{T(K(1 + \mathtt{i}\xi) - \Lambda_D)\right\}\mathbf{1}\right](j). \qquad (1.40)$$

Then if $\Im(\xi) < 0$ it holds that

$$c_T^*(\xi) = D(\xi,x,j)$$

where $x = \log S_0$ is the log-price at the current time and $Z_0 = j$ the initial level of the volatility.

Remarks. (i) The call option price can now be calculated using the method described in Carr-Madan [13] by evaluating the integral

$$\begin{aligned}
C_T(K) &= \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}sk} c_T^*(s - \mathtt{i}\alpha)\mathrm{d}s \\
&= \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} \Re\left[e^{-\mathrm{i}sk} D(s - \mathtt{i}\alpha, \log S_0, Z_0)\right]\mathrm{d}s,
\end{aligned} \qquad (1.41)$$

for $k = \log(K)$ and any strictly positive α. The integral in (1.41) can be approximated efficiently by a finite sum using the FFT algorithm (see [13]). Since in our model we have an explicit formula for the transform $c_T^*(s)$ given by (1.40), the pricing of European call options is immediate.

(ii) A simple calculation shows that the put option price $P_T(K) = \mathbb{E}_{x,j}\left[(B_T^D)^{-1}(K - S_T)^+\right]$ can be expressed in terms of the formula for $D(\xi,x,j)$ in (1.40) and any strictly negative constant α in the following way:

$$P_T(K) = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} \Re\left[e^{-\mathrm{i}sk} D(s - \mathtt{i}\alpha, \log S_0, Z_0)\right]\mathrm{d}s, \quad \text{where} \quad k = \log(K).$$

---

[13] Both applications follows from the fact that conditions (I) and (II) are necessary and sufficient for the relative compactness of $(Y^{(n)})_{n\in\mathbb{N}}$ (Ethier and Kurtz [15, Theorem 3.6.3]).

Proof. To find the European call option price $C_T(K) = \mathbb{E}_{x,j}\left[(B_T^D)^{-1}(S_T - K)^+\right]$ in model (1.19) we first need to find the Fourier transform in the log-strike $k = \log(K)$ of the function

$$c_T(k) = \exp(\alpha k)\mathbb{E}_{x,j}\left[(B_T^D)^{-1}(S_T - \exp(k))^+\right],$$

where α is some strictly positive constant. Fubini's theorem and the form of the characteristic function (1.31) imply the following for $\xi = v - \mathtt{i}\alpha$:

$$\begin{aligned}
c_T^*(\xi) &= \int_{\mathbb{R}} \exp((\mathtt{i}v + \alpha)k)\mathbb{E}_{x,j}\left[(B_T^D)^{-1}(S_T - \exp(k))^+\right]\mathrm{d}k \\
&= \mathbb{E}_{x,j}\left[(B_T^D)^{-1}\int_{\mathbb{R}} \exp((\mathtt{i}v + \alpha)k)(S_T - \exp(k))^+\mathrm{d}k\right] \\
&= \mathbb{E}_{x,j}\left[(B_T^D)^{-1}\exp((1 + \alpha + \mathtt{i}v)X_T)\right]/(\alpha^2 + \alpha - v^2 + \mathtt{i}(2\alpha + 1)v) \\
&= \frac{\mathrm{e}^{x(1+\alpha+\mathtt{i}v)}}{\alpha^2 + \alpha - v^2 + \mathtt{i}(2\alpha + 1)v}\left[\exp(T(K(1 + \alpha + \mathtt{i}v) - \Lambda_D))\mathbf{1}\right](j).
\end{aligned}$$

This concludes the proof. $\qquad\square$

1.4.2 Implied volatility at extreme strikes

The *implied volatility* $\sigma_i(K, T)$ for a given strike K and maturity T is uniquely defined by the identity

$$C^{\mathrm{BS}}(S_0, K, T, \sigma_{x,i}(K, T)) = \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T - K)^+\right], \tag{1.42}$$

where $C^{\mathrm{BS}}(S_0, K, T, \sigma)$ is the Black-Scholes formula and $S_0 = \exp(x)$. The results in Lee [23] and refinements in Benaim and Friz [6] imply that in model (1.19) the slope of the volatility smile is uniquely determined by the quantities $\alpha_i^{\pm}$, $i = 1, \ldots, n$, defined in (1.21). In the particular case where the distribution of jumps is double exponential, $\alpha_i^{\pm}$ are in fact the reciprocals of mean-jump sizes in model (1.19).

In order to state the precise result, define

$$\begin{aligned}
q_+ &:= \sup\left\{u : \mathbb{E}_{x,i}\left[S_T^{1+u}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}, \\
q_- &:= \sup\left\{u : \mathbb{E}_{x,i}\left[S_T^{-u}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}.
\end{aligned}$$

If the chain Z is irreducible, the quantities $q_{\pm}$ can be identified explicitly to be equal to

$$q_+ = \min\{\alpha_i^+ - 1 : i \in \{1, \ldots, N_0\} \;\&\; p_i\lambda_i > 0\}, \tag{1.43}$$

$$q_- = \min\{\alpha_i^- : i \in \{1, \ldots, N_0\} \;\&\; (1 - p_i)\lambda_i > 0\}. \tag{1.44}$$

As noted above the quantities q_+ and q_- depend only on the mean-jump sizes of the compound Poisson processes in model (1.19).

Denote the forward price by $F_T := \mathbb{E}_{x,i}[S_T]$. Then the asymptotic behaviour for the implied volatility is described as follows:

Proposition 7 *Suppose that Z is irreducible. For $T > 0$ and $K > 0$ and with $q_\pm$ given in (1.43)–(1.44), it holds that*

$$\lim_{K \to \infty} \frac{T\sigma_{x,i}(K,T)^2}{\log(K/F_T)} = 2 - 4\left(\sqrt{q_+^2 + q_+} - q_+\right),$$

$$\lim_{K \to 0} \frac{T\sigma_{x,i}(K,T)^2}{|\log(K/F_T)|} = 2 - 4\left(\sqrt{q_-^2 + q_-} - q_-\right).$$

Remark. Note that, if the Markov chain Z is irreducible, the asymptotic slope of the implied volatility smile for large and small strikes depends neither on the spot $S_0 = e^x$ nor on the starting volatility regime i.

1.4.3 Forward starting options and the forward smile

A forward starting call option is a call option whose strike is fixed at a later date as a proportion of the value of the underlying at that moment. More precisely, the pay-off of a T_1-forward starting call option at maturity $T_2 > T_1$ is given by

$$(S_{T_2} - \kappa S_{T_1})^+, \qquad \kappa \in \mathbb{R}_+.$$

Denote the current value of this forward-starting option by $F_{T_1,T_2}(\kappa)$ and let

$$F_{T_1,T_2}^*(\xi) = \int_{\mathbb{R}} e^{i\xi k} F_{T_1,T_2}(e^k) dk, \quad \text{where} \quad \Im(\xi) < 0,$$

be its Fourier transform in the forward log-strike $k = \log \kappa$.

Proposition 8 *For ξ with $\Im(\xi) < 0$ it holds that*

$$F_{T_1,T_2}^*(\xi) = \frac{e^{(1+i\xi)x}}{i\xi - \xi^2} \cdot \left[\exp(T_1(Q - \Lambda_F)) \exp\left\{(T_2 - T_1)(K(1 + i\xi) - \Lambda_D)\right\}\mathbf{1}\right](j), \tag{1.45}$$

where $x = \log S_0$ is the log-spot price and $Z_0 = j$ the initial level of the volatility.

Remark. An inversion formula, analogous to the one in (1.41), can be used to obtain the value $F_{T_1,T_2}(\kappa)$ from Proposition 8.

Proof. The price of a T_1-forward starting option is given by the expression

$$F_{T_1,T_2}(\kappa) = \mathbb{E}_{x,i}\left[(B_{T_2}^D)^{-1}(S_{T_2} - \kappa S_{T_1})^+\right], \quad \kappa \in \mathbb{R}_+.$$

The process (X, Z) is Markov and therefore, by conditioning on the σ-algebra generated by the process up to time T_1, employing the form (1.31) of the characteristic function of X_T and using the spatial homogeneity of the log-price $X_t = \log S_t$ in our model, we obtain the following expression for the price of the forward starting option

$$F_{T_1,T_2}(\kappa) = \mathbb{E}_{x,i}\left[(B_{T_2}^D)^{-1}(S_{T_2} - \kappa S_{T_1})^+\right] \tag{1.46}$$

$$= \mathbb{E}_{x,i}\left[\frac{S_{T_1}}{B_{T_1}^D}\mathbb{E}_{0,Z_{T_1}}\left[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+\right]\right]$$

$$= \sum_{j\in E^0}\mathbb{E}_{x,i}\left[\frac{S_{T_1}}{B_{T_1}^D}I_{\{Z_{T_1}=j\}}\right]\mathbb{E}_{0,j}[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+]$$

$$= S_0\sum_{j\in E^0}e_i'\exp(T(K(-\mathbf{i}) - \Lambda_D))e_j\mathbb{E}_{0,j}[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+]$$

$$= S_0 e_i'\exp(T(K(-\mathbf{i}) - \Lambda_D))C_{T_2-T_1}(\kappa;1), \tag{1.47}$$

where $C_{T_2-T_1}(\kappa;1)$ is a vector (of call option prices) with j-th component equal to $\mathbb{E}_{0,j}[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+]$. The martingale condition in (1.25) and Proposition 6 conclude the proof. $\qquad\square$

Remarks. (i) A quantity of great interest in the derivatives markets is the *forward implied volatility* $\sigma_{x,i}^{fw}(S_T,\kappa,T)$ at a future time T implied by the model. It is defined as the unique solution to the equation

$$C^{\mathrm{BS}}(S_{T_1},\kappa S_{T_1},T_2 - T_1,\sigma_{x,i}^{fw}(S_{T_1},\kappa,T_1)) = \mathbb{E}_{x,i}\left[\frac{B_{T_1}^D}{B_{T_2}^D}(S_{T_2} - \kappa S_{T_1})^+\middle| S_{T_1}\right], \tag{1.48}$$

where the left-hand side denotes the Black-Scholes formula with strike κS_{T_1} and spot S_{T_1}. The reason for the importance of the forward implied volatility $\sigma_{x,i}^{fw}(S_T,\kappa,T)$ lies in the problem of hedging of exotic derivatives using vanilla options. If at a future time T the spot trades at the level S_T, then the trader needs to know where, according to the model, would the vanilla surface be trading at. This is of particular importance when hedging a barrier contract that knocks out at the level S_T, because conditional on this event the trader is left with a portfolio of vanilla options that was created as a semi-static hedge for the exotic derivative.

(ii) In the model given by (1.18) we can compute the right-hand side of (1.48). In view of the Markov property of the process (X,Z), this equation is equivalent to

$$C^{\mathrm{BS}}(S_{T_1},\kappa S_{T_1},T_2 - T_1,\sigma_{x,i}^{fw}(S_{T_1},\kappa,T_1)) = S_{T_1}\mathbb{E}_{x,i}\left[\mathbb{E}_{0,Z_{T_1}}\left[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+\right]\middle| S_{T_1}\right]$$

$$= S_{T_1}\sum_{j\in E^0}\mathbb{P}_{x,i}\left[Z_{T_1} = j\middle| S_{T_1}\right]\mathbb{E}_{0,j}[(B_{T_2-T_1}^D)^{-1}(S_{T_2-T_1} - \kappa)^+]$$

$$= S_{T_1}f^{x,i}(X_{T_1},T_1)'C_{T_2-T_1}(\kappa,1),$$

where the coordinates of the vector $f^{x,i}(y,T)$ are defined by

$$f_j^{x,i}(y,T) := \mathbb{P}_{x,i}\left[Z_T = j\middle| X_T = y\right] \tag{1.49}$$

and $C_{T_2-T_1}(\kappa,1)$ is as defined in the line following equation (1.47).

(iii) The vector $C_{T_2-T_1}(\kappa,1)$ can be computed by formula (1.41) and, in the light of definition (1.49), Proposition 3 and formulae (1.29) and (1.30) for $q_T^{x,i}(y,j)$ and $q_T^{x,i}(y)$, it follows that

$$f_j^{x,i}(y,T)q_T^{x,i}(y) = q_T^{x,i}(y,j).$$

This yields the quantity in (1.49) and hence a formula for the forward implied volatility in our model.

1.4.4 Volatility derivatives

An option on the realized variance is a derivative security that delivers $\phi(\Sigma_T)$ at expiry T, where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is some measurable payoff function and Σ_T is the quadratic variation up to time T of the process $\log S = X$. More formally, for a refining sequence of partitions[14] $(\Pi_n)_{n \in \mathbb{N}}$ of the interval $[0, T]$, Σ_T is given by

$$\Sigma_T := \lim_{n \to \infty} \sum_{t_i^n \in \Pi_n, i \geq 1} \log\left(\frac{S_{t_i^n}}{S_{t_{i-1}^n}}\right)^2$$

It is well-known that the sequence on the right-hand side converges in probability, uniformly on compact time intervals (see Jacod & Shiryaev [20], Theorem 4.47) and the limit is given by

$$\Sigma_T = \int_0^T \sigma(Z_t)^2 \mathrm{d}t + \sum_{i \in E^0} \sum_{t \leq T} I_{\{Z_t = i\}} (\Delta J_t^i)^2. \tag{1.50}$$

where $\Delta J_t^i := J_t^i - J_{t-}^i$. The process $\{\Sigma_t\}_{t \geq 0}$ is called the quadratic variation or realized variance process of X, and its law is explicitly characterised as follows:

Proposition 9 *(i) The process $\{(\Sigma_t, Z_t)\}_{t \geq 0}$ is a Markov process with*

$$\Sigma_t = \int_0^t \sigma(Z_s)^2 \mathrm{d}s + \sum_{i \in E^0} \int_0^t I_{\{Z_s = i\}} d\widetilde{J}_s^i,$$

where $\widetilde{J}^i$, $i \in E^0$, is a compound Poisson process with intensity λ_i and positive jump sizes K_i with probability density

$$g_i(x) = \frac{1}{2\sqrt{x}} \left[p_i \beta_i^+ e^{\sqrt{x} B_i^+} (-B_i^+) \mathbf{1} + (1 - p_i) \beta_i^- e^{\sqrt{x} B_i^-} (-B_i^-) \mathbf{1} \right] I_{(0, \infty)}(x).$$

(ii) The discounted Laplace transform of Σ_t is given by

$$\mathbb{E}_i \left[\frac{\exp(-u\Sigma_t)}{B_t^D} \right] = \left[\exp(t(K_\Sigma(u) - \Lambda_D)) \mathbf{1} \right](i), \qquad u > 0, \tag{1.51}$$

where $K_\Sigma(u) = Q + \Lambda_\Sigma(u)$ with $\Lambda_\Sigma(u)$ an $N_0 \times N_0$ diagonal matrix with i-th element given by

$$\psi_i^\Sigma(u) := -u\sigma_i^2 + \lambda_i \left(\mathbb{E}\left[\exp(-uK_i)\right] - 1 \right), \tag{1.52}$$

with

$$\mathbb{E}\left[\exp(-uK_i)\right] = \sqrt{\frac{\pi}{u}} \left(p_i \beta_i^+ \Phi\left(\frac{1}{\sqrt{2u}} B_i^+\right)(-B_i^+) + (1 - p_i) \beta_i^- \Phi\left(\frac{1}{\sqrt{2u}} B_i^-\right)(-B_i^-) \right) \mathbf{1},$$

where $\Phi(x) := \exp(x^2/2) \mathcal{N}(x)$, with the cumulative normal distribution function $\mathcal{N}$.

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[14] The sets $\Pi_n = \{t_0^n, t_1^n, \ldots, t_n^n\}$, $n \in \mathbb{N}$, consist of increasing sequences of times such that $t_0^n = 0$, $t_n^n = T$, $\Pi_n \subset \Pi_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \max\{|t_i^n - t_{i-1}^n| : i = 1, \ldots, n\} = 0$.

Remarks. (i) As a given matrix M in practice typically[15] admits a spectral decomposition $M = UDU^{-1}$ where D is a diagonal matrix, $\Phi(M)$ can be evaluated by $\Phi(M) = U\Phi(D)U^{-1}$ where $\Phi(D)$ is the diagonal matrix with i-th element $\Phi(D_{ii})$.

(ii) It is important to note that the realized variance process Σ does not possess exponential moments of any order. This follows directly from the fact that the distribution of jumps g_i given in (1.59) decays at the rate $e^{-c\sqrt{x}}$, for some positive constant c, and implies that the left-hand side in formula (1.51) will be infinite for complex numbers u with negative imaginary part.

(iii) The expression (1.51) for the discounted Laplace transform can be employed to obtain explicit results for the values of volatility derivatives. The buyer of a swap on the realized variance pays premiums at a certain rate (the swap rate) to receive at maturity a pay-off $\phi(\Sigma_T)$ that is a function ϕ of the realized variance Σ_T, with as most common examples the volatility and the variance swap. In the case of a variance swap this function is linear ($\phi(x) = x/T$) whereas for a volatility swap it is a square root ($\phi(x) = \sqrt{x/T}$). The swap rates are determined such that at initiation the value of the swap is nil.

Corollary 1 *Suppose that $Z_0 = j$. Then the variance and volatility swap rates $\varsigma_{var}(T,j)$ and $\varsigma_{vol}(T,j)$ are given as follows:*

$$\varsigma_{vol}(T,j) = \frac{1}{2\sqrt{\pi T}} \int_0^\infty \left\{ \left[\exp(T(Q - \Lambda_D)) - \exp(T(K_\Sigma(u) - \Lambda_D)) \right] \mathbf{1} \right\}(j) \frac{\mathrm{d}u}{u^{3/2}}, \tag{1.53}$$

$$\varsigma_{var}(T,j) = \frac{1}{T} \int_0^T \left[\exp\{t(Q - \Lambda_D)\} \Lambda_V \exp\{(T - t)(Q - \Lambda_D)\}\mathbf{1}\right](j)\mathrm{d}t \tag{1.54}$$

$$= \frac{1}{Th} \left[\{\exp(T(Q - \Lambda_D)) - \exp(T(K_\Sigma(h) - \Lambda_D))\} \mathbf{1} \right](j) + o(h), \ h \downarrow 0, \tag{1.55}$$

where Λ_V is a $N_0 \times N_0$ diagonal matrix with i-th element given by

$$V(i) = \sigma_i^2 + 2\lambda_i \left(p_i(\beta_i^+)'(B_i^+)^{-2} + (1 - p_i)(\beta_i^-)'(B_i^-)^{-2} \right) \mathbf{1}. \tag{1.56}$$

Remarks. (i) The Laplace transform $\widehat{\sigma}_{var}(q,j)$ of $\sigma_{var}(\cdot, j) : T \mapsto T\varsigma_{var}(T,j)$ is explicitly given by

$$\widehat{\sigma}_{var}(q,j) = \left[(qI - \Lambda_D - Q)^{-1}\Lambda_V(qI - \Lambda_D - Q)^{-1}\mathbf{1} \right](j). \tag{1.57}$$

(ii) It is clear from the definition of $K_\Sigma(u)$ that the integral in (1.53) converges at the rate proportional to $1/\sqrt{U}$ where U is an arbitrary upper bound used in the numerical integration in (1.53).

Proof of Proposition 9. It is clear from the representation (1.50), that the increment $\Sigma_t - \Sigma_s$, for any $t > s \geq 0$, satisfies the equation

$$\Sigma_t - \Sigma_s = \int_s^t \sigma^2(Z_u)\mathrm{d}u + \sum_{i \in E^0} \int_s^t I_{\{Z_u = i\}}\mathrm{d}\widetilde{J}_u^i, \tag{1.58}$$

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[15] This is the case since the set of all square matrices that do not possess a diagonal decomposition is of co-dimension one in the space of all square matrices and therefore has Lebesgue measure zero.

where $\widetilde{J}^i$, $i \in E^0$, is a compound Poisson process with intensity λ_i and positive jump sizes K_i distributed as $(U^i)^2$ where U_i follows a $DPH(p_i, \beta_i^+, B_i^+, \beta_i^-, B_i^-)$ distribution. In particular, K_i is distributed according to the density

$$g_i(x) = \frac{1}{2\sqrt{x}}(f_i(\sqrt{x}) + f_i(-\sqrt{x})), \quad x > 0, \tag{1.59}$$

where the probability density function f_i is given by

$$f_i(x) = p_i(\beta_i^+)'\mathrm{e}^{xB_i^+}(-B_i^+)\mathbf{1}I_{(0,\infty)}(x) + (1-p_i)(\beta_i^-)'\mathrm{e}^{xB_i^-}(-B_i^-)\mathbf{1}I_{(-\infty,0)}(x).$$

As the $\widetilde{J}^i$ have independent increments, and Z is a Markov chain, it directly follows from (1.58) that (Σ_t, Z_t) is a Markov process, and moreover, a regime-switching subordinator. The form of the discounted Laplace transform can be derived as in Theorem 1. $\qquad\square$

Proof of Corollary 1. Employing the following representation for the square root

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_0^\infty [1 - \exp(-ux)] \frac{du}{u^{3/2}}, \quad \text{for any} \quad x \geq 0,$$

as well as the form of the discounted characteristic function given in (1.51) and Fubini's theorem yields that the volatility swap rate can be calculated via a single one-dimensional integral

$$\mathbb{E}_i\left[\frac{\sqrt{\frac{1}{T}\Sigma_T}}{B_T^D}\right] = \frac{1}{2\sqrt{\pi T}} \int_0^\infty e_i' \left[\exp(T(Q - \Lambda_D)) - \exp(T(K_\Sigma(u) - \Lambda_D))\right] \mathbf{1} \frac{du}{u^{3/2}}.$$

The derivation of the variance swap rate formula (1.54) rests on a conditioning argument. Indeed, by conditioning on the sigma algebra $\mathcal{F}_T^Z = \sigma(\{Z_t\}_{t \leq T})$ generated by Z up to time T, it follows that

$$\mathbb{E}_{x,i}\left[\frac{1}{B_T^D}\left\{\int_0^t \sigma(Z_s)^2 \mathrm{d}s + \sum_{i \in E^0} \int_0^t I_{\{Z_s=i\}}\mathrm{d}\widetilde{J}_s^i\right\}\right] = \sum_{j \in E^0} \mathbb{E}_{x,i}\left[\frac{1}{B_T^D}\int_0^T I_{\{Z_s=j\}}\mathrm{d}s\right] w(j),$$

where $w(j) := \sigma^2(j) + \mathbb{E}[\widetilde{J}_1^j]$. From the definition of $\widetilde{J}^j$ it is easily checked that $\mathbb{E}[\widetilde{J}_1^j]$ is equal to λ_j times the second moment of the density f_i. One verifies by a straightforward calculation that $w(j)$ is equal to $V(j)$ given in (1.56). Furthermore, the Markov property of Z applied at time t yields that

$$\mathbb{E}_{x,i}\left[\frac{1}{B_T^D}\int_0^T I_{\{Z_s=j\}}\mathrm{d}s\right] = e_i' \exp(t(Q - \Lambda_D))e_j'e_j \exp((T - t)(Q - \Lambda_D))\mathbf{1}.$$

Equation (1.54) follows then by an application of Fubini's theorem. Furthermore, Equation (1.55) follows by noting that the expectation $\mathbb{E}_{x,i}[\Sigma_T/B_T^D]$ can also be obtained by calculating the negative of the derivative of the Laplace transform at zero:

$$\mathbb{E}_{x,i}\left[\frac{\Sigma_T}{B_T^D}\right] = -\frac{\mathrm{d}}{\mathrm{d}u}\left\{[\exp(T(K_\Sigma(u) - \Lambda_D))\mathbf{1}](i)\right\}\Big|_{u=0}.$$

$\qquad\square$

1.5 First passage times for regime-switching processes

1.5.1 Three key matrices

The characteristics of the process (X, Z) can be summarised in terms of three matrices Q_0, Σ and V that will shortly be specified. Given those three matrices we will show how to reconstruct (X, Z) in Section 1.7.

The matrices Q_0, Σ and V will be specified in nine-block matrices using block-notation; the middle block of Q_0, Σ and V describes the rates of regime-switches, and the volatility and drift of the process in the different regimes, while the upper left and lower right block of Q_0 specify the distribution of up-ward and down-ward jumps in the different regimes, in terms of the (phase-type) generators. More precisely, we define the three key matrices Q_0, Σ and V, in block notation, by

$$Q_0 := \begin{pmatrix} B^+ & b^+ & O \\ A^+ & Q - \Lambda_\lambda & A^- \\ O & b^- & B^- \end{pmatrix}, \tag{1.60}$$

$$\Sigma := \begin{pmatrix} O & O & O \\ O & \Lambda_S & O \\ O & O & O \end{pmatrix}, \qquad V := \begin{pmatrix} I & O & O \\ O & \Lambda_M & O \\ O & O & -I \end{pmatrix}. \tag{1.61}$$

Here Q is the generator matrix of the chain Z, and Λ_λ, Λ_S, Λ_M denote $N_0 \times N_0$ diagonal matrices with elements

$$\Lambda_\lambda(i,i) := \lambda_i, \qquad \Lambda_S(i,i) := \sigma(i) \qquad \text{and} \qquad \Lambda_M(i,i) := \mu(i). \tag{1.62}$$

Further, O and I are zero and identity matrices of appropriate sizes such that Q_0, Σ and V are square matrices of the same dimension. In block-notation $A^\pm$, $B^\pm$ and $b^\pm$ are given by

$$A^\pm := \begin{pmatrix} \lambda_1^\pm \beta_1^{\pm\prime} & & \\ & \ddots & \\ & & \lambda_N^\pm \beta_N^{\pm\prime} \end{pmatrix}, \qquad B^\pm := \begin{pmatrix} B_1^\pm & & \\ & \ddots & \\ & & B_N^\pm \end{pmatrix}, \qquad b^\pm := \begin{pmatrix} -B_1^\pm \mathbf{1} & & \\ & \ddots & \\ & & -B_N^\pm \mathbf{1} \end{pmatrix}, \tag{1.63}$$

where $\lambda_i^+ := \lambda_i p_i$ and $\lambda_i^- := \lambda_i(1 - p_i)$.

Remark. The matrix Q_0 is in fact the generator matrix of a Markov chain, as it has non-negative off-diagonal elements and zero row sums. We denote the state-space of this Markov chain by E. In the sequel we will frequently use the following partition of the set E:

$$E^\pm = \{i \in E : \Sigma_{ii} \neq 0\}, \qquad E^+ = \{i \in E : \Sigma_{ii} = 0, V_{ii} > 0\}, \qquad E^- = \{i \in E : \Sigma_{ii} = 0, V_{ii} < 0\}. \tag{1.64}$$

Note further that $E^\pm$ can and will be identified with the state-space E^0 of the chain Z. See Section 1.7 for further properties of the Markov chain defined by the generator Q_0.

1.5.2 Matrix Wiener-Hopf factorisation

For a given vector of discount rates $h : E \to \mathbb{C}$, the matrix Wiener-Hopf factorisation associates to the matrix

$$Q_h := Q_0 - \Lambda_h,$$

where Λ_h is a diagonal matrix with i-th diagonal element $\Lambda_h(i,i) = h(i)$, a quadruple of matrices which, as we will show below, characterises the distributions of the running maximum and minimum of X.

Let us briefly describe the sets of matrices of which this quadruple are elements. Denote by $\mathbb{D}(n)$ the set of $n \times n$ square matrices whose eigenvalues all have non-positive real part. Note that by Lemma 2, $\mathbb{D}(n)$ includes the set $\mathbb{G}(n)$ of $n \times n$ sub-generator matrices (i.e. matrices with non-negative off-diagonal elements and non-positive rows). Recall that $\mathbb{C}^{n \times m}$ denotes the set of $n \times m$ matrices with complex entries. Denote by $\mathbb{H}(n,m)$ the set of $n \times m$ sub-probability matrices (i.e. matrices with non-negative elements and row sums smaller or equal to one).

Denote by N, N^+, $\underline{N}^+$, N^- and $\underline{N}^-$ the number of elements of the sets E, $E^0 \cup E^+$, E^+, $E^0 \cup E^-$, and E^-, respectively. Also, let $\mathcal{H}$ denote the set

$$\mathcal{H} = \left\{ h : E \to \mathbb{C} \quad : \quad \min_{i \in E} \Re(h(i)) \geq 0, \quad \min_{i \in E^0} \Re(h(i)) > 0 \right\}.$$

Definition. Let $h \in \mathcal{H}$ and let W^+, G^+, W^- and G^- be elements of the sets $\mathbb{C}^{N \times N^+}$, $\mathbb{D}(N^+)$, $\mathbb{C}^{N \times N^-}$ and $\mathbb{D}(N^-)$, respectively. A quadruple (W^+, G^+, W^-, G^-) is called a *matrix Wiener-Hopf factorisation* of Q_h if the following matrix equations are satisfied:

$$\frac{1}{2} \Sigma^2 W^+ (G^+)^2 - V W^+ G^+ + Q_h W^+ = O^+, \tag{1.65}$$

$$\frac{1}{2} \Sigma^2 W^- (G^-)^2 + V W^- G^- + Q_h W^- = O^-, \tag{1.66}$$

where O^+ and O^- are zero matrices of size $N \times N^+$ and $N \times N^-$.

Theorem 4 *(i) For any $h \in \mathcal{H}$, there exists a unique matrix Wiener-Hopf factorisation of Q_h, denoted by $(\eta_h^+, Q_h^+, \eta_h^-, Q_h^-)$.*

(ii) If $h = \Re(h)$, then $Q_h^+ \in \mathbb{G}(N^+)$ and $Q_h^- \in \mathbb{G}(N^-)$ are sub-generator matrices, and $\eta_h^+ \in \mathbb{H}(N, N^+)$ and $\eta_h^- \in \mathbb{H}(N, N^-)$ are in block-notation given by

$$\eta_h^+ = \begin{pmatrix} I_+ \\ \underline{\eta}_h^+ \end{pmatrix}, \qquad and \qquad \eta_h^- = \begin{pmatrix} \underline{\eta}_h^- \\ I_- \end{pmatrix}, \tag{1.67}$$

for some matrices $\underline{\eta}_h^+ \in \mathbb{H}(\underline{N}^-, N^+)$ and $\underline{\eta}_h^- \in \mathbb{H}(\underline{N}^+, N^-)$, and identity matrices I_+ and I_- of sizes $N^+ \times N^+$ and $N^- \times N^-$.

Remarks. (i) For h given by $h(i) = q I_{E^0}(i)$ with $q > 0$ we will also write $(\eta_q^+, Q_q^+, \eta_q^-, Q_q^-)$. The proof of Theorem 4 will be given in Section 1.8.

(ii) We allow the vector h to take complex values to be able to deal with a Laplace inversion using a Browmich integral, which involves the integration of the resulting first-passage quantities over a curve in the complex plane. Note that, for any real-valued $h \in \mathcal{H}$, i.e. $h = \Re(h)$, the matrix Q_h is a transient generator matrix, since the off-diagonal elements of Q_h are non-negative and $Q_h \mathbf{1} \neq 0$.

1.5.3 First-passage into a half-line

The marginal distributions of the maximum and minimum as well as the distributions of the first-passage times into a half-line can be described explicitly in terms of the matrix Wiener-Hopf factorisation. Denote by $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$ the running maximum of X at time t and by $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ the corresponding running minimum, and let T_a^+ and T_a^- be the first passage times of X into a half-line,

$$T_a^+ = \inf\{t \geq 0 : X_t \in (a, \infty)\}, \qquad T_a^- = \inf\{t \geq 0 : X_t \in (-\infty, a)\}.$$

The distributions of those random variables are related via

$$\Gamma^+(t) = \mathbb{P}_{x,i}(T_a^+ < t) = \mathbb{P}_{x,i}(\overline{X}_t > a),$$
$$\Gamma^-(t) = \mathbb{P}_{x,i}(T_a^- < t) = \mathbb{P}_{x,i}(-\underline{X}_t > a).$$

If $\mathbf{e}_q$ denotes a random time that is exponentially distributed with parameter $q > 0$ and that is independent of (X, Z), then the running maximum and minimum at time $\mathbf{e}_q$ follows a phase-type distribution, with parameters explicitly given in terms of the matrix Wiener-Hopf factorisation $(\eta_q^+, Q_q^+, \eta_q^-, Q_q^-)$.

For any $h \in \mathcal{H}$, define the $N \times N^+$ and $N \times N^-$ matrices $\Phi_{h,a}^+(x)$ and $\Phi_{h,a}^-(x)$ by

$$\Phi_{h,a}^+(x, i, j) = \left[\eta_h^+ \exp\left((a - x) Q_h^+ \right) \right](i, j) I_{(-\infty, a]}(x) + \delta_{ij} I_{(a, \infty)}(x), \qquad i \in E, j \in E^+ \cup E^0, \quad (1.68)$$
$$\Phi_{h,a}^-(x, i, j) = \left[\eta_h^- \exp\left((x - a) Q_h^- \right) \right](i, j) I_{[a, \infty)}(x) + \delta_{ij} I_{(-\infty, a)}(x), \qquad i \in E, j \in E^- \cup E^0, \quad (1.69)$$

where δ_{ij} is a Kronecker delta (i.e. $\delta_{ij} = I_{\{i\}}(j)$). For h given by $h(i) = q I_{E^0}(i)$ with $q > 0$, we will also denote $\Phi_{h,a}^\pm(x)$ by $\Phi_{q,a}^\pm(x)$.

Proposition 10 *For $q > 0$ it holds that under $\mathbb{P}_{x,i}$*

$$\overline{X}_{\mathbf{e}_q} - x \sim PH(\eta_q^+(i), Q_q^+), \qquad x \in \mathbb{R}, i \in E^0,$$
$$-\underline{X}_{\mathbf{e}_q} + x \sim PH(\eta_q^-(i), Q_q^-), \qquad x \in \mathbb{R}, i \in E^0,$$

where $\eta_q^+(i)$ and $\eta_q^-(i)$ are the i-th rows of η_q^+ and η_q^-. In particular the Laplace transforms $\widehat{\Gamma}^\pm(q) := \int_0^\infty e^{-qt} \Gamma^\pm(t) dt$, for $q > 0$, are given by the formulae

$$\widehat{\Gamma}^+(q) = \frac{1}{q} \sum_{j \in E^0} \mathbb{E}_{x,i}[e^{-qT_a^+} I_{\{Z_{T_a^+} = j\}}] = \frac{1}{q} \sum_{j \in E^0} \Phi_{q,a}^+(x, i, j), \qquad x, a \in \mathbb{R}, i \in E^0,$$

$$\widehat{\Gamma}^-(q) = \frac{1}{q} \sum_{j \in E^0} \mathbb{E}_{x,i}[e^{-qT_a^-} I_{\{Z_{T_a^-} = j\}}] = \frac{1}{q} \sum_{j \in E^0} \Phi_{q,a}^-(x, i, j), \qquad x, a \in \mathbb{R}, i \in E^0.$$

Remarks. (i) The proof of Proposition 10 will be given in Section 1.8.

(ii) Denote by $\Phi_{q,\upsilon}^+(x,i)$ and $\Phi_{q,\ell}^-(x,i)$ the N^+-dimensional and N^--dimensional row vectors with j-th elements $\Phi_{q,\upsilon}^+(x,i,j)$ and $\Phi_{q,\ell}^-(x,i,j)$, respectively. Then Proposition 10 implies that, under the probability measure $\mathbb{P}_{x,i}$, $x \in \mathbb{R}$, $i \in E^0$, the processes $M^+ = \{M_t^+\}_{t\geq 0}$ and $M^- = \{M_t^-\}_{t\geq 0}$, defined by

$$M_t^+ = \mathrm{e}^{-q(t\wedge T_\upsilon^+)}\Phi_{q,\upsilon}^+\left(X_{t\wedge T_\upsilon^+}, Z_{t\wedge T_\upsilon^+}\right), \qquad M_t^- = \mathrm{e}^{-q(t\wedge T_\ell^-)}\Phi_{q,\ell}^-\left(X_{t\wedge T_\ell^-}, Z_{t\wedge T_\ell^-}\right),$$

are row-vectors of bounded martingales. Indeed, the Markov property of (X,Z) implies that

$$\mathbb{E}_{x,i}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}\Big|\mathcal{F}_t^{(X,Z)}\right]$$

$$= I_{\left\{t<T_\upsilon^+\right\}}\mathrm{e}^{-qt}\mathbb{E}_{X_t,Z_t}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}\right] + I_{\left\{t\geq T_\upsilon^+\right\}}\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}$$

$$= I_{\left\{t<T_\upsilon^+\right\}}\mathrm{e}^{-qt}\mathbb{E}_{X_t,Z_t}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}\right] + I_{\left\{t\geq T_\upsilon^+\right\}}\mathrm{e}^{-qT_\upsilon^+}\mathbb{E}_{X_{T_\upsilon^+},Z_{T_\upsilon^+}}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}\right]$$

$$= \mathrm{e}^{-q(t\wedge T_\upsilon^+)}\mathbb{E}_{X_{t\wedge T_\upsilon^+},Z_{t\wedge T_\upsilon^+}}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\left\{Z_{T_\upsilon^+}=j\right\}}\right]$$

$$= \mathrm{e}^{-q(t\wedge T_\upsilon^+)}\Phi_{q,\upsilon}^+\left(X_{t\wedge T_\upsilon^+}, Z_{t\wedge T_\upsilon^+}\right)e_j^+ = M_t^+ e_j^+,$$

where e_j^+ denotes the j-th standard basis vector in $\mathbb{R}^{N^+}$ and we used that $\mathbb{E}_{x,i}\left[\mathrm{e}^{-qT_\upsilon^+} I_{\{Z_{T_\upsilon^+}=j\}}\right] = e_j^+(i)$ if $x \geq \upsilon$ and $i \in E^0$, which directly follows from the definitions (1.68) and .

(iii) Suppose that $Q_0 = O$. This corresponds to a model in which there are no jumps and no switches between the regimes (i.e. with probability one the process stays in the starting regime and evolves as a Brownian motion with drift). In this case we can identify the matrix Wiener-Hopf factorisation in closed form. Note that we have $N^+ = N^- = N$ and hence the matrices $G^\pm$ and $W^\pm$ are of dimension $N \times N$. If we take $W^\pm$ to be equal to the identity matrix and $h(i) = R_D(i) + q$, the matrix equations (1.65)–(1.66) reduce to

$$\frac{1}{2}\Sigma^2(G^+)^2 - VG^+ = (\Lambda_D + qI) = \frac{1}{2}\Sigma^2(G^-)^2 + VG^-,$$

where I is an $N \times N$ identity matrix (recall that the discount rate matrix Λ_D is diagonal and satisfies $\Lambda_D(i,i) = R_D(i)$ for all $i \in E^0$). These equations are satisfied by the diagonal matrices

$$G^+ = \mathrm{diag}(-\omega_i^+, i=1,\ldots,N), \qquad G^- = \mathrm{diag}(-\omega_i^-, i=1,\ldots,N), \tag{1.70}$$

where

$$\omega_i^\pm = \mp\frac{\mu_i}{\sigma_i^2} + \sqrt{\left(\frac{\mu_i}{\sigma_i^2}\right)^2 + \frac{2(q+r_i)}{\sigma_i^2}} \qquad \text{and} \quad r_i := R_D(i). \tag{1.71}$$

In particular, we obtain the well known fact that the maximum of a Brownian motion with drift at an independent exponential time is exponentially distributed:

$$\mathbb{P}_{x,i}\left(\overline{X}_{\mathbf{e}_{q+r_i}} > a\right) = \mathrm{e}^{-\omega_i^+(a-x)} \qquad \text{for } x < a.$$

1.5.4 Joint distribution of the maximum and minimum

In the previous section we have shown how the marginal distribution of the maximum as well as of the minimum can be explicitly expressed in terms of the matrix Wiener-Hopf factorisation. Also the *joint* distribution of the running maximum and minimum,

$$\psi_{x,i}(t) = \mathbb{P}_x(\underline{X}_t > \ell, \overline{X}_t < \upsilon),$$

can be explicitly identified in terms of the matrix Wiener-Hopf factorisation, by considering appropriate linear combinations of the functions Φ^+ and Φ^- defined in (1.68).

To formulate the result, introduce the matrices $Z^+ \in \mathbb{C}^{N^- \times N^+}$ and $Z^- \in \mathbb{C}^{N^+ \times N^-}$ by

$$Z^+(i,j) = \left[\eta_h^+ e^{Q_h^+(\upsilon - \ell)}\right](i,j), \qquad i \in E^0 \cup E^-, \quad j \in E^0 \cup E^+,$$

$$Z^-(i,j) = \left[\eta_h^- e^{Q_h^-(\upsilon - \ell)}\right](i,j), \qquad i \in E^0 \cup E^+, \quad j \in E^0 \cup E^-,$$

and define, for any $h \in \mathcal{H}$ and any $x \in \mathbb{R}$, the $N \times N^+$ and $N \times N^-$ matrices $\Psi_{h,\ell,\upsilon}^+(x)$ and $\Psi_{h,\ell,\upsilon}^-(x)$

$$\Psi_{h,\ell,\upsilon}^+(x) = \left(\eta_h^+ e^{Q_h^+(\upsilon - x)} - \eta_h^- e^{Q_h^-(x - \ell)} Z^+\right)\left(I - Z^- Z^+\right)^{-1} I_{[\ell,\upsilon]}(x) \tag{1.72}$$
$$+ \Delta^+ I_{(\upsilon,\infty)}(x),$$

$$\Psi_{h,\ell,\upsilon}^-(x) = \left(\eta_h^- e^{Q_h^-(x - \ell)} - \eta_h^+ e^{Q_h^+(\upsilon - x)} Z^-\right)\left(I - Z^+ Z^-\right)^{-1} I_{[\ell,\upsilon]}(x) \tag{1.73}$$
$$+ \Delta^- I_{(-\infty,\ell)}(x),$$

where Δ^+ and Δ^- are $N \times N^+$ and $N \times N^-$ matrices with elements

$$\Delta^+(i,j) = \delta_{ij}, \qquad \Delta^-(i,k) = \delta_{ik}, \qquad\qquad i \in E, j \in E^+ \cup E^0, k \in E^0 \cup E^-.$$

For h given by $h(i) = q I_{E^0}(i)$ with $q > 0$, we will also write $\Psi_{h,\ell,\upsilon}^\pm(x) = \Psi_{q,\ell,\upsilon}^\pm(x)$.

Proposition 11 *Let $i \in E^0$ and*

$$\psi_{x,i}^+(t) = \mathbb{P}_{x,i}(T_{\ell,\upsilon} < t, X_{T_{\ell,\upsilon}} \geq \upsilon), \qquad \psi_{x,i}^-(t) = \mathbb{P}_{x,i}(T_{\ell,\upsilon} < t, X_{T_{\ell,\upsilon}} \leq \ell),$$

where

$$T_{\ell,\upsilon} := \inf\{t \geq 0 : X_t \notin [\ell,\upsilon]\} = T_\ell^- \wedge T_\upsilon^+.$$

For any $q > 0$ the Laplace transforms in t of $\psi_{x,i}^+$ and $\psi_{x,i}^-$ are given by

$$\widehat{\psi}_{x,i}^+(q) = \frac{1}{q}(\Psi_{q,x}^+ \mathbf{1})(i), \qquad \widehat{\psi}_{x,i}^-(q) = \frac{1}{q}(\Psi_{q,x}^- \mathbf{1})(i).$$

The Laplace transform of $\psi_{x,i}(t) = \mathbb{P}_{x,i}(\underline{X}_t > \ell, \overline{X}_t < \upsilon)$ is hence of the form

$$\widehat{\psi}_{x,i}(q) = \widehat{\psi}_{x,i}^+(q) + \widehat{\psi}_{x,i}^-(q). \tag{1.74}$$

Remarks. (i) If $Q_0 = O$, then $\eta_q^{\pm}$ are identity matrices and the following identities hold:

$$\Psi_{q,x}^+ = (\mathrm{e}^{Q_q^+(v-x)} - \mathrm{e}^{Q_q^-(x-\ell)}\mathrm{e}^{Q_q^+(v-\ell)})(I - \mathrm{e}^{Q_q^-(v-\ell)}\mathrm{e}^{Q_q^+(v-\ell)})^{-1},$$

$$\Psi_{q,x}^- = (\mathrm{e}^{Q_q^+(v-x)} - \mathrm{e}^{Q_q^+(x-\ell)}\mathrm{e}^{Q_q^-(v-\ell)})(I - \mathrm{e}^{Q_q^+(v-\ell)}\mathrm{e}^{Q_q^-(v-\ell)})^{-1},$$

where $Q_q^{\pm}$ are diagonal matrices given in (1.70). In particular, we find the well known two-sided exit identity for Brownian motion with drift:

$$\widehat{\psi}_{x,i}(q) = \mathbb{E}_{x,i}[\mathrm{e}^{-q\tau_{\ell,v}}I_{\{\tau_v < \tau_\ell\}}] = \frac{\mathrm{e}^{\omega_i^+(x-\ell)} - \mathrm{e}^{\omega_i^-(x-\ell)}}{\mathrm{e}^{\omega_i^+(v-\ell)} - \mathrm{e}^{\omega_i^-(v-\ell)}},$$

where $\omega_i^{\pm}$ is given in (1.71) with $r_i = 0$.

(ii) In the case that no volatility is present ($\Sigma \equiv 0$) the identities simplify and we find the expressions in [5].

(iii) Under $\mathbb{P}_{x,i}$, $x \in \mathbb{R}$, $i \in E^0$, the process $\widetilde{M}^+ = \{\widetilde{M}_t^+\}_{t \geq 0}$ defined by

$$\widetilde{M}_t^+ = \mathrm{e}^{-q(t \wedge T_{\ell,v})}\Psi_{q,\ell,v}^+\left(X_{t \wedge T_{\ell,v}}, Z_{t \wedge T_{\ell,v}}\right),$$

is a row-vector of bounded martingales, where we denoted by $\Psi_{q,\ell,v}^+(x,i)$ the N^+-dimensional row vector with j-th element $\Psi_{q,\ell,v}^+(x,i,j)$.

Proof. Let $q > 0$ and define

$$g^+(x,i) = \Psi_{q,\ell,v}^+(x,i).$$

In view of the fact that $\widetilde{M}^+$ is a bounded martingale, it holds that

$$\begin{aligned}
g^+(x,i) &= \sum_{j \in E^0 \cup E^+} \mathbb{E}_{x,i}\left[\mathrm{e}^{-q\tau}\Psi_{q,\ell,v}^+(X_\tau, Z_\tau, j)I_{\{\tau < \infty\}}\right] \\
&= \sum_{j \in E^0 \cup E^+} \mathbb{E}_{x,i}\left[\mathrm{e}^{-q\tau}I_{\{X_\tau \geq v, Z_\tau = j\}}\right] \\
&= q\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_{x,i}(\tau < t, X_\tau \geq v)\mathrm{d}t = q\widehat{\psi}_{x,i}^+(q).
\end{aligned}$$

Here we used that, in view of the definitions (1.72) and (1.67) of $\Psi_{q,\ell,v}^+$ and η_q^+, the function g^+ satisfies:

$$\begin{aligned}
g^+(x,i) &= 1 \qquad \text{if } x \geq v,\ i \in E^0 \\
g^+(x,i) &= 0 \qquad \text{if } x \leq \ell,\ i \in E^0.
\end{aligned}$$

The expression for $\widehat{\psi}_{x,i}^-$ can be derived by a similar reasoning. $\qquad\square$

1.5.5 Valuing a double-barrier rebate option

A double-barrier rebate option pays a constant rebate L at the moment τ one of the barrier levels is crossed, if this happens before maturity T. By standard arbitrage pricing arguments the Laplace transform $\widehat{v}_{reb}$ in maturity of the price of v_{reb} such an option is given by

$$\widehat{v}_{reb}(q) = \frac{1}{q}\mathbb{E}_{x,i}\left[\left(B_\tau^D\right)^{-1}\exp\left(-q\tau\right)\right] \tag{1.75}$$

We will find below the following more general quantity, which will also be employed in the sequel:

$$H^{x,i}(q,u,j) := \mathbb{E}_{x,i}\left[\left(B_\tau^D\right)^{-1}\exp\left(\mathrm{i}uX_\tau - q\tau\right)I_{\{Z_\tau=j\}}\right] \quad \text{for} \quad j = 1,\ldots,N. \tag{1.76}$$

In the following result an explicit expression is given for the quantity $H_j^{x,i}$ in terms of the matrix Wiener-Hopf factorisation, which is an extension of the identity (1.74).

We denote by E_i^+ and E_i^-, for $i \in E^0 = \{1,\ldots,N\}$, the parts of the state-space E corresponding to the blocks B_i^+ and B_i^- in the matrix Q_0 in (1.60). Recall that the definition of $\alpha_i^\pm$ was given in (1.21).

Theorem 5 *For any $h \in \mathcal{H}$, $i,j \in E^0$, and $u \in \mathbb{C}$ that satisfies $\Im(u) \in (-\alpha_j^+, \alpha_j^-)$ it holds that*

$$H_j^{x,i}(h,u) = \left(\Psi_{h,x}^+ k_{j,u}^+\right)(i) + \left(\Psi_{h,x}^- k_{j,u}^-\right)(i), \tag{1.77}$$

where the column vectors $k_{j,u}^+ = (k_{j,u}^+(i), i \in E^+ \cup E^0)$ and $k_{j,u}^- = (k_{j,u}^-(i), i \in E^0 \cup E^-)$ are given by

$$k_{j,u}^+(i) = \mathrm{e}^{uv} \cdot \begin{cases} 1, & \text{if } i = j \in E^0, \\ \left((-uI_j^+ - B_j^+)^{-1}(-B_j^+)\mathbf{1}\right)(i), & \text{if } i \in E_j^+, \\ 0, & \text{otherwise} \end{cases}$$

$$k_{j,u}^-(i) = \mathrm{e}^{u\ell} \cdot \begin{cases} 1, & \text{if } i = j \in E^0, \\ \left((uI_j^- - B_j^-)^{-1}(-B_j^-)\mathbf{1}\right)(i), & \text{if } i \in E_j^-, \\ 0, & \text{otherwise} \end{cases}$$

where I_j^+ and I_j^- are $|E_j^+| \times |E_j^+|$ and $|E_j^-| \times |E_j^-|$ identity matrices.

The proof is given in Section 1.8.

1.6 Double-no-touch and other barrier options

In this section we show how the prices of double-no-touch and double knock-out call options can be expressed in terms characteristic function H of the process at the corresponding first passage time, which was identified in Theorem 5.

A *double-no-touch* is a derivative security that pays one unit of the underlying asset at expiry T if the underlying asset price does not leave the interval $[\exp(\ell), \exp(v)]$ during the time period $[0,T]$, where $\ell < v$. Similarly a *double knock-out call option* struck at K delivers the payoff $(S_T - K)^+ := \max\{S_T - K, 0\}$ if throughout the life of the option the asset price stays within the interval $[\exp(\ell), \exp(v)]$. The arbitrage free prices for a double-no-touch and a double knock-out call option are respectively given by

$$D_{x,i}(T) = \mathbb{E}_{x,i}\left[\frac{I_{\{\tau>T\}}}{B_T^D}\right], \tag{1.78}$$

$$C_{x,i}(k,T) = \mathbb{E}_{x,i}\left[\frac{I_{\{\tau>T\}}}{B_T^D}(S_T - K)^+\right], \tag{1.79}$$

where $k = \log K$ is the log-strike and τ is the first time the process S leaves the interval $[\exp(\ell), \exp(v)]$ or equivalently

$$\tau := \inf\{t \geq 0 : X_t \notin [\ell, v]\}. \tag{1.80}$$

We will find it more convenient to consider the double-touch-in and the knock-in call whose values v_{dti} and v_{kic}, as function of maturity T and log strike $k = \log K$, are given by

$$v_{dti}(T) := \mathbb{E}_{x,i}\left[\frac{I_{\{\tau \leq T\}}}{B_T^D}\right] = \mathbb{E}_i\left[(B_T^D)^{-1}\right] - D_{x,i}(T), \tag{1.81}$$

$$v_{kic}(T, k) := \mathbb{E}_{x,i}\left[\frac{I_{\{\tau \leq T\}}}{B_T^D}(S_T - K)^+\right] = \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T - \mathrm{e}^k)^+\right] - C_{x,i}(k, T). \tag{1.82}$$

Since the zero-coupon bond price as well as the call option price have already been identified, the problem of calculating the double-no-touch and knock-out call prices thus reduces to identifying v_{dti} and v_{kic}. In general, no closed form expressions are known for these function in terms of elementary functions. Below we show that the Laplace transform in T of $v_{dti}(T)$ as well as the joint Fourier-Laplace transform in (k, T) of v_{kic} can be identified explicitly in terms of the parameters that define the log-price process X. Both transforms will be identified in terms of Laplace transform $\widehat{F}_{x,i}(u, q)$ in T of the function $T \mapsto F_{x,i}(u, T)$ given by

$$F_{x,i}(u, T) := \mathbb{E}_{x,i}\left[\frac{I_{\{\tau \leq T\}}}{B_T^D}\exp(\mathtt{i}uX_T)\right]. \tag{1.83}$$

The Laplace transform $\widehat{F}_{x,i}(u, q)$ in turn will be given in terms of the N_0-vector $H^{x,i}(q, u)$ whose coordinates are given by

$$H_j^{x,i}(q, u) := \mathbb{E}_{x,i}\left[\exp\left(\mathtt{i}uX_\tau - \int_0^\tau (R_D(Z_s) + q)ds\right) I_{\{Z_\tau = j\}}\right] \quad \text{for} \quad j = 1, \ldots, N_0.$$

Theorem 6 *For any $q > 0$ and ξ with $\Im(\xi) < 0$ it holds that*

$$\widehat{v}_{dti}(q) = \widehat{F}_{x,i}(0, q), \tag{1.84}$$

$$\widehat{v}_{kic}^*(q, \xi) = \frac{1}{\mathtt{i}\xi - \xi^2}\widehat{F}_{x,i}(\xi - \mathtt{i}, q). \tag{1.85}$$

Here, $\widehat{F}_{x,i}(u, q)$ is given by

$$\widehat{F}_{x,i}(u, q) = \left(H^{x,i}(q, u)\right)'(qI + \Lambda_D - K(u))^{-1}\mathbf{1}, \tag{1.86}$$

for all $q \in \mathbb{C}$, such that $\Re(q) > q^* = \max\{\Re(\psi_i(u)) - R_D(i) : i = 1, \ldots, N_0\}$.

Remarks. (i) Note that if $u = 0$, then $q^* \leq 0$, as the interest rates $R_D(i)$ is assumed to be non-negative.

(ii) The Laplace transform in (1.86) can be inverted by evaluating the Bromwich integral

$$F_{x,i}(u, t) = \frac{1}{2\pi}\int_{c-\mathtt{i}\infty}^{c+\mathtt{i}\infty} \mathrm{e}^{tq}\left(H^{x,i}(q, u)\right)'(qI + \Lambda_D - K(u))^{-1}\mathbf{1}\mathrm{d}q, \tag{1.87}$$

for any $c > q^*$. An efficient algorithm to approximate this integral can e.g. be found in Abate and Whitt [1].

Proof. First note that by a similar calculation to the one in the proof of Proposition 6 we find that the Fourier-transform in log-strike k of $e^{\alpha k} v_{kic}(T, k)$ can be expressed in terms of F by

$$\int_{\mathbb{R}} e^{(iv + \alpha)k} v_{kic}(T, k) dk = \frac{1}{(\alpha + iv)(1 + \alpha + iv)} \mathbb{E}_{x,i}\left[\frac{I_{\{\tau \leq T\}}}{B_T^D} \exp\{(1 + \alpha + iv)X_T\}\right],$$

for any $\alpha > 0$.

Assume next that $q > 0$ is real with $q > q^*$. The Laplace transform $\widehat{F}_{x,i}(u, q)$ has the following well-known equivalent probabilistic representation

$$\widehat{F}_{x,i}(u, q) = \mathbb{E}[F_{x,i}(u, e_q)]/q, \tag{1.88}$$

where e_q is an exponential random variable with parameter q which is independent of the process (X, Z).

By conditioning on $\mathcal{F}_\tau$ and applying strong Markov property at τ of the process (X, Z) together with Lemma 4 we find that

$$
\begin{aligned}
\mathbb{E}[F_{x,i}(u, e_q)] &= \mathbb{E}_{x,i}\left[\frac{\exp(iuX_\tau)}{B_\tau^D} \mathbb{E}\left[I_{\{\tau \leq e_q\}} \exp\left(-\int_\tau^{e_q} R_D(Z_s)ds + iu(X_{e_q} - X_\tau)\right) \Big| \mathcal{F}_\tau\right]\right] \\
&= \mathbb{E}_{x,i}\left[\frac{\exp(iuX_\tau)}{B_\tau^D} h(Z_\tau, q, u)e^{-q\tau}\right] \\
&= \sum_j H_j^{x,i}(q, u)h(j, q, u), \tag{1.89}
\end{aligned}
$$

where the value $h(j, q, u)$ for each state $j \in E^0$ of the Markov chain Z is given by

$$
\begin{aligned}
h(j, q, u) &:= \mathbb{E}_{0,j}\left[\exp\left(-\int_0^{e_q'} R_D(Z_s)ds + iuX_{e_q'}\right)\right] \tag{1.90} \\
&= \left[(qI + \Lambda_D - K(u))^{-1}\mathbf{1}\right](j)
\end{aligned}
$$

for some exponential random variable e_q' with parameter q that is independent of the Markov process (X, Z), where the second equality follows from Lemma 3 and Theorem 1.

A key observation that follows from the representation (1.89) is that the function

$$q \mapsto \mathbb{E}[F_{x,i}(u, e_q)]/q$$

has a holomorphic extension to the complex half-plane $\{q \in \mathbb{C} : \Re(q) > q^*\}$ which therefore[16] coincides on this domain with the Laplace transform. Thus, Equation (1.89) holds for q in this domain, and the proof is complete. $\qquad\square$

---

[16] If two holomorphic functions defined on a connected open set Ω in $\mathbb{C}$ coincide on a subset with at least one accumulation point in Ω, then they coincide on the entire Ω. For a proof of this well-known statement see [29], page 208, Theorem 10.18.

1.7 Embedding of the process (X, Z)

In this section and the next we will provide a proof of the matrix Wiener-Hopf factorisation and its corollaries derived in previous sections. We will proceed in two steps:

(i) Reduction of the first-passage problems of X over constant levels, which will involve overshoots and undershoots due to the jumps of X, to the first-hitting problem of a constant level by a regime-switching Brownian motion, employing a classical argument for the embedding of phase-type jumps (see Asmussen [2]), which we review in this section.

(ii) Solution of the first-passage problem of regime-switching Brownian motion via a characterisation of the dynamics of the ladder processes, which is carried out in Section 1.8.

Let Y be a continuous time Markov chain with finite state space $E \cup \partial$ and generator restricted to E given by Q_0, where ∂ is an absorbing cemetery state and E and Q_0 will be specified shortly, and denote by $\xi = \{\xi_t, t \geq 0\}$ the Markov modulated Brownian motion given by

$$\xi_t = x + \int_0^t s(Y_s) \mathrm{d}W_s + \int_0^t m(Y_s) \mathrm{d}s, \tag{1.91}$$

where $x \in \mathbb{R}$ is the starting point of ξ and s and m are functions from $E \cup \partial$ to $\mathbb{R}$ also to be specified shortly. The couple (ξ, Y) defined as such is a two-dimensional strong Markov process. In the sequel we will denote by $\widetilde{\mathbb{P}}_{x,i}$ and $\widetilde{\mathbb{E}}_{x,i}$ the conditional probability $\widetilde{\mathbb{P}}_{x,i} = \widetilde{\mathbb{P}}[\cdot | \xi_0 = x, Y_0 = i]$ and conditional expectation $\widetilde{\mathbb{E}}_{x,i} = \widetilde{\mathbb{E}}[\cdot | \xi_0 = x, Y_0 = i]$ respectively.

Let the state-space E be as in Section 1.5. In other words $E = E^- \cup E^0 \cup E^+$ where E^0 is the state-space of the chain Z and E^+ and E^- are given in (1.64). The generator Q_0 is given in (1.60) and $m(i) := \Lambda_V(i,i)$, $s(i) := \Lambda_\Sigma(i,i)$ for all $i \in E$, where the matrices Λ_V, Λ_Σ are defined in (1.61). Thus, while the chain Y is in state $j \in E^0$, ξ evolves as a Brownian motion with drift $m(j)$ and volatility $s(j)$, and while Y takes values in E^+ and E^-, the path of ξ is linear with slope $+1$ or -1. Informally, a path of X can be obtained from a path of ξ by replacing these stretches of unit slope by jumps of the same length, as is illustrated in Figure 1.1. These linear increasing and decreasing stretches of path of ξ thus correspond to the positive and negative jumps of X, respectively.

More formally, by appropriately time-changing (ξ, Y) a stochastic process can be constructed that has the same law as (X, Z). Denote by

$$T_0(t) = \int_0^t I_{\{Y_s \in E^0\}} \mathrm{d}s \qquad \text{and} \qquad T_0^{-1}(u) = \inf\{t \geq 0 : T_0(t) > u\},$$

the time before t spent by the chain Y in E^0 and its right-continuous inverse respectively. It is clear from the definition of the generator Q_0 in (1.60) that when the chain Y jumps from any of the states in $E^+ \cup E^-$ to a state $j \in E^0$, it must have been in the state j just before it had left E^0. Therefore the form of the matrix (1.60) implies that the process $Y \circ T_0^{-1}$, which simply ignores all excursions of Y into $E^+ \cup E^-$, is a Markov chain with the same generator as Z. Furthermore it is straightforward to verify that $\xi \circ T_0^{-1}$ is a
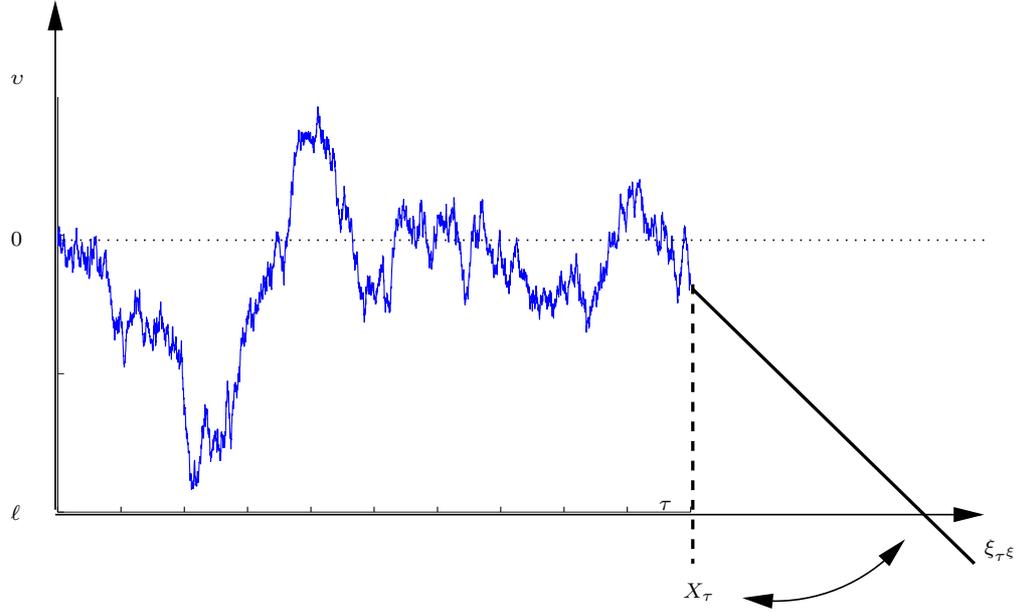
Fig. 1.1. Shown is a sample path of X until the first time τ that X exits the interval $[\ell, v]$. The process ξ has no positive jumps and always hits a level at first-passage.

regime-switching jump-diffusion and the law of the process $(\xi \circ T_0^{-1}, Y \circ T_0^{-1})$ under $\widetilde{\mathbb{P}}_{x,i}$ is equal to that of (X, Z) under $\mathbb{P}_{x,i}$, for all $x \in \mathbb{R}$ and $i \in E^0$.

If we define the first-passage time

$$\tau^\xi = \inf\{t \geq 0 : \xi_t \notin [\ell, v] \quad \text{and} \quad Y_t \in E^0\} \tag{1.92}$$

it follows that random variables $T_0(\tau^\xi)$ and the stopping time τ defined in (1.80) have the same distribution. In particular, with the extension of h to E that puts $h(i) = 0$ for $i \notin E^0$ and that we will also denote by h, it holds that

$$\left(\xi_{\tau^\xi}, \int_0^{\tau^\xi} h(Y_s)\mathrm{d}s, Y_{\tau^\xi}\right) \quad \text{under } \widetilde{\mathbb{P}}_{x,i} \text{ has the same distribution as } \left(X_\tau, \int_0^\tau h(Z_s)\mathrm{d}s, Z_\tau\right) \text{ under } \mathbb{P}_{x,i}$$

for $x \in \mathbb{R}$ and $i \in E^0$. The function $H_j^{x,i}$ defined in (1.76) can thus be expressed in terms of the embedding (ξ, Y) as follows:

$$H_j^{x,i}(h, u) = \widetilde{\mathbb{E}}_{x,i}\left[\exp\left(u\xi_{\tau^\xi} - \int_0^{\tau^\xi} h(Y_s)\mathrm{d}s\right) I_{\{Y_{\tau^\xi} = j\}}\right] \qquad i, j \in E^0. \tag{1.93}$$

1.8 Ladder processes

This section is devoted to the proof of existence and uniqueness of the h-matrix Wiener-Hopf factorisation, for any $h \in \mathcal{H}$. For real $h \in \mathcal{H}$ the vector h has the probabilistic interpretation of a vector of state-dependent

rates of discounting, and in this case the matrix Wiener-Hopf factorisation has a probabilistic interpretation in terms of ladder processes. The existence and uniqueness results are extended to the case of general complex $h \in \mathcal{H}$ by analytical continuation arguments. It is of interest to consider the case of complex entries, as the Laplace transforms of barrier option prices are expressed in terms of the matrix Wiener-Hopf factorisation as we saw above and some widely used Laplace transform inversion algorithms are based on the Bromwich (complex) integral representation.

A classical probabilistic approach to characterisation of the joint distribution of τ_a^+ and the position of Y at τ_a^+ is to consider the up-crossing ladder process of (ξ, Y), defined as follows:

Definition. The *up-crossing ladder process* $Y^+ = \{Y_t^+\}_{t \geq 0}$ of (ξ, Y) is given by

$$Y_a^+ := \begin{cases} Y(\tau_a^+), & \text{if } \tau_a^+ < \infty, \\ \partial, & \text{otherwise,} \end{cases} \tag{1.94}$$

where ∂ is a graveyard state and

$$\tau_a^+ = \inf\{s \geq 0 : \xi_s > a\}$$

with $\inf \emptyset = \infty$.

Remark. In the case that the original chain is killed at (state-dependent, real-valued) rate h, the up-crossing ladder process can be defined as follows. Recall from Markov chain theory that the chain Y^h with state-space $E \cup \{\partial\}$ and generator matrix

$$\begin{pmatrix} Q_h & -\Lambda_h \mathbf{1} \\ \mathbf{0} & 0 \end{pmatrix} \qquad \text{where } Q_h := Q_0 - \Lambda_h \text{ and } \Lambda_h \text{ is the diagonal matrix with } (\Lambda_h)_{ii} = h_i$$

has the same distribution as the chain Y killed (i.e. sent to the graveyard state ∂) independently at rate $h(i)$ when $Y_t = i$. In particular, for $i, j \in E$ and $t \geq 0$ it holds that

$$\widetilde{\mathbb{E}}_i \left[e^{-\int_0^t h(Y_s)ds} I_{\{Y_t=j\}} \right] = \widetilde{\mathbb{P}}(Y_t^h = j | Y_0^h = i) = \exp(Q_h t)(i, j). \tag{1.95}$$

If we denote by $\xi^h = \{\xi_t^h\}_{t \geq 0}$ the process defined by (1.91) driven by the "killed" chain Y^h instead of Y, then the up-crossing ladder process Y^{h+} is given by

$$Y_t^{h+} = \begin{cases} Y_{\tau_t^{h+}}^h & \text{if } \tau_t^{h+} < \infty, \\ \partial & \text{otherwise,} \end{cases}$$

where τ_a^{h+} is the stopping time defined by

$$\tau_a^{h+} = \inf\{s \geq 0 : \xi_s^h > a\}$$

with $\inf \emptyset = \infty$.

Proposition 12 *The process Y^+ is a Markov chain with state-space $E^0 \cup E^+ \cup \{\partial\}$.*

Remarks. (i) We denote by Q_0^+ the generator of Y^+ restricted to $E^0 \cup E^+$ and by η_0^+ the initial distribution, given by

$$\eta_0^+(i,j) = \widetilde{\mathbb{P}}_{0,i}\left[Y_0^+ = j, \tau_0^+ < \infty\right] \qquad \text{for } i \in E, \ j \in E^+ \cup E^0 \tag{1.96}$$

(ii) More generally, for any real valued $h \in \mathcal{H}$ it holds that Y^{h+} is a Markov chain, with generator, restricted to $E^0 \cup E^+$ denoted by Q_h^+ and initial (sub-)probability distribution by $\eta_h^+ = (\eta_h^+(i,j), i \in E, j \in E^+ \cup E^0)$ with

$$\eta_h^+(i,j) = \widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_0^+} h(Y_s)\mathrm{d}s} I_{\{Y_0^+ = j, \tau_0^+ < \infty\}}\right] \quad \text{for } i \in E, \ j \in E^+ \cup E^0. \tag{1.97}$$

We have the following identities analogous to (1.95):

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[e^{-\int_0^{\tau_t^+} h(Y_s)\mathrm{d}s} I_{\{Y_{\tau_t^+} = j, \tau_t^+ < \infty\}}\Big| Y_0 = i\right] &= \widetilde{\mathbb{E}}\left[e^{-\int_{\tau_0^+}^{\tau_t^+} h(Y_s)\mathrm{d}s} I_{\{Y_{\tau_t^+} = j, \tau_t^+ < \infty\}}\Big| Y_{\tau_0^+} = i\right] \\
&= \widetilde{\mathbb{P}}\left(Y_{\tau_t^+}^h = j, \tau_t^{h+} < \infty \big| Y_{\tau_0^+}^h = i\right) \\
&= \widetilde{\mathbb{P}}(Y_t^{h+} = j | Y_0^{h+} = i) \\
&= \exp(Q_h^+ t)(i,j), \qquad i,j \in E^0 \cup E^+, \tag{1.98}
\end{aligned}
$$

where we wrote $\widetilde{\mathbb{P}}[\cdot|A] := \widetilde{\mathbb{P}}[\cdot|A \cap \{\xi_0 = 0\}]$ and $\widetilde{\mathbb{E}}[\cdot|A] := \widetilde{\mathbb{E}}[\cdot|A \cap \{\xi_0 = 0\}]$ to simplify the notation. Note that the first equality holds since $\tau_0^+ = 0$ if the chain Y starts at $i \in E^0 \cup E^+$. In particular, From (1.98) we find that for $i,j \in E^0 \cup E^+$

$$Q_h^+(i,j) = \lim_{t \downarrow 0} \frac{1}{t} \widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_t^+} h(Y_s)\mathrm{d}s} I_{\{Y_{\tau_t^+} = j, \tau_t^+ < \infty\}}\right], \qquad i \neq j, \tag{1.99}$$

$$Q_h^+(i,i) = \lim_{t \downarrow 0} \frac{1}{t}\left\{\widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_t^+} h(Y_s)\mathrm{d}s} I_{\{Y_{\tau_t^+} = i, \tau_t^+ < \infty\}}\right] - 1\right\}. \tag{1.100}$$

Proof. Let f be any bounded real-valued Borel function and let $b < a$. Denote by $\mathcal{F}_a = \sigma\{Y_u^+\}_{u \leq a}$ and $\mathcal{G}_t = \sigma\{Y_s\}_{s \leq t}$ the sigma algebras generated by Y_u^+ up to time a and by Y_s up to time t. Observing that $\mathcal{F}_b \subset \mathcal{G}_{\tau_b}$ and using the spatial homogeneity and continuity of ξ and the strong Markov property of (ξ, Y) we find that

$$
\begin{aligned}
\widetilde{\mathbb{E}}_{x,i}[f(Y_a^+)|\mathcal{F}_b] &= \widetilde{\mathbb{E}}_{x,i}\left[\widetilde{\mathbb{E}}_{x,i}\left[f(Y_{\tau_a}^+)|\mathcal{G}_{\tau_b}\right]\Big|\mathcal{F}_b\right] \\
&= \widetilde{\mathbb{E}}_{x,i}\left[\widetilde{\mathbb{E}}_{b,Y_{\tau_b}^+}[f(Y_{\tau_a}^+)]\Big|\mathcal{F}_b\right] \\
&= \widetilde{\mathbb{E}}_{x,i}\left[\widetilde{\mathbb{E}}_{Y_b^+}[f(Y_{a-b}^+)]\Big|\mathcal{F}_b\right] = \widetilde{\mathbb{E}}_{Y_b^+}[f(Y_{a-b}^+)]
\end{aligned}
$$

where we wrote $\widetilde{\mathbb{E}}_j = \widetilde{\mathbb{E}}_{0,j}$. $\qquad \square$

For any $h \in \mathcal{H}$, we define the square-matrix (resolvent) functions $u \mapsto R_h^+(u)$ of dimension N^+ on the complex half-plane $\mathbb{C}_{>0}$ by

$$R_h^+(u)(i,j) = \widetilde{\mathbb{E}}_{0,i}\left[\int_0^\infty e^{-uy - \int_0^{\tau_y^+} h(Y_s)ds} I_{\{Y_y^+ = j, \tau_y^+ < \infty\}} dy\right] \quad \text{for } i,j \in E^+ \cup E^0 \text{ and } u \in \mathbb{C}_{>0}. \tag{1.101}$$

The matrix Q_h^+ can then be defined for any $h \in \mathcal{H}$, as follows:

Lemma 5 *Let $h \in \mathcal{H}$ and $a \geq 0$ and define the matrix $Q_h^+ \in \mathbb{D}(N^+)$ by*

$$Q_h^+(i,j) = -\omega_i^+ I_{\{i=j\}} + \begin{cases} \frac{2}{\sigma_i^2}(\widetilde{Q}\eta_h^+ R_h^+(\omega_i^-))(i,j) & \text{if } i \in E^0, \\ (\widetilde{Q}\eta_h^+)(i,j) & \text{if } i \in E^+, \end{cases} \tag{1.102}$$

where $q_i = -Q_0(i,i)$, $h_i = h(i)$, $\widetilde{Q} = Q_0 + \text{diag}\{q_i : i \in E\}$ and

$$\omega_i^+ = \begin{cases} F\left(\dfrac{\mu_i}{\sigma_i^2}, \dfrac{q_i + h_i}{\sigma_i^2}\right) & \text{if } i \in E^0 \\ q_i + h_i & \text{if } i \in E^\pm \end{cases} \tag{1.103}$$

where

$$F(\nu,\theta) := -\nu + \sqrt{\nu^2 + 2\theta} \qquad \nu \in \mathbb{R}, \; \theta \in \mathbb{C}_{>0}. \tag{1.104}$$

Then it holds that

$$\widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_a^+} h(Y_s)ds} I_{\{Y_a^+ = j, \tau_a^+ < \infty\}}\right] = \exp(Q_h^+ a)(i,j) \qquad i,j \in E^0 \cup E^+. \tag{1.105}$$

Remarks. (i) Note that Proposition 10 follows as a direct consequence of Equation (1.105) since

$$\mathbb{P}_{x,i}\left(\overline{X}_{\mathbf{e}_q} > a\right) = \mathbb{P}_{x,i}\left(Y_\zeta^{q+} > a\right),$$

where ζ is the life time of the killed up-crossing ladder process.

(ii) Equation (1.105) yields in particular the joint Laplace transform of the vector

$$\mathbf{Z}_a^+ = \left(\int_0^{\tau_a^+} I_{\{Y_s = i\}} ds, \; i \in E\right), \tag{1.106}$$

whose components record the length of time spent by Y in each of the states in E, until the moment τ_a^+ of first-passage.

(iii) The *down-crossing ladder process* Y^-, defined as the up-crossing ladder process of $(-\xi, Y)$, is a Markov chain with generator restricted to $E_0 \cup E^-$ denoted by Q_0^-. Analogously to equation (1.102), for any $h \in \mathcal{H}$ a matrix Q_h^- can be defined, which satisfies

$$\widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_a^-} h(Y_s)ds} I_{\{Y_a^- = j, \tau_a^- < \infty\}}\right] = \exp(Q_h^- a)(i,j) \qquad i,j \in E^0 \cup E^-. \tag{1.107}$$

1.8.1 Proof of Lemma 5

We will show that the limits in (1.99)–(1.100) in fact exist for any $h \in \mathcal{H}$, and identify these. Let

$$\rho = \inf\{t \geq 0 : Y_t \neq Y_0\}$$

be the first time that the chain Y jumps and $\tau_a^i = \inf\{t \geq 0 : X_t^i > a\}$ the first-passage time of $X_t^i := \mu_i t + \sigma_i W_t$ (recall that if $i \in E^\pm$, then $\mu_i = \pm 1$ and $\sigma_i = 0$) over the level a and let $\mathbf{e}_i$ be an exponential random time with mean $1/q_i$ (with $q_i = -Q_0(i,i)$) that is independent of X^i. In view of the definition of ξ (i.e. the first jump time ρ of Y is independent of X^i), it follows that

$$\widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_t^+} h(Y_s)ds}I_{\{Y_t^+=j,\tau_t^+<\infty,\tau_t^+<\rho\}}\right] = I_{\{i=j\}}\widetilde{\mathbb{E}}\left[e^{-\tau_t^i h(i)}I_{\{\tau_t^i<\mathbf{e}_i\}}\right] = I_{\{i=j\}}\widetilde{\mathbb{E}}\left[e^{-\tau_t^i(h_i+q_i)}\right]$$

$$= I_{\{i=j,i\in E^+\}}\exp\left(-t(h_i+q_i)\right)$$

$$\quad + I_{\{i=j,i\in E^0\}}\exp\left(-t\sigma_i^{-2}\left(-\mu_i + \sqrt{\mu_i^2 + 2(q_i+h_i)\sigma_i^2}\right)\right)$$

$$= I_{\{i=j\}}[1 - \omega_i^+ t + o(t)] \qquad \text{as } t \downarrow 0,$$

where ω_i^+ was defined in Lemma 5 and we used the fact that the Laplace transform of τ_a^i is given by

$$\widetilde{\mathbb{E}}[e^{-q\tau_a^i}] = \exp\left(-a\sigma_i^{-2}\left(-\mu_i + \sqrt{\mu_i^2 + 2q\sigma_i^2}\right)\right) \qquad a > 0, \ q \in \mathbb{C}_{>0}.$$

For every $h \in \mathcal{H}$ we can define the following matrices

$$\widetilde{P}_t^{h+}(i,j) := \widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_t^+} h(Y_s)ds}I_{\{Y_t^+=j,\tau_t^+<\infty\}}\right] \qquad \text{for } i,j \in E^+ \cup E^0. \tag{1.108}$$

Note that definiton in (1.101) implies that the identity $R_h^+(u)(i,j) = \int_0^\infty e^{-uy}\widetilde{P}_y^{h+}(i,j)dy$ holds for all $u \in \mathbb{C}_{>0}$. For every $h \in \mathcal{H}$, $x \in (-\infty, t]$ and $m \in E$ we have the follwing identity

$$\widetilde{\mathbb{E}}_{x,m}\left[e^{-\int_0^{\tau_t^+} h(Y_s)ds}I_{\{Y_t^+=j,\tau_t^+<\infty\}}\right] = \widetilde{\mathbb{E}}_{0,m}\left[e^{-\int_0^{\tau_{t-x}^+} h(Y_s)ds}I_{\{Y_{t-x}^+=j,\tau_{t-x}^+<\infty\}}\right]$$

$$= \left(\eta_h^+ \widetilde{P}_{t-x}^{h+}\right)(m,j) \quad \text{for } j \in E^+ \cup E^0, \tag{1.109}$$

where η_h^+ is defined in (1.97). The first equality in (1.109) is a consequence of the spacial homogeneity of the process ξ.

Let $\overline{\xi}_u = \sup_{s\leq u}\xi_s$ denote the running supremum of the stochastic process ξ. The strong Markov property applied at the first jump time ρ of the chain Y and (1.109) imply that

$$\widetilde{\mathbb{E}}_{0,i}\left[e^{-\int_0^{\tau_t^+} h(Y_s)ds}I_{\{Y_t^+=j,\tau_t^+<\infty,\tau_t^+\geq\rho\}}\right] = \widetilde{\mathbb{E}}_{0,i}\left[e^{-h(i)\rho}I_{\{\overline{\xi}_\rho\leq t\}}\left(\eta_h^+ \widetilde{P}_{t-\xi_\rho}^{h+}\right)(Y_\rho,j)\right]$$

$$= \sum_{m\in E\setminus\{i\}}\frac{Q_0(i,m)}{q_i}\widetilde{\mathbb{E}}_{0,i}\left[e^{-h(i)\rho}I_{\{\overline{\xi}_\rho\leq t\}}\left(\eta_h^+ \widetilde{P}_{t-\xi_\rho}^{h+}\right)(m,j)\right] \tag{1.110}$$

The last equality is a consequence of the fact that Y_ρ is independent of the random vector $(\rho, \xi_\rho, \overline{\xi}_\rho)$ and takes values in the set $E \backslash \{i\}$ with $\widetilde{\mathbb{P}}_{0,i}(Y_\rho = m) = Q_0(i, m)/q_i$. Since ρ is the first jump time of the chain Y, the vector $(\rho, \xi_\rho, \overline{\xi}_\rho)$ has the same distribution as the vector $(\mathbf{e}_i, X^i_{\mathbf{e}_i}, \overline{X}^i_{\mathbf{e}_i})$, where as above $\mathbf{e}_i$ is an exponential random variable with mean $1/q_i$, independent of the Brownian motion with drift $X^i_t = \mu_i t + \sigma_i W_t$. The symbol $\overline{X}^i_{\mathbf{e}_i}$ denotes the maximum of X^i at the independent exponential time $\mathbf{e}_i$. Note that if $i \in E^+$, then $\overline{X}^i_t = X^i_t = t$ for all $t \in \mathbb{R}_+$ and the expectation in (1.110) is very easy to compute.

Assume now that $i \in E^0$. Then the Wiener-Hopf factorisation implies that the random variables $\overline{X}^i_{\mathbf{e}_u}$ and $\overline{X}^i_{\mathbf{e}_u} - X^i_{\mathbf{e}_u}$ are exponentially distributed with parameters

$$\sigma_i^{-2}\left(-\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right) \quad \text{and} \quad \sigma_i^{-2}\left(\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right)$$

respectively and independent for any exponential random variable $\mathbf{e}_u$ with parameter $u > 0$, which is independent of X^i. Therefore the joint density $f_{\overline{X}^i_t, \overline{X}^i_t - X^i_t}$ satisfies the following identity for any $x, y \in (0, \infty)$

$$\int_0^\infty u \mathrm{e}^{-ut} f_{\overline{X}^i_t, \overline{X}^i_t - X^i_t}(x, y) \mathrm{d}t = \int_0^\infty u \mathrm{e}^{-ut} f_{\overline{X}^i_t}(x) \mathrm{d}t \int_0^\infty u \mathrm{e}^{-ut} f_{\overline{X}^i_t - X^i_t}(y) \mathrm{d}t \quad \text{for all} \quad u > 0, \quad (1.111)$$

where $f_{\overline{X}^i_t - X^i_t}$ and $f_{\overline{X}^i_t}$ are the densities of the corresponding random variables. It is clear that there exists a unique extension to the complex half-plane $\mathbb{C}_{>0}$ of both sides of the formula in (1.111) and that the following formulae hold

$$\int_0^\infty \mathrm{e}^{-ut} f_{\overline{X}^i_t}(x) \mathrm{d}t = \frac{\sigma_i^{-2}\left(-\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right)}{u} \mathrm{e}^{-x\sigma_i^{-2}\left(-\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right)} \qquad x > 0, u \in \mathbb{C}_{>0}, (1.112)$$

$$\int_0^\infty \mathrm{e}^{-ut} f_{\overline{X}^i_t - X^i_t}(y) \mathrm{d}t = \frac{\sigma_i^{-2}\left(\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right)}{u} \mathrm{e}^{-y\sigma_i^{-2}\left(\mu_i + \sqrt{\mu_i^2 + 2u\sigma_i^2}\right)} \qquad y > 0, u \in \mathbb{C}_{>0}. \quad (1.113)$$

By substituting the identity (1.111) into the expectation (1.110), applying the formulae in (1.112) and (1.113) and taking the limit as t tends to zero we see that the limits in (1.99) and (1.100) are as stated for any $h \in \mathcal{H}$. Furthermore the formula in (1.102) holds.

The strong Markov property of (ξ, Y) next implies that for any $h \in \mathcal{H}$ and any $t > 0$ the matrices $\widetilde{P}^{h+}_t = (\widetilde{P}^{h+}_t(i, j)_t, i, j \in E^0 \cup E^+)$ with $\widetilde{P}^{h+}_t(i, j)$ defined by (1.108) satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{P}^{h+}_t = \widetilde{P}^{h+}_t Q^+_h, \quad t > 0, \qquad P^{h+}_0 = \mathbb{I},$$

where $\mathbb{I}$ denotes an $N^+ \times N^+$ identity matrix, the unique solution of which is given by

$$\widetilde{P}^{h+}_t = \exp(Q^+_h t).$$

This proves that the matrix Q^+_h identified above satisfies (1.105). Since $\widetilde{P}^{h+}_t(i, j) \to 0$ for all $i, j \in E^0 \cup E^+$ as $t \to \infty$, it follows that all eigenvalues of Q^+_h must have non-positive real part and therefore $Q^+_h \in \mathbb{D}(N^+)$. The proof of the existence of a matrix $Q^-_h \in \mathbb{D}(N^-)$ satisfying (1.107) is similar and omitted. $\qquad \square$

1.8.2 Proof of Theorem 4

(Existence) For any $h \in \mathcal{H}$ and $x, \ell \in \mathbb{R}$ define the matrices $\Phi_\ell^\pm(x)$ by

$$\Phi_\ell^+(x) = \eta_h^+ \exp\left(Q_h^+(\ell - x)\right) \qquad \Phi_\ell^-(x) = \eta_h^- \exp\left(Q_h^-(x - \ell)\right), \tag{1.114}$$

The proof of existence rests on the martingale property of $M^+ = \{M_t^+\}_{t \geq 0}$ and $M^- = \{M_t^-\}_{t \geq 0}$ given by

$$M_t^+ = \mathrm{e}^{-\int_0^{t \wedge \tau_\ell^+} h(Y_s)\mathrm{d}s} f_+\left(Y_{t \wedge \tau_\ell^+}, \xi_{t \wedge \tau_\ell^+}\right) \quad \text{and} \quad M_t^- = \mathrm{e}^{-\int_0^{t \wedge \tau_\ell^-} h(Y_s)\mathrm{d}s} f_-\left(Y_{t \wedge \tau_\ell^-}, \xi_{t \wedge \tau_\ell^-}\right) \tag{1.115}$$

with

$$f_+(i, x) := e_i' \Phi_\ell^+(x) k_+, \qquad f_-(i, x) := e_i' \Phi_\ell^-(x) k_-, \tag{1.116}$$

where k_+ and k_- are N^+- and N^-- column vectors, respectively.

The martingale property of M^+ follows from the equality

$$M_t^+ = \widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} k_+\left(Y_\ell^+\right) I_{\{\tau_\ell^+ < \infty\}} | \mathcal{G}_t\right]$$

where $\{\mathcal{G}_t\}_{t \geq 0}$ denotes the filtration generated by (ξ, Y). To verify this identity observe first that the Markov property of (ξ, Y) yields that

$$\widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} k_+\left(Y_\ell^+\right) I_{\{\tau_\ell^+ < \infty\}} | \mathcal{G}_t\right]$$

$$= \mathrm{e}^{-\int_0^{t \wedge \tau_\ell^+} h(Y_s)\mathrm{d}s} \widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} k_+\left(Y_\ell^+\right) I_{\{\tau_\ell^+ < \infty\}}\right]\Bigg|_{(x,i) = \left(\xi_{t \wedge \tau_\ell^+}, Y_{t \wedge \tau_\ell^+}\right)}.$$

Further, in view of the strong Markov property and spatial homogeneity of ξ, the expectation on the right-hand side of the previous display is for $x \leq \ell$ given by

$$\widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} k_+\left(Y_\ell^+\right) I_{\{\tau_\ell^+ < \infty\}}\right]$$

$$= \widetilde{\mathbb{E}}_{0,i}\left[\mathrm{e}^{-\int_0^{\tau_{\ell-x}^+} h(Y_s)\mathrm{d}s} k_+\left(Y_{\ell-x}^+\right) I_{\{\tau_{\ell-x}^+ < \infty\}}\right]$$

$$= \sum_{j \in E^0 \cup E^+} \widetilde{\mathbb{E}}_{0,i}\left[\mathrm{e}^{-\int_0^{\tau_0^+} h(Y_s)\mathrm{d}s} I_{\{Y_0^+ = j, \tau_0^+ < \infty\}}\right] \widetilde{\mathbb{E}}\left[\mathrm{e}^{-\int_0^{\tau_{\ell-x}^+} h(Y_s)\mathrm{d}s} k_+\left(Y_{\ell-x}^+\right) I_{\{\tau_{\ell-x}^+ < \infty\}} | Y_0 = j, \xi_0 = 0\right]$$

$$= e_i' \eta_h^+ \exp\left(Q_h^+(\ell - x)\right) = e_i' \Phi_\ell^+(x) k_+ = f_+(i, x) \tag{1.117}$$

where the last line follows by the definitions (1.105) and (1.97) of Q_h^+ and η_h^+.

As M^+ is a martingale, an application of Itô's lemma shows that $f_+ = (f_+(i, u), i \in E)$ satisfies for all $u < \ell$

$$\tfrac{1}{2} s(i)^2 f_+''(i, u) + m(i) f_+'(i, u) + \sum_j q_{ij}(f_+(j, u) - f_+(i, u)) = 0, \tag{1.118}$$

where f'_+ and f''_+ denote the first and second derivatives of f_+ with respect to u. By substituting the expressions (1.114) – (1.116) into equation (1.118) we find, since k^+ was arbitrary, that Q_h^+ and η_h^+ satisfy the first set of equations of the system (1.65). The proof for Q_h^- and η_h^- is analogous and omitted.

(Uniqueness) Now we turn to the proof of the uniqueness of the Wiener-Hopf factorization. To this end, let (W^+, G^+, W^-, G^-) be a complex matrix Wiener-Hopf factorization and define the function $\widetilde{f}$ as f_+ in (1.116), but replacing η^+ and Q^+ by W^+ and G^+ respectively. Since the pair (W^+, G^+) satisfies equation (1.65), it follows by an application of Itô's lemma, that $M'_t = \mathrm{e}^{-\int_0^t h(Y_s)\mathrm{d}s} \widetilde{f}(Y_t, \xi_t)$ is a local martingale. In view of the facts that $G^+ \in \mathbb{D}(N^+)$ and $h \in \mathcal{H}$ it follows that M' is in fact bounded on $\{t \leq \tau_\ell^+\}$. An application of Doob's optional stopping theorem then yields that

$$
\begin{aligned}
\widetilde{f}(j, x) &= \widetilde{\mathbb{E}}_{x,j}\left[\mathrm{e}^{-\int_0^{t\wedge\tau_\ell^+} h(Y_s)\mathrm{d}s} \widetilde{f}(Y_{t\wedge\tau_\ell^+}, \xi_{t\wedge\tau_\ell^+})\right] \\
&= \widetilde{\mathbb{E}}_{x,j}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} \widetilde{f}(Y_{\tau_\ell^+}^+, \xi_{\tau_\ell^+}) I_{\{\tau_\ell^+ < \infty\}}\right] + \lim_{t\to\infty} \widetilde{\mathbb{E}}_{x,j}\left[\mathrm{e}^{-\int_0^t h(Y_s)\mathrm{d}s} \widetilde{f}(Y_t, \xi_t) I_{\{\tau_\ell^+ = \infty\}}\right]. \quad (1.119)
\end{aligned}
$$

By the definition of $\widetilde{f}$, the absence of positive jumps of ξ and (1.117), the first expectation in (1.119) is equal to

$$
\widetilde{\mathbb{E}}_{x,j}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} \widetilde{f}(Y_\ell^+, \ell) I_{\{\tau_\ell^+ < \infty\}}\right] = \widetilde{\mathbb{E}}_{x,j}\left[\mathrm{e}^{-\int_0^{\tau_\ell^+} h(Y_s)\mathrm{d}s} k_+(Y_\ell^+) I_{\{\tau_\ell^+ < \infty\}}\right] = f_+(j, x)
$$

for $x \leq \ell$. The second term in (1.119) is zero, since $\int_0^t I_{\{Y_s \in E_0\}}\mathrm{d}s \to \infty$ $\widetilde{\mathbb{P}}_{x,j}$ almost surely on the event $\{\tau_\ell^+ = \infty\}$ for all $j \in E$ and $x \in \mathbb{R}$, and $\min_{i\in E_0} \Re(h(i)) > 0$. Thus $f_+ = \widetilde{f}$ for all N^+-column vectors k_+ and we deduce that $G^+ = Q_h^+$ and $W^+ = \eta_h^+$. Similarly, one can show that $G^- = Q_h^-$ and $W^- = \eta_h^-$ and the uniqueness is proved. $\qquad\square$

1.8.3 Proof of Theorem 5

Applying the strong Markov property at $\bar{\tau}$ and noting that $\bar{\tau} \leq \tau^\xi$ yields, in view of the representation (1.93) that

$$
\begin{aligned}
H_j^{x,i}(h, u) &= \widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\bar{\tau}} h(Y_s)\mathrm{d}s} I_{\{Y_{\bar{\tau}}=j\}} F(\xi_{\bar{\tau}}, Y_{\bar{\tau}})\right] \\
&= \widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\bar{\tau}} h(Y_s)\mathrm{d}s} I_{\{Y_{\bar{\tau}}=j, \tau_\upsilon^+ < \tau_\ell^-\}} F(\upsilon, Y_{\bar{\tau}})\right] + \widetilde{\mathbb{E}}_{x,i}\left[\mathrm{e}^{-\int_0^{\bar{\tau}} h(Y_s)\mathrm{d}s} I_{\{Y_{\bar{\tau}}=j, \tau_\upsilon^+ > \tau_\ell^-\}} F(\ell, Y_{\bar{\tau}})\right]
\end{aligned}
$$

where

$$
F(x, i) = \widetilde{\mathbb{E}}_{x,i}[\mathrm{e}^{-\int_0^{\tau^\xi} \widetilde{h}(Y_s)\mathrm{d}s + u\xi_{\tau^\xi}} I_{\{Y_{\tau^\xi}=j\}}] = \widetilde{\mathbb{E}}_{x,i}[\mathrm{e}^{u\xi_{\tau^\xi}} I_{\{Y_{\tau^\xi}=j\}}]
$$

since $\widetilde{h}(i) = h(i) I_{\{i\in E^0\}}$, by definition of $\widetilde{h}$. From the definition of τ^ξ it is straightforward to check that

$$
\widetilde{\mathbb{P}}_{x,i}[\tau^\xi = 0] = 1 \qquad \text{for } i \in E^0 \text{ and } x \in \{\ell, \upsilon\},
$$

so that $F(\upsilon, i) = \mathrm{e}^{u\upsilon}\delta_{ij}$ and $F(\ell, i) = \mathrm{e}^{u\ell}\delta_{ij}$ if $i \in E^0$, where δ_{ij} denotes the Kronecker delta.

Moreover, in view of the form (1.60)–(1.61) of Q_h and the definition of phase-type distribution, it is clear that, conditionally on $Y_0 = i \in E_j^+$ and $\xi_0 = \upsilon$, $Y_{\tau^\xi} = j$ and $\tau^\xi \sim \mathrm{PH}(\delta_i, B_j^+)$, where δ_i is the vector with elements $\delta_i = (\delta_{ik})$. Therefore we find, using (1), that for u with $\Re(u) < \alpha_j^+$ and $i \in E_j^+$

$$F(v, i) = \widetilde{\mathbb{E}}_{v,i}[\mathrm{e}^{u\xi_{\tau_\xi}}]$$
$$= \mathrm{e}^{uv}\widetilde{\mathbb{E}}_{v,i}[\mathrm{e}^{u\tau^\xi}] = \mathrm{e}^{uv}\left[(-uI_j^+ - B_j^+)^{-1}(-B_j^+)\mathbf{1}\right](i).$$

Similarly, it follows that for $i \in E_j^-$ and u with $\Re(u) > -\alpha_j^-$

$$F(\ell, i) = \mathrm{e}^{u\ell}\left[(uI_j^- - B_j^-)^{-1}(-B_j^-)\mathbf{1}\right](i).$$

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