Questions from the Lévy processes reading group 1

1. Check that if $(X_t)_{t\geq 0}$ is a Lévy process and Ψ is the characteristic exponent of X_1, i.e. $\mathbf{E}(\exp(i\langle\lambda, X_1\rangle)) = \exp(-\Psi(\lambda))$ for $\lambda \in \mathbb{R}^d$, then
$$\mathbf{E}\left(e^{i\langle\lambda, X_t\rangle}\right) = e^{-t\Psi(\lambda)}.$$

2. Let E be a Polish space and ν be a measure on E with finite mass, i.e. $c := \nu(E) < \infty$. Check that if $(\xi_n)_{n\geq 1}$ is an i.i.d. sequence of random variables with law $c^{-1}\nu$, and N has a mean c Poisson distribution, then
$$\varphi := \sum_{n=1}^{N} \delta_{\xi_n}$$
defines a Poisson measure with intensity ν.

3. How can we construct a Poisson measure on E in the case that ν is σ-finite, rather than finite?

4. Let φ be a Poisson measure on $E \times [0,\infty)$ with intensity $\mu = \nu \otimes dx$. Check that $\varphi(E \times \{t\})$ is equal to 0 or 1 for all $t \geq 0$, almost-surely.

5. Make sense of
$$\varphi = \sum_{t\geq 0} \delta_{(e(t),t)}.$$