Questions from the Lévy processes reading group 2

1. Determine a, Q and π in the Lévy-Khintchine formula for Gaussian and Poisson random variables.

2. Suppose $(\xi_n)_{n\geq 1}$ are i.i.d. with law ν on $\mathbb{R}^d\backslash\{0\}$, independent of a Poisson process $N = (N_t)_{t\geq 0}$ of rate c. Check that if we define

$$e(t) := \begin{cases} \xi_n & \text{if } N_{t-} < n = N_t, \\ 0 & \text{otherwise,} \end{cases}$$

then $e = (e(t))_{t\geq 0}$ is a Poisson point process with characteristic measure $c\nu$ (in the sense of Section 0.5 of the book).

3. In the setting of question 2, check that the compound Poisson process

$$\sum_{n=1}^{N_t} \xi_n = \sum_{s\leq t} e(s)$$

is a Lévy process with characteristic exponent

$$\psi(\lambda) = c\int_{\mathbb{R}^d} (1 - e^{i\langle\lambda,x\rangle})\nu(dx).$$

4. Continue in the setting of question 2, but now suppose that the finite measure $\pi := c\nu$ is supported only upon $[-1,1]\backslash\{0\}$. Check that $M = (M_t)_{t\geq 0}$, where

$$M_t := \sum_{s\leq t} e(s) - t\int_{[-1,1]\backslash\{0\}} x\pi(dx),$$

is a martingale. Show that its variance at time t is given by

$$t\int_{[-1,1]\backslash\{0\}} x^2\pi(dx).$$

5. In the setting of question 4, use Doob's L^2 (sub-)martingale inequality to confirm the result from the reading group that

$$\mathbf{E}\left(\sup_{s\leq t}\left|\sum_{r\leq s} e(r) - s\int_{[-1,1]\backslash\{0\}} x\pi(dx)\right|^2\right) \leq 4t\int_{[-1,1]\backslash\{0\}} x^2\pi(dx).$$