Questions from the Lévy processes reading group 3

In all the following questions, you may assume that $X = (X_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}^d$ with characteristic exponent Ψ.

1. Make sure you understand the proof of the fact that: Ψ bounded $\Rightarrow \Pi(\mathbb{R}^d) < \infty$.
2. (Markov property) Show that
$$\mathbf{E}\left(f(X_{t+s})|\mathcal{F}_t\right) = \mathbf{E}\left(f(X_{t+s})|\sigma(X_t)\right)$$
for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$.
3. Fix $t \geq 0$. Check that $X' = (X'_s)_{s\geq 0}$, where $X'_s := X_{t+s} - X_t$, is independent of $\mathcal{F}_t$ and has the same distribution as X.
4. Let T be a stopping time. Check that
$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$
defines a σ-algebra.
5. Let T be a stopping time such that $\mathbf{P}(T < \infty) > 0$. On $\{T < \infty\}$, define $X' = (X'_s)_{s\geq 0}$ by setting $X'_s := X_{T+s} - X_T$. Check that, for $0 \leq u \leq v \leq s \leq t$, $\lambda_1, \lambda_2 \in \mathbb{R}$,
$$\mathbf{E}\left(e^{i\langle\lambda_1, X'_t - X'_s\rangle + i\langle\lambda_2, X'_v - X'_u\rangle}\mathbf{1}_{A\cap\{T<\infty\}}\right) = e^{i(t-s)\Psi(\lambda_1) + i(v-u)\Psi(\lambda_2)}\mathbf{P}\left(A \cap \{T < \infty\}\right).$$
6. In the setting of the previous question, explain why the following statements hold (interpret the final three as holding conditionally on $\{T < \infty\}$):
   (a) X' is independent of $\mathcal{F}_T$,
   (b) $X'_t - X'_s$ is independent of $X'_v - X'_u$,
   (c) $X'_t - X'_s \sim X'_{t-s}$,
   (d) the characteristic exponent of X' is Ψ.
7. Use the previous results to establish the Strong Markov property: If T is a stopping time such that $\mathbf{P}(T < \infty) > 0$, then conditionally on $\{T < \infty\}$, $X' = (X'_s)_{s\geq 0}$, where $X'_s := X_{T+s} - X_T$, is independent of $\mathcal{F}_T$ and has the same distribution as X.