Implications for Hedging of the choice of driving process for one-factor Markov-functional models

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Abstract

In this paper, we study the implications for hedging Bermudan swaptions of the choice of the instantaneous volatility for the driving Markov process of the one-dimensional swap Markov-functional model. We find that there is a strong evidence in favour of what we term "parametrization by time" as opposed to "parametrization by expiry". We further propose a new parametrization by time for the driving process which takes as inputs into the model the market correlations of relevant swap rates. We show that the new driving process enables a very effective vega-delta hedge with a much more stable gamma profile for the hedging portfolio compared with the existing ones.

Key words: one-dimensional swap Markov-functional model, Bermudan swaption, correlation, hedging, vega, gamma, parametrization by time and by expiry.

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1 Introduction

The problem of pricing and hedging a Bermudan swaption has been of great practical concern in the fixed income quantitative research. The product itself is among the most common exotic interest rate derivatives. However, opinions differ as to what constitutes an effective modeling framework for pricing and hedging Bermudan swaptions. One of the biggest debates is whether it is necessary to use a multi-factor model. A good summary of the current literature on this topic is given in (Pietersz & Pelsser, 2010). In (Pietersz & Pelsser, 2010), the authors carry out a comparison of the hedging performance of a single factor Markov-functional model and multi-factor market models in relation to Bermudan swaptions and their findings support the claim that if a single factor Markov-functional model is appropriately calibrated to "terminal correlations" of swap rates that are relevant to the Bermudan swaption then the hedging performance of both the multi-factor and single factor models are comparable.

In this paper, we restrict attention to the pricing and hedging of Bermudan swaptions within the context of a one factor Markov-functional model driven by a Gaussian process. The contribution we make here is to study the implications for hedging of the choice of the instantaneous volatility for the driving Markov process. This is a topic which seems to have received little attention in the literature for one factor Markov-functional models or equivalently for one factor separable market models (see (Bennett & Kennedy, 2005) and (Pietersz, Pelsser, & Regenmortel, 2004)). One popular choice is to take a Gaussian process with exponential instantaneous volatility, referred to as the mean reversion process (MR), as is done in (Pietersz & Pelsser, 2010). We begin our investigation by comparing this candidate with one based on the Hull-White short-rate model, referred to as the Hull-White process (HW) which was first introduced in (Bennett & Kennedy, 2005).

For these two candidate processes the vega profiles of a Bermudan swaption under the swap Markov-functional model turn out to have some key differences (see also (Pertursson, 2008) for a comparison of their vega profiles under different market scenarios). These differences can be linked back to the difference in nature of the two parametrizations for the driving process. The mean reversion process (MR) is an example of what we term "parametrization by expiry". Here the auto-correlations of the driving process are chosen at the outset and controlled by parameters which are user inputs. As such the changes in the correlations of swap rates at their setting dates relevant to the pricing of a Bermudan are not hedged. In contrast, the Hull-White process (HW) is an example of "parametrization by time". In this type of parametrization, the auto-correlations of the driving process are linked explicitly to market implied volatilities and it is this feature which allows the possibility of hedging against moves in market correlations of relevant swap rates.

Based on the insight gained by our study of the MR and HW processes, we propose a new parametrization by time for the driving process. This new parametrization takes as inputs into the model the market correlations of relevant swap rates. These market correlations are estimated via a full rank LIBOR market model using a two-step procedure involving a global and local fit to the swaption matrix. This new parametrization has a vega response spread over the swaption matrix but interestingly the total vega for each expiry (row of the swaption matrix) is approximately the same as for the HW model. We give an explanation for why this is the case.

The different vega profiles of the parametrizations by expiry and by time have a direct consequence for hedging. We find that when the driving process is parameterized by time the "total" gamma (sum of all gammas) of a vega-delta neutral portfolio for a Bermudan swaption is stabilized. In contrast, it is not possible to control the "total" gamma for this portfolio with the vega profile associated with parametrization by expiry. We further find that the proposed parametrization by time for the driving process with a vega response spread over the swaption matrix leads to a more stable "parallel" gamma profile (sum of each row of the gamma matrix) than that of the HW process.

The paper is organized as follows. In section 2, we review the preliminaries and set up the notations. In section 3, we first describe the one-dimensional swap Markov-functional model and analyze the difference between parametrizations by expiry and by time. After that, we construct a new parametrization by time for the driving process. In section 4, we compute the vegas of a Bermudan and analyze them theoretically. A hedging result with an emphasis on the gamma risks will be addressed in section 5. Section 6 concludes the paper.

2 Notations and preliminaries

Consider a general tenor structure

$$0 = T_0 < T_1 < \cdots < T_{n+1},$$

where $\alpha_i = T_{i+1} - T_i$ are the accrual factors for $i = 0, \dots, n$.

Let D_{tT} denote the time-t value of a zero-coupon discount bond that matures at time T. We denote by L^i the forward LIBOR that sets (expires) at T_i and settles (matures) at T_{i+1}. Forward LIBORs and discount bonds can be linked via the relation

$$L^i_t = \frac{D_{tT_i} - D_{tT_{i+1}}}{\alpha_i D_{tT_{i+1}}}, \qquad t \le T_i, \tag{1}$$

for $i = 0, \dots, n$. We denote by $y^{i,j}$ the forward swap rate of an interest rate swap with setting dates $T_i, T_{i+1}, \dots, T_{i+j-1}$ and settlement dates $T_{i+1}, T_{i+2}, \dots, T_{i+j}$. Similar to forward LIBORs, forward swap rates can also be written in terms of discount bonds

$$y^{i,j}_t = \frac{D_{tT_i} - D_{tT_{i+j}}}{\sum_{k=i}^{i+j-1} \alpha_k D_{tT_{k+1}}}, \qquad t \le T_i, \tag{2}$$

for $i = 0, \ldots, n$. It is clear that $y^{i,1}$ coincides with L^i. For each swap rate $y^{i,j}$, we further introduce the corresponding at the money (ATM) Black implied volatility $\widetilde{\sigma}_{i,j}$.

The type of Bermudan swaption we consider in this paper is the co-terminal version, as opposed to other non-standard types of Bermudan swaption. The holder of a co-terminal Bermudan swaption has the right, on any of the swap exercise dates to enter the remaining swap which ends at the pre-determined terminal date T_{n+1}. The underlying swap at T_i consists of a number of coupons that set at T_j and settle at T_{j+1} for $j = i, \ldots, n$. We further denote the notional amount by N and the strike by K. Suppose that the Bermudan swaption is to be exercised at time T_i. In case of a pay fixed type, the holder will then receive the corresponding coupons from the underlying swap, i.e. at T_{j+1} for each $j = i, \ldots, n$ he or she will receive the floating leg $N\alpha_j L^j_{T_j}$ and pay the fixed leg $N\alpha_j K$. In case of a receive fixed type, the holder will receive the fixed legs in exchange for the floating legs. Although the coupons depend on the values of the LIBORs at their setting dates, the exercise value of the underlying swap at each exercise date T_i depends on the corresponding co-terminal swap rate at its setting date $y^{i,n+1-i}_{T_i}$. For a pay fixed Bermudan, the holder will only exercise at time T_i if $y^{i,n+1-i}_{T_i}$ is above the strike level K. Nevertheless, even when the immediate exercise value is positive, the holder can decide to hold on to the swaption in view of a more favourable co-terminal swap rate $y^{j,n+1-j}_{T_i}$ for $j > i$. It was noted in (Pietersz & Pelsser, 2010) that although the joint distribution of the random variables $\{y^{j,n+1-j}_{T_i}; j = i, \ldots, n; i = 1, \ldots, n\}$ fully determines the price of a Bermudan swaption, the main contribution (up to first order approximation) actually comes from the joint distribution of the co-terminal swap rates at their setting dates $\{y^{i,n+1-i}_{T_i}; i = 1, \ldots, n\}$ (see also (Piterbarg, 2004)). This is why we are interested in their correlation structure.

3 Pricing Bermudan swaptions under the one-dimensional swap Markov-functional model

The defining characteristic of the standard Markov-functional model (MF) is that discount bond prices are assumed to be at any time functions of some low-dimensional (usually one or two) Markov process x, which is Markovian in some specified martingale measure. The exact forms are only determined at the exercise dates, i.e. $D_{T_iT_j}(x_{T_i})$ for $0 \le i \le j \le n$, since this is all that is typically needed in practice. Depending on the application, the functional forms are derived numerically from relevant market prices and the martingale property necessary to maintain the arbitrage-free property of the model. Note that the functional forms of the discount bonds implicitly imply the functional forms of all forward swap/LIBOR rates and vice versa. Given the functional forms, the conditional expected value under the specified martingale measure of a payoff at any exercise date T_i can be derived numerically. Hence, the value of an exotic product can be calculated by backward induction on a grid.

Here, we restrict attention to the development of the one-dimensional swap Markov-functional model (SMF) for the pricing and hedging of Bermudan swaptions. This section starts by reviewing the one-dimensional SMF model and its current choices of driving Markov process. We then propose an alternative choice which is more suitable for our current application.

3.1 The one-dimensional swap Markov-functional model

In the one-dimensional SMF model, the functional forms of the discount bonds are chosen so that accurate calibration to the market prices of the co-terminal vanilla swaptions is achieved.

We assume that these market prices are given by the Black's formula with the corresponding co-terminal implied volatilities $\{\widetilde{\sigma}_{i,n+1-i}\}_{i=1,\dots,n}$. The freedom to specify the law of x allows the model to capture well some features of the real market relevant to the exotic products. For a Bermudan swaption, those features are the correlations of the co-terminal forward swap rates at their setting dates as we discussed in section 2.

In our model, we choose to work with the terminal measure $\mathbb{S}^{n+1}$ which takes the terminal discount bond $D_{\cdot T_{n+1}}$ as the numeraire. Details of the implementation of the SMF model under the terminal measure can be found in (Hunt, Kennedy, & Pelsser, 2000) and (Hunt & Kennedy, 2004). We assume the driving process x is a Gaussian process satisfying

$$x_t := \int_0^t \sigma(u)\mathrm{d}W_u,$$

where W denotes a standard Brownian motion under $\mathbb{S}^{n+1}$ and $\sigma(\cdot)$ is a deterministic function of time. For the implementation of the model, we only need to specify the law of x at each exercise date T_i for $i = 1, \dots, n$.

An important result that was observed in (Bennett & Kennedy, 2005) is the approximate linear relationship between the logarithms of the co-terminal forward swap rates and x

$$\ln y_t^{i,n+1-i} \approx \underbrace{\gamma_i}_{\text{constant}} x_t + \underbrace{\eta_t^i}_{\text{deterministic}} . \tag{3}$$

Consequently, the joint distributions of the log of the co-terminal forward swap rates can be captured via our choice of x since correlation is invariant under the linear transformation

$$\mathrm{Corr}(x_{T_i}, x_{T_j}) = \sqrt{\frac{\xi_{\min(T_i,T_j)}}{\xi_{\max(T_i,T_j)}}} \approx \mathrm{Corr}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j}),$$

where $\xi_{T_i} := \mathrm{Var}(x_{T_i}) = \int_0^{T_i} \sigma^2(t)\mathrm{d}t$ and the superscript "mo" denotes model quantities.

We further note that by matching the model to the Black's formula for the co-terminal vanilla swaptions, we have the following approximation in the terminal measure (exact in the associated swaption measure)

$$\mathrm{Var}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}) \approx \widetilde{\sigma}_{i,n+1-i}^2 T_i. \tag{4}$$

Hence, once x is chosen the γ_i's are implicitly determined and from (3) and (4) we have

$$\gamma_i^2 \xi_{T_i} \approx \widetilde{\sigma}_{i,n+1-i}^2 T_i. \tag{5}$$

Note that each γ_i is matched specifically to the associated co-terminal swap rate $y^{i,n+1-i}$ and once the model is calibrated it stays constant from today till expiry. In that sense, the γ_i's are expiry-dependent quantities.

We now present in the following two current candidates for x before exploring other choices in the later sections.

Current candidates:

- **MR:** The first choice is referred to as the mean reversion (MR) driving process with $\sigma(t) = e^{at}$, where $a > 0$ is the mean reversion parameter. It follows that one can write the variance of x at each exercise date T_i as

$$\xi_{T_i} = \int_0^{T_i} e^{2at}\mathrm{d}t = \frac{1}{2a}(e^{2aT_i} - 1).$$

For this choice of parametrization, one can see that any changes in the market implied volatilities will not influence x and its auto-correlations once we fix the parameter a. However, the expiry-dependent quantities γ_i's may change as we can see from (5). In that sense, the MR process is an example of "parametrization by expiry".

- **HW:** An alternative choice of x is motivated by considering the Hull-White short-rate model which was first introduced in (Bennett & Kennedy, 2005). We refer to it as the Hull-White (HW) process. For each $i = 1, \dots, n$, we have the following specification for the HW process

$$\xi_{T_i} = \left(\frac{T_{n+1} - T_i}{(1 + \alpha_i y_0^{i,n+1-i})(\psi_{T_{n+1}} - \psi_{T_i})} \right)^2 \widetilde{\sigma}^2_{i,n+1-i} T_i, \tag{6}$$

  where $\psi_{T_i} = \frac{1}{a}(1 - e^{-aT_i}), a > 0$. In contrast to the MR process, any changes in the market co-terminal implied volatilities will have an immediate effect on x and its auto-correlations. From the linear approximation in (3), we see that the instantaneous volatilities of the co-terminal swap rates will be altered in certain time periods. On the other hand, the expiry-dependent quantities γ_i's will stay the same as we can see from substitution of the expression (6) into (5). In that sense, the HW process is an example of "parametrization by time".

For both the above model parametrizations, an increase in the implied volatility of one of the co-terminal swap rates tends to increase the value of a Bermudan. This is not surprising as the value of the associated vanilla swaption has increased. But the optionality of a Bermudan provides extra value in addition to the value of the underlying vanilla options. This extra value is highly dependent on the correlations between the co-terminal swap rates at their setting dates and for the above two models these correlations behave very differently in response to changes in the co-terminal implied volatilities. This leads to very different hedging profiles as we shall see in the later sections. In the next section we investigate the essential difference in nature between the two parametrizations by considering the underlying LIBORs.

3.2 Parametrizations by time and by expiry

In the previous subsection, we discussed the idea of parametrizations by expiry and by time in terms of the responses of the instantaneous volatilities of the co-terminal swap rates to a shift in the implied volatilities. Here we explore how this idea carries over to the LIBORs as they are the basic building blocks of any interest rate model.

For all choices of x, the linear approximation in (3) implies that the instantaneous volatility of the log of the co-terminal forward swap rate $y_t^{i,n+1-i}$ is approximately $\gamma_i \sigma(t)$ under the terminal measure. In order to gain insight into the effect of shifting the implied volatilities, we make the simplifying assumption that each log-LIBOR $\ln L^i$ has a positive and deterministic volatility function $\sigma_i(t), t \leq T_i$. Under this assumption, we can use the approximation described in appendix A. In a one factor model instead of the multi-factor setting in appendix A, the instantaneous volatility of the log of the co-terminal forward swap rate $y^{i,n+1-i}$ can be linked to the instantaneous volatilities of the log-LIBORs by the following approximation

$$\gamma_i \sigma(t) \approx \sum_{k=i}^{n} \zeta_k^{i,n+1-i}(0) \sigma_k(t), \tag{7}$$

where $\{\zeta_k^{i,n+1-i}(0)\}_{k=i,\dots,n}$ are constant empirical weights that depend on the initial discount curve. This can be seen from SDE (35) in appendix A.2.

Since the last LIBOR $L^n = y^{n,1}$, we have that $\sigma_n(t) \approx \gamma_n\sigma(t)$. Using the derived form for $\sigma_n(t)$, we can deduce $\sigma_{n-1}(t)$ by considering the approximation in (7) for $\gamma_{n-1}\sigma(t)$

$$\sigma_{n-1}(t) \approx \sigma_{n-1}\sigma(t),$$

where we let

$$\sigma_{n-1} := \frac{\gamma_{n-1} - \zeta_n^{n-1,2}(0)\gamma_n}{\zeta_{n-1}^{n-1,2}(0)}.$$

We assume $\sigma_{n-1} > 0$ so that $\sigma_{n-1}(\cdot)$ will also be positive as we assumed earlier. Inductively, assume we have that $\sigma_k(t) \approx \sigma_k\sigma(t)$ where σ_k is a positive constant for each $k = i+1, \dots, n$. When $k = n$, σ_n is the same as γ_n. We can then derive $\sigma_i(t)$ by considering the approximation in (7) for $\gamma_i\sigma(t)$. We rewrite (7) in the following form

$$\begin{aligned}
\gamma_i\sigma(t) &\approx \zeta_i^{i,n+1-i}(0)\sigma_i(t) + \sum_{k=i+1}^{n} \zeta_k^{i,n+1-i}(0)\sigma_k\sigma(t) \\
\Longleftrightarrow \sigma_i(t) &\approx \left(\frac{\gamma_i - \sum_{k=i+1}^{n}\zeta_k^{i,n+1-i}(0)\sigma_k}{\zeta_i^{i,n+1-i}(0)}\right)\sigma(t).
\end{aligned}$$

This again reduces $\sigma_i(t)$ approximately to the form $\sigma_i\sigma(t)$ where

$$\sigma_i := \frac{\gamma_i - \sum_{k=i+1}^{n}\zeta_k^{i,n+1-i}(0)\sigma_k}{\zeta_i^{i,n+1-i}(0)}. \tag{8}$$

This concludes that $\sigma_i(t) \approx \sigma_i\sigma(t)$ for all $i = 1, \dots, n$ where each constant σ_i can be derived inductively by (8) and is assumed to be positive. One can see that each σ_i depends on $\{\gamma_k\}_{k=i,\dots,n}$ and the empirical weights $\{\zeta_k^{i,n+1-i}(0)\}_{k=i,\dots,n}$, $\{\zeta_k^{i+1,n-i}(0)\}_{k=i+1,\dots,n}$, $\dots$, $\{\zeta_k^{n-1,2}(0)\}_{k=n-1,n}$. Since these empirical weights do not depend on the implied volatilities, we will safely ignore their involvements in the next discussion.

We now analyze how shifting the implied volatilities affects the instantaneous volatility functions $\{\sigma_i(\cdot)\}_{i=1,\dots,n}$ of the log-LIBORs for each choice of x.

Parametrization by expiry: For the MR process, by the approximation in (5) we have that

$$\gamma_i \approx \frac{\widetilde{\sigma}_{i,n+1-i}\sqrt{T_i}}{\sqrt{\frac{e^{2aT_i}-1}{2a}}} \qquad i = 1, \dots, n. \tag{9}$$

Suppose we want to bump the co-terminal implied volatility $\widetilde{\sigma}_{i,n+1-i}$ and keep the rest the same. It is clear that the instantaneous volatility $\sigma(t) = e^{at}$ of the MR process will not be affected. We observe other effects and summarize them below:

- γ_j for $j \neq i$ are unchanged as we can see from (9). This then follows from (8) that the constants $\{\sigma_j\}_{j=i+1,\dots,n}$ and hence $\{\sigma_j(\cdot)\}_{j=i+1,\dots,n}$ also remain unchanged.
- γ_i will increase directly as a result of (9). From (8), we see that σ_i and hence $\sigma_i(\cdot)$ will increase as $\{\sigma_j\}_{j=i+1,\dots,n}$ are unchanged.

- Since γ_{i-1} and $\{\sigma_j\}_{j=i+1,\dots,n}$ stay the same but σ_i increases, again we can see from (8) that σ_{i-1} and hence $\sigma_{i-1}(\cdot)$ will decrease.
- The effects on $\{\sigma_k\}_{k=1,\dots,i-2}$ and hence $\{\sigma_k(\cdot)\}_{k=1,\dots,i-2}$ will be quite small. This is because the increase in σ_i and decrease in σ_{i-1} tend to cancel each other out in the sum in (8) when we consider σ_k for $k < i-1$.

Note that all the above effects are (global) from today till expiry (illustrated in figure 3.1). In that sense, the instantaneous volatilities of the log-LIBORs are clearly parameterized by expiry.

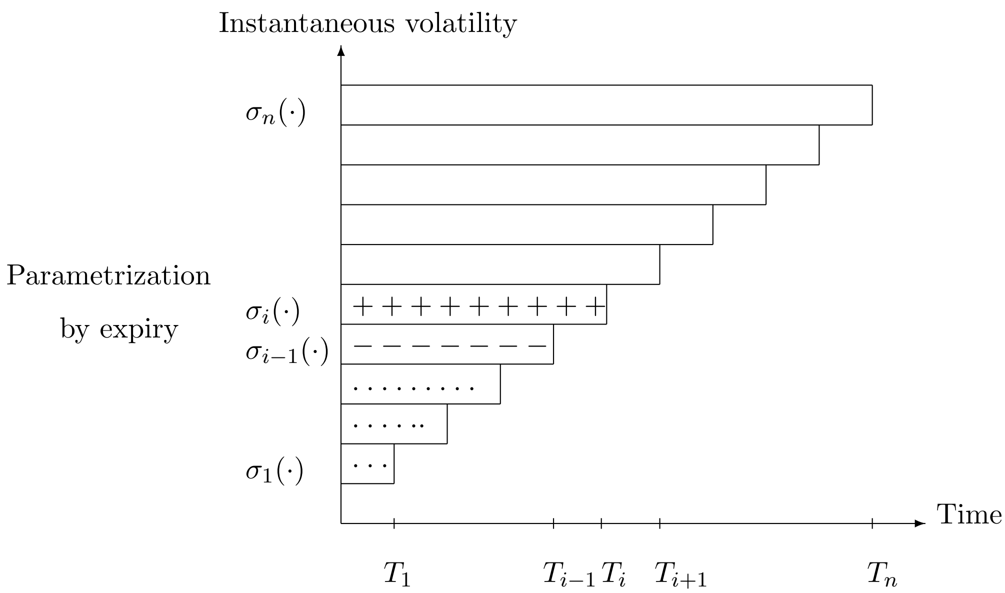

Figure 3.1: Global effect of bumping $\widetilde{\sigma}_{i,n+1-i}$ on the instantaneous volatility functions of the log-LIBORs. The dots represent a very small effect.

Parametrization by time: For the HW process where ξ_{T_i} is given by (6), it can be seen from (5) that the γ_i's are independent of the implied volatilities

$$\gamma_i \approx \frac{(1+\alpha_i y_0^{i,n+1-i})(\psi_{T_{n+1}} - \psi_{T_i})}{T_{n+1} - T_i}, \qquad \psi_{T_i} = \frac{1}{a}(1 - e^{-aT_i}), \qquad a > 0. \tag{10}$$

Although the instantaneous volatility function $\sigma(\cdot)$ of x is not defined explicitly, we know it exists such that $\int_0^{T_i} \sigma^2(t)\mathrm{d}t = \xi_{T_i}$ is given by (6).

We now bump the co-terminal implied volatility $\widetilde{\sigma}_{i,n+1-i}$ and keep the rest unchanged. It is clear from (10) that γ_j will stay the same for $j = 1, \dots, n$. Hence, it follows from (8) that the constants σ_j will also remain unchanged for all j. The only effect of the bump is on the function $\sigma(\cdot)$ (see (6)). One can see that the variance of x at T_i is shifted but those at the other exercise dates remain unchanged. Consequently, we have that the only effect on x is the following

$$\xi_{T_i} - \xi_{T_{i-1}} = \int_{T_{i-1}}^{T_i} \sigma^2(t)\mathrm{d}t \qquad \text{increases,} \qquad \xi_{T_{i+1}} - \xi_{T_i} = \int_{T_i}^{T_{i+1}} \sigma^2(t)\mathrm{d}t \qquad \text{decreases.}$$

The above effect implies that on average the instantaneous volatility function $\sigma(\cdot)$ of x is increased during the time period $(T_{i-1}, T_i]$ but is decreased during the next one $(T_i, T_{i+1}]$. Since $\sigma_j(t) \approx \sigma_j \sigma(t)$ is only defined during the corresponding LIBOR's life, i.e. $t \in [0, T_j]$, the effect on $\sigma(\cdot)$ only carries over to $\{\sigma_j(\cdot)\}_{j=i,\dots,n}$. It is then clear that on average the collection of instantaneous volatilities $\{\sigma_j(\cdot)\}_{j=i,\dots,n}$ will increase and decrease during the two consecutive time intervals $(T_{i-1}, T_i]$ and $(T_i, T_{i+1}]$ respectively (figure 3.2). For the last co-terminal implied volatility $\widetilde{\sigma}_{n,1}$, the equivalent effect is that $\sigma_n(\cdot)$ will only increase during the period $(T_{n-1}, T_n]$. Note that the above effects are local as the instantaneous volatilities of the log-LIBORs are only shocked locally for some particular time periods in response to the movement of the corresponding implied volatility. In that sense, the instantaneous volatilities of the log-LIBORs are clearly parameterized by time.

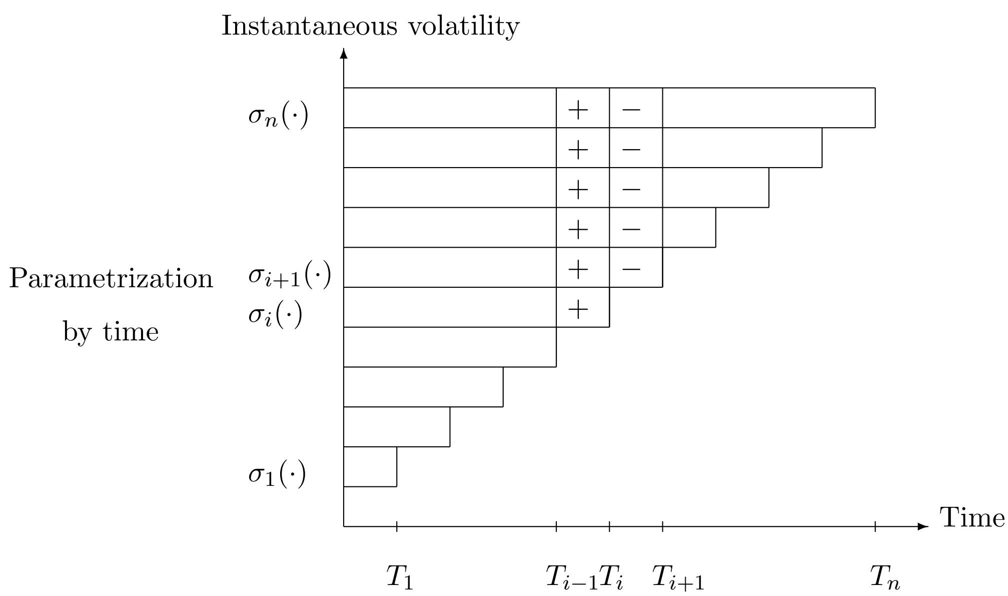

Figure 3.2: Local effects of bumping the $\widetilde{\sigma}_{i,n+1-i}$ on the instantaneous volatility functions of the log-LIBORs.

The difference in parametrizations mentioned above has a fundamental effect on the hedging of a Bermudan swaption. Specifically, the global and local effects of the parametrizations by expiry and by time influence the correlations of the forward LIBORs and the co-terminal forward swap rates in very different ways. This fact, in turn, leads to very different hedging behaviours of correlation-dependent products like the Bermudan swaption. We will investigate further the difference in their vega profiles in section 4.2. Since the "parametrization by time" outperforms the other type as we explore later in section 5, we will next propose an alternative for this parametrization.

3.3 An alternative parametrization of time

We recall that the correlation of the MR process is fixed from the outset while the HW specification links the correlation structure of the model/co-terminal swap rates explicitly to the market implied volatilities. However at each exercise date T_i, the HW process only takes into account the co-terminal implied volatility $\widetilde{\sigma}_{i,n+1-i}$. In this section, we explore alternative ways to specify the x process which link the model's correlation structure to implied volatilities of different tenors (see table 3.1).

Tenor		1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
Expiry	1	…	…	…	…	…	…	…	…	…	$\widetilde{\sigma}_{1,10}$	$\widetilde{\sigma}_{1,1}$	$\widetilde{\sigma}_{1,2}$	$\widetilde{\sigma}_{1,3}$	$\widetilde{\sigma}_{1,4}$	$\widetilde{\sigma}_{1,5}$	$\widetilde{\sigma}_{1,6}$	$\widetilde{\sigma}_{1,7}$	$\widetilde{\sigma}_{1,8}$	$\widetilde{\sigma}_{1,9}$	$\widetilde{\sigma}_{1,10}$
	2	…	…	…	…	…	…	…	…	$\widetilde{\sigma}_{2,9}$	…	$\widetilde{\sigma}_{2,1}$	$\widetilde{\sigma}_{2,2}$	$\widetilde{\sigma}_{2,3}$	$\widetilde{\sigma}_{2,4}$	$\widetilde{\sigma}_{2,5}$	$\widetilde{\sigma}_{2,6}$	$\widetilde{\sigma}_{2,7}$	$\widetilde{\sigma}_{2,8}$	$\widetilde{\sigma}_{2,9}$	…
	3	…	…	…	…	…	…	…	$\widetilde{\sigma}_{3,8}$	…	…	$\widetilde{\sigma}_{3,1}$	$\widetilde{\sigma}_{3,2}$	$\widetilde{\sigma}_{3,3}$	$\widetilde{\sigma}_{3,4}$	$\widetilde{\sigma}_{3,5}$	$\widetilde{\sigma}_{3,6}$	$\widetilde{\sigma}_{3,7}$	$\widetilde{\sigma}_{3,8}$	…	…
	4	…	…	…	…	…	…	$\widetilde{\sigma}_{4,7}$	…	…	…	$\widetilde{\sigma}_{4,1}$	$\widetilde{\sigma}_{4,2}$	$\widetilde{\sigma}_{4,3}$	$\widetilde{\sigma}_{4,4}$	$\widetilde{\sigma}_{4,5}$	$\widetilde{\sigma}_{4,6}$	$\widetilde{\sigma}_{4,7}$	…	…	…
	5	…	…	…	…	…	$\widetilde{\sigma}_{5,6}$	…	…	…	…	$\widetilde{\sigma}_{5,1}$	$\widetilde{\sigma}_{5,2}$	$\widetilde{\sigma}_{5,3}$	$\widetilde{\sigma}_{5,4}$	$\widetilde{\sigma}_{5,5}$	$\widetilde{\sigma}_{5,6}$	…	…	…	…
	6	…	…	…	…	$\widetilde{\sigma}_{6,5}$	…	…	…	…	…	$\widetilde{\sigma}_{6,1}$	$\widetilde{\sigma}_{6,2}$	$\widetilde{\sigma}_{6,3}$	$\widetilde{\sigma}_{6,4}$	$\widetilde{\sigma}_{6,5}$	…	…	…	…	…
	7	…	…	…	$\widetilde{\sigma}_{7,4}$	…	…	…	…	…	…	$\widetilde{\sigma}_{7,1}$	$\widetilde{\sigma}_{7,2}$	$\widetilde{\sigma}_{7,3}$	$\widetilde{\sigma}_{7,4}$	…	…	…	…	…	…
	8	…	…	$\widetilde{\sigma}_{8,3}$	…	…	…	…	…	…	…	$\widetilde{\sigma}_{8,1}$	$\widetilde{\sigma}_{8,2}$	$\widetilde{\sigma}_{8,3}$	…	…	…	…	…	…	…
	9	…	$\widetilde{\sigma}_{9,2}$	…	…	…	…	…	…	…	…	$\widetilde{\sigma}_{9,1}$	$\widetilde{\sigma}_{9,2}$	…	…	…	…	…	…	…	…
	10	$\widetilde{\sigma}_{10,1}$	…	…	…	…	…	…	…	…	…	$\widetilde{\sigma}_{10,1}$	…	…	…	…	…	…	…	…	…

Table 3.1: Market data from the swaption matrix to be incorporated into the driving process x for a 11 years annual Bermudan swaption. HW's approach (left), alternative approach (right).

3.3.1 One step covariance

One way to view a Bermudan swaption is as the right to choose between the associated European swaptions. In setting up a model, one might choose to try to capture the correlations between the co-terminal swap rates at their setting dates. These are the correlations that matter when pricing a Bermudan swaption. In a one factor model, we cannot capture all these correlations. One choice is to consider the one step correlations, i.e. the correlation of $y_{T_i}^{i,n+1-i}$ with its nearest neighbour $y_{T_{i+1}}^{i+1,n-i}$ for each i. Note that we will work with the log of the swap rates because it allows for a direct connection with the driving process x as we shall see in the model's setup.

In the approach adopted here, we estimate the one step covariances $\mathrm{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_{i+1}}^{i+1,n-i})$ for $i = 1, \ldots, n-1$ using the swaption matrix from the market. This is a two-step procedure. The first step is to approximate the correlations of the log-LIBORs at each exercise date by a global fit to the swaption matrix. The second step is to deduce the corresponding covariances of the log-LIBORs by performing a local fit to each row of the swaption matrix and using the correlations from the first step. We then use these covariances to derive the required one step covariances (see equation (36)). In fact, we only need $\mathrm{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{i+1,n-i})$ as

$$\mathrm{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j}) \approx \mathrm{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{j,n+1-j}),$$

for $i < j \leq n$ (see appendix A.2). Details of the whole approximation procedure can be found in appendix A. In what follows we will use the superscript "ma" to denote quantities estimated from the market.

Model's setup: Once we have estimated the one step covariances from the market, we set up the model as follows. Recall the linear approximation under the terminal measure in (Bennett &

Kennedy, 2005), for $i < n$

$$\begin{aligned}
\gamma_i x_t + \eta_t^i &\approx \ln y_t^{i,n+1-i} \\
\Rightarrow \mathrm{Corr}(x_{T_i}, x_{T_{i+1}}) &\approx \mathrm{Corr}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_{i+1}}^{i+1,n-i}) \\
\Longleftrightarrow \sqrt{\frac{\xi_{T_i}}{\xi_{T_{i+1}}}} &\approx \frac{\mathrm{Cov}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{i+1,n-i})}{\sqrt{\mathrm{Var}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i})\mathrm{Var}^{\mathrm{mo}}(\ln y_{T_{i+1}}^{i+1,n-i})}}.
\end{aligned} \tag{11}$$

As for each $i = 1, \ldots, n$ $\mathrm{Var}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i})$ can be inferred from the corresponding Black implied volatility $\widetilde{\sigma}_{i,n+1-i}$, we can now incorporate the one step covariances into the model as described below.

- Without loss of generality, fix $\xi_{T_n} = \widetilde{\sigma}_{n,1}^2 T_n$.
- At T_{n-1}, by knowledge of $\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1})$ and hence $\mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1}) = \frac{\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1})}{\widetilde{\sigma}_{n-1,2}\sqrt{T_{n-1}}\widetilde{\sigma}_{n,1}\sqrt{T_n}}$ that we have estimated from the market, we can recover $\xi_{T_{n-1}}$ by fixing

$$\begin{aligned}
\sqrt{\frac{\xi_{T_{n-1}}}{\xi_{T_n}}} &= \mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1}) \\
\Longleftrightarrow \sqrt{\xi_{T_{n-1}}} &= \mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1})\sqrt{\xi_{T_n}},
\end{aligned}$$

  where we use the relation in (11).

- Inductively, assume we are at T_i and have derived ξ_{T_j} for $j > i$ from the previous steps. By the approximation for $\mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_{i+1}}^{i+1,n-i})$ from the market and the knowledge of $\xi_{T_{i+1}}$, we can fix ξ_{T_i}

$$\sqrt{\xi_{T_i}} = \mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_{i+1}}^{i+1,n-i})\sqrt{\xi_{T_{i+1}}},$$

  where we again use (11).

- We have now fixed ξ_{T_i} for $i = 1, \ldots, n$ and the SMF model can be implemented on the grid.

The above model is an example of parametrization by time and its overall vega profile (in terms of a Bermudan swaption) has a close connection to that of the HW model as we shall see in section 4. As implied volatilities change, the implied correlations in the market change. The one step covariance model attempts to hedge this risk but with the focus just on the one step covariance with the next co-terminal swap rate. Clearly, we are ignoring some important market information. This is exactly the motivation for our next proposed model.

3.3.2 Weighted covariance

We propose another alternative choice of x based on the following intuition. In order to make a decision whether to exercise the option at T_i, the one step correlation $\mathrm{Corr}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_{i+1}}^{i+1,n-i})$

will be the most important as T_{i+1} is immediately after T_i. The correlations with the later rates are less significant since we can wait until later to decide.

For each $i = 1, \dots, n-1$, in addition to the one step covariance we can choose to take into account the significance of the covariances $\text{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$ for all $j > i$ but with different levels of impact on the Bermudan swaption's price. Again, as there is only one factor in the model we cannot capture everything. One way is to consider the weighted covariance $\text{Cov}(\ln y_{T_i}^{i,n+1-i}, \sum_{j=i+1}^{n} p^{T_j - T_i} \ln y_{T_j}^{j,n+1-j})$ at each exercise date T_i where the weight is chosen to be a monotonically decreasing function in $T_j - T_i$

$$p^{T_j - T_i} = \exp[-\alpha(T_j - T_i)], \qquad j > i,$$

for some $\alpha > 0$. Note that the weighted covariance can be estimated in exactly the same way as we estimate the one step covariance from the market (see appendix A). As we shall see in section 4.3, the weighted covariance model spreads the vega responses over the swaption matrix while the one step covariance model only assigns a significant contribution to the first column and the reverse diagonal. It is this feature that gives the weighted covariance model a potential hedging advantage over the one step covariance model.

Model's setup: For $i = 1, \dots, n-1$, we calibrate the model to the following market quantity

$$\text{Cov}^{\text{ma}}(\ln y_{T_i}^{i,n+1-i}, \sum_{j=i+1}^{n} p^{T_j - T_i} \ln y_{T_j}^{j,n+1-j}) \approx \sum_{j=i+1}^{n} p^{T_j - T_i} \text{Cov}^{\text{ma}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{j,n+1-j}). \qquad (12)$$

For ease of exposition, we denote this market quantity by B_i. One can incorporate the B_i's into the model as follows

- Without loss of generality, fix $\xi_{T_n}^{\alpha} = \widetilde{\sigma}_{n,1}^2 T_n$*.
- At T_{n-1}, we only need to estimate from the market the one step covariance $\text{Cov}^{\text{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_{n-1}}^{n,1})$ for B_{n-1} and consequently the one step correlation $\text{Corr}^{\text{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1})$. Since we want to calibrate the model's correlation structure to B_{n-1}, we then require that

$$\begin{aligned} p\sqrt{\frac{\xi_{T_{n-1}}^{\alpha}}{\xi_{T_n}^{\alpha}}} \widetilde{\sigma}_{n-1,2} \sqrt{T_{n-1}} \widetilde{\sigma}_{n,1} \sqrt{T_n} &= B_{n-1} \\ \iff \sqrt{\xi_{T_{n-1}}^{\alpha}} &= \text{Corr}^{\text{ma}}(\ln y_{T_{n-1}}^{n-1,2}, \ln y_{T_n}^{n,1}) \sqrt{\xi_{T_n}^{\alpha}}, \end{aligned}$$

  where we use the relation in (11). Note that this step is the same as that in the one step covariance model.

- Inductively, assume we are at T_i and have derived $\{\xi_{T_j}^{\alpha}\}_{j=i+1,\dots,n}$ from the previous steps, $\xi_{T_i}^{\alpha}$ will then be obtained by backward induction. Since we have estimated B_i from the market and want to calibrate the model's correlation structure to B_i, we require that

$$\sum_{j=i+1}^{n} p^{T_j - T_i} \sqrt{\frac{\xi_{T_i}^{\alpha}}{\xi_{T_j}^{\alpha}}} \widetilde{\sigma}_{i,n+1-i} \sqrt{T_i} \widetilde{\sigma}_{j,n+1-j} \sqrt{T_j} = B_i, \qquad (13)$$

*The superscript α is added to emphasize the dependence.

where we use $\mathrm{Corr}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$ for $j > i$ instead of just $j = i + 1$ as in the one step covariance case. Hence, from (13) we fix

$$\sqrt{\xi_{T_i}^{\alpha}} = \frac{B_i}{\widetilde{\sigma}_{i,n+1-i}\sqrt{T_i}\sum_{j=i+1}^{n} p^{T_j-T_i}\sqrt{\frac{1}{\xi_{T_j}^{\alpha}}}\widetilde{\sigma}_{j,n+1-j}\sqrt{T_j}}. \tag{14}$$

- We have now fixed $\xi_{T_i}^{\alpha}$ for $i = 1, \ldots, n$ and the SMF model can be implemented on the grid.

One can immediately see that the one step covariance process is a special case of the general weighted covariance process when α is very large. The reason is the following. When α is sufficiently large, $p^{T_j-T_i}$ will decay exponentially fast and all the weights will then become insignificant compared with $p^{T_{i+1}-T_i}$. Consequently, only the one step covariance matters in the market quantity B_i and the weighted covariance process is reduced to the one step covariance one.

Remark 1 *For both the one step and weighted covariance models, one can view the vectors of Black implied volatilities* $(\widetilde{\sigma}_{i,1}, \widetilde{\sigma}_{i,2}, \ldots, \widetilde{\sigma}_{i,n+1-i})$ *for* $i = 1, \ldots, n$ *as the model's initial inputs (see the global and local fits in appendix A). It follows from the constructions of both the one step and weighted covariance processes that one can write*

$$\xi_{T_i}^{\alpha} = f^i(\xi_{T_{i+1}}^{\alpha}, \ldots, \xi_{T_n}^{\alpha}; \{\widetilde{\sigma}_{i,k}\}_{k=1,\ldots,n+1-i}; \{\widetilde{\sigma}_{j,n+1-j}\}_{j=i+1,\ldots,n}), \tag{15}$$

where f^i *is some deterministic function (see appendix A.2 for more details). This cascade structure of the model clearly has an important implication on the response of the Bermudan price to changes in implied volatilities.*

Although the one step and weighted covariance processes appear to be more complicated than the HW process, it turns out that they are still quite similar in some context. In the next section, we will explore further their connection through the Bermudan swaption's vegas.

4 Vegas

In this section, we study the vegas of a Bermudan swaption produced by the different models. While the deltas and the gammas do not vary so much from model to model as we shall see in section 5, the vegas prove to be the most influential in the hedging of a Bermudan. In addition, the underlying parametrization of the model has an important implication for the vegas. It is, therefore, worthwhile to investigate the behaviour of the vegas from different perspectives to explore the model's structure. In subsection 4.1, we review analytically the vegas under different models. We then study the vegas numerically and investigate further the link between them.

4.1 The vega computation under the swap Markov-functional model

In order to compute the price of a Bermudan swaption in a SMF model, we need two sets of initial inputs and a covariance/correlation structure (captured through the process x). The first set of inputs are the initial discount bonds which can be safely ignored in this discussion as we are only concerned with the vegas here. The second set of inputs are the co-terminal implied volatilities which are used to recover the prices of the underlying co-terminal vanilla swaptions and fix the functional forms of the corresponding co-terminal forward swap rates at their setting dates. In the implementation of the SMF model, this is the calibration to the market marginals and it is done for all different specifications of x. In general, one can view the value of a Bermudan $\hat{V}_{T_0}$ as a price function which maps (the square root of) the variances of x and the second set of inputs to a real positive value:

$$\begin{aligned} \hat{V}_{T_0} : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^+ \\ \hat{V}_{T_0}(\overline{\xi}, \overline{\sigma}) &:= v_0, \end{aligned} \tag{16}$$

where $\overline{\xi} = (\sqrt{\xi_{T_1}}, \sqrt{\xi_{T_2}}, \ldots, \sqrt{\xi_{T_n}})$ and $\overline{\sigma} = (\widetilde{\sigma}_{1,n}, \widetilde{\sigma}_{2,n-1}, \ldots, \widetilde{\sigma}_{n,1})$. For the weighted covariance process, the equivalent input from x is $\overline{\xi}^{\alpha} = (\sqrt{\xi^{\alpha}_{T_1}}, \sqrt{\xi^{\alpha}_{T_2}}, \ldots, \sqrt{\xi^{\alpha}_{T_n}})$. Note that the notation $\overline{\xi}^{\alpha}$ when $\alpha \to \infty$ indicates the one step covariance case as we explained in section 3.3.2. For the data we are working with, it is observed that the vectors $\overline{\xi}^{\alpha}$ are quite similar for all α and hence different choices for α result in similar prices for the Bermudan.

Define the vega $\nu_{i,k}$ to be the total derivative of the Bermudan swaption's price with respect to $\widetilde{\sigma}_{i,k}$ for each $i = 1, \ldots, n$ and $k = 1, \ldots, n+1-i$:

$$\nu_{i,k} := \frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}\widetilde{\sigma}_{i,k}}. \tag{17}$$

We apply the finite difference/bumping-revaluation method to calculate these derivatives numerically.

We now consider the vegas on a particular i^{th} row of the swaption matrix. In order to distinguish the vegas produced by different models, we denote $\nu^{\alpha}_{i,k}$ for the weighted covariance process. Note again that the notation $\nu^{\alpha}_{i,k}$ as $\alpha \to \infty$ indicates the vegas for the one step covariance process. For the HW and MR models, we denote the vegas by $\nu^{\mathrm{hw}}_{i,k}$ and $\nu^{\mathrm{mr}}_{i,k}$ respectively. Note that only the co-terminal vegas $\nu^{\mathrm{hw}}_{i,n+1-i}$ and $\nu^{\mathrm{mr}}_{i,n+1-i}$ matter in the HW and MR models which follows directly from their setups. The other vegas are all zero for these models.

1. $k = 1, \ldots, n-i$ (**off reverse diagonal**): For the one step and weighted covariance models, as $\widetilde{\sigma}_{i,k}$ is only involved in $\overline{\xi}^{\alpha}$, by the chain rule and equation (15) we have that

$$\nu^{\alpha}_{i,k} = \frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}\widetilde{\sigma}_{i,k}} = \sum_{s=1}^{i} \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi^{\alpha}_{T_s}}} \times \frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\widetilde{\sigma}_{i,k}}. \tag{18}$$

   The term $\frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi^{\alpha}_{T_s}}}$ should be interpreted as the partial derivative of the price function $\hat{V}_{T_0}$ with respect to the s^{th} coordinate of the vector $\overline{\xi}^{\alpha}$. For the total derivatives $\frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\widetilde{\sigma}_{i,k}}$ where $1 \le s < i$, by equation (15) it is clear that the dependence of $\sqrt{\xi^{\alpha}_{T_s}}$ on $\widetilde{\sigma}_{i,k}$ is through $\{\sqrt{\xi^{\alpha}_{T_{j^*}}}\}_{j^*=s+1,\ldots,i}$.

   Note that the price of a Bermudan swaption is not sensitive to these implied volatility inputs under the HW and MR models. Hence, as noted above we have zero values for the corresponding vegas in these models.

2. $k = n+1-i$ (**reverse diagonal**): For the one step and weighted covariance models, as $\widetilde{\sigma}_{i,n+1-i}$ is involved in both $\overline{\xi}^{\alpha}$ and $\overline{\sigma}$ we have that

$$\nu^{\alpha}_{i,n+1-i} = \frac{\partial \hat{V}_{T_0}}{\partial \widetilde{\sigma}_{i,n+1-i}} + \sum_{s=1}^{i} \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi^{\alpha}_{T_s}}} \times \frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}}. \tag{19}$$

Note that the first term on the right hand side of (19) is approximately the i$^{\text{th}}$ bucket vega $\nu^{\text{mr}}_{i,n+1-i}$ of the MR process when the prices are comparable between models. This term only reflects the change of the marginal distribution of $y^{i,n+1-i}_{T_i}$ under its own swaption measure. For the HW process, the equivalent vega is

$$\begin{aligned}\nu^{\text{hw}}_{i,n+1-i} &= \frac{\partial \hat{V}_{T_0}}{\partial \widetilde{\sigma}_{i,n+1-i}} + \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi_{T_i}}} \times \frac{\mathrm{d}\sqrt{\xi_{T_i}}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}} \\ &\approx \nu^{\text{mr}}_{i,n+1-i} + \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi_{T_i}}} \times \frac{\mathrm{d}\sqrt{\xi_{T_i}}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}}.\end{aligned} \tag{20}$$

One can see that the above differences between $\nu^{\text{hw}}_{i,n+1-i}$ and $\nu^{\alpha}_{i,n+1-i}$ on one hand and $\nu^{\text{hw}}_{i,n+1-i}$ and $\nu^{\text{mr}}_{i,n+1-i}$ on the other hand come from the differences in their correlation structures. Since the HW and the one step and the weighted covariance models are different examples of parametrization by time, their vegas are closely connected as we shall see in section 4.4. For the MR and HW models which respond to the reverse diagonal only, the difference in their correlation structures follow directly from the difference between the parameterizations by expiry and by time. We will review this difference in the next section.

4.2 The Bermudan swaption's vegas under the HW and MR models

The example we consider here is a 11 years annual Bermudan swaption with fixed rate $K = 5\%$, notional $N = 100$ million and the following initial data:

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	13.12	13.19	13.21	13.21	13.22	13.00	12.78	12.58	12.39	12.17
	2	13.16	13.16	13.09	13.04	12.92	12.72	12.51	12.31	12.12	...
	3	13.06	12.97	12.91	12.82	12.66	12.42	12.20	12.03	...	...
	4	12.95	12.82	12.72	12.51	12.32	12.14	11.93	...	...	...
	5	12.76	12.57	12.43	12.24	12.03	11.87	...	...	...	...
	6	12.38	12.19	12.12	11.89	11.71	...	...	...	...	...
	7	12.10	11.89	11.77	11.59	...	...	...	...	...	...
	8	11.69	11.56	11.46	...	...	...	...	...	...	...
	9	11.48	11.31	...	...	...	...	...	...	...	...
	10	11.19	...	...	...	...	...	...	...	...	...

Table 4.1: Black implied volatilities (%) of the ATM swaptions on October 17, 2007.

and

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	4.54	4.55	4.56	4.58	4.60	4.63	4.66	4.70	4.73	4.76
	2	4.55	4.57	4.59	4.62	4.65	4.69	4.73	4.76	4.79	...
	3	4.58	4.62	4.65	4.68	4.72	4.76	4.80	4.83	...	...
	4	4.65	4.68	4.72	4.76	4.80	4.84	4.87	...	...	...
	5	4.71	4.75	4.80	4.84	4.88	4.92	...	...	...	...
	6	4.80	4.84	4.89	4.93	4.97	...	...	...	...	...
	7	4.89	4.94	4.98	5.01	...	...	...	...	...	...
	8	4.99	5.03	5.06	...	...	...	...	...	...	...
	9	5.06	5.09	...	...	...	...	...	...	...	...
	10	5.13	...	...	...	...	...	...	...	...	...

Table 4.2: Initial swap rates (%) on October 17, 2007.

Here we compare the vegas of the Bermudan swaption under the HW and MR models. The mean reversion parameter a is fixed at 3% for both two driving processes so that the Bermudan prices produced by the two models are close and also comparable to those that are produced by the one step and weighted covariance models. We display the vegas in tables 4.3 and 4.4. The position of each vega corresponds to the implied volatility in the swaption matrix. Recall that the off reverse diagonal entries are all zero since the Bermudan swaption's price is not sensitive to the corresponding implied volatility inputs here.

Remark 2 *In practice, traders usually quote vega as the change in price when implied volatility increases by* 100 *basis points (bp) or* 1% *so we will scale the "true" vega by a factor of* 0.01*, i.e.* $\nu_{i,k} \to 0.01\nu_{i,k}$. *For example, the entry* 4.09 *in the first row and the last column of table 4.3 means that when* $\widetilde{\sigma}_{1,10}$ *increases by* 1% *the Bermudan price (with notional* 100 *million) will increase by* $40,900$.

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0	0	0	0	0	0	0	0	0	4.09
	2	0	0	0	0	0	0	0	0	10.00	...
	3	0	0	0	0	0	0	0	9.48	...	...
	4	0	0	0	0	0	0	7.76	...	...	...
	5	0	0	0	0	0	6.23	...	...	...	...
	6	0	0	0	0	4.69	...	...	...	...	...
	7	0	0	0	3.74	...	...	...	...	...	...
	8	0	0	2.94	...	...	...	...	...	...	...
	9	0	2.16	...	...	...	...	...	...	...	...
	10	1.56	...	...	...	...	...	...	...	...	...

Table 4.3: The Bermudan swaption's scaled vegas (in 10^4) under the HW model.

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0	0	0	0	0	0	0	0	0	6.28
	2	0	0	0	0	0	0	0	0	14.54	...
	3	0	0	0	0	0	0	0	12.12	...	...
	4	0	0	0	0	0	0	8.48	...	...	...
	5	0	0	0	0	0	5.62	...	...	...	...
	6	0	0	0	0	3.37	...	...	...	...	...
	7	0	0	0	1.86	...	...	...	...	...	...
	8	0	0	0.83	...	...	...	...	...	...	...
	9	0	0.22	...	...	...	...	...	...	...	...
	10	-0.02	...	...	...	...	...	...	...	...	...

Table 4.4: The Bermudan swaption's scaled vegas (in 10^4) under the MR model.

It is seen in figure 4.1 that the vegas for both models as a function of expiry display "humped" shapes whose peaks are attained at the same exercise date. We also observe that the HW vegas are lower than the MR vegas at the early exercise dates but higher at the later ones. Recall from subsection 4.1 that this difference in vegas is actually caused by the difference in the correlation structures. This is rather important for a strongly correlation-dependent product like the Bermudan swaption. In the following, we will give a crude explanation on how a change in one of the co-terminal implied volatilities can affect the correlations of the co-terminal swap rates under the two models which will then clearly indicate their vegas.

Parametrization by expiry (MR): We recall that the MR process is unaltered by bumping any co-terminal implied volatility. This then implies that the correlation structure of the model/co-terminal forward swap rates is unaltered, i.e. $\mathrm{Corr}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$ which are approximately $\sqrt{\frac{\xi_{\min(T_i,T_j)}}{\xi_{\max(T_i,T_j)}}}$ are unchanged for all $i, j \leq n$.

Parametrization by time (HW): Bumping $\widetilde{\sigma}_{i,n+1-i}$ has an immediate effect on the HW process and hence the correlation structure of the model/co-terminal forward swap rates. Specifically, $\mathrm{Corr}^{\mathrm{mo}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j}) \approx \sqrt{\frac{\xi_{\min(T_i,T_j)}}{\xi_{\max(T_i,T_j)}}}$ will increase for $j > i$ but decrease for $j < i$. Heuristically, we have the following overall effect: on average the co-terminal forward swap rates will tend to be more correlated if i is small and less correlated if i is large.

We employ the following heuristic argument for the correlation's effects on the Bermudan price. The optionality of a Bermudan implies that the lower the correlations of the co-terminal swap rates get, the higher the price of the Bermudan swaption becomes. For more details of how correlations affect the Bermudan price, see (Andersen & Piterbarg, 2010) and (Rebonato, 2004) for example for reference. Hence, we draw the following conclusion on the vegas. If x is parameterized by time and i is small (early exercise date), the co-terminal forward swap rates will tend to be more correlated on average. This effect will cause the HW price to increase less than the MR price and make $\nu_{i,n+1-i}^{\mathrm{hw}}$ lower than $\nu_{i,n+1-i}^{\mathrm{mr}}$. On the other hand, if i is large (late exercise date), the co-terminal forward swap rates will tend to be less correlated on average. This then causes $\nu_{i,n+1-i}^{\mathrm{hw}}$ to be higher than $\nu_{i,n+1-i}^{\mathrm{mr}}$. This fundamental difference is the key observation which leads to very different hedging profiles as we shall see later.

Remark 3 *The MR vegas become very small or even negative at the end of the option which is possible under some circumstances in practice. See appendix B in (Pietersz & Pelsser, 2004) for an explanation of negative vega for a two stock Bermudan option example.*

4.3 The Bermudan swaption's vegas under the one step and weighted covariance models

We test the one step and weighted covariance processes with different values of α and display their vega matrices in tables 4.5, 4.6, 4.7 and 4.8.

We first look at the vegas for the one step covariance model in table 4.5. The first thing to notice from this table is that the vega response starts shifting away from the reverse diagonal entries. We obtain a vega profile which assigns a significant contribution to the first column of the swaption matrix. The other vega entries are seen to be much smaller and very close to zero except for the co-terminal ones. The vega behaviour of the first column can be seen from the local fit in appendix A.2. In this local fit step, apart from the reverse diagonal entries we see that shifting the first column has the most distinctive effect on the one step covariance terms that we estimate from the market. The one step covariance model is set up such that it responds to the changes in the one step covariances only (not the other covariances as considered in the weighted covariance model). Thus it is a shift in the first column or a reverse diagonal entry of the swaption matrix that has the largest vega response. Note that the swaptions corresponding to the first column are fairly illiquid so it would be desirable to use more implied volatilities to moderate the response to any inaccurate market signals. The results produced by the weighted covariance model indeed have this feature.

For the weighted covariance model, we observe different patterns for the vega response depending on different values of α. For example, when $\alpha = 0.05$ we observe a bigger response in the central part of the table compared with that when we use a much higher value of α, for instance $\alpha = 5$. For some particular rows, it is seen that those central entries even dominate the reverse diagonal and the first column. For larger values of α such as 0.3 in table 4.7, we observe a clearer trend in the vega entries. They tend to increase in tenor for each expiry. When α gets much higher (table 4.8), it is clear that the entries look very similar to the one step covariance case where the vegas from the central part become much more insignificant and dominated by the reverse diagonal and the first column. This is predictable as one step covariance is a special case of the weighted covariance model when α is very large.

Remark 4 *We get a few negative vega entries when* $\alpha = 0.05$*. Note that those in the reverse diagonal (the first two rows) are quite large in magnitude. One reason for this behaviour is the following. When we shift the co-terminal implied volatility* $\widetilde{\sigma}_{i,n+1-i}$*, the approximations from the market for the correlations* $Corr^{ma}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$ *tend to increase for* $j > i$*. For a very low value of* α*, the model will take into account all these increases in correlations with high levels of impact on the Bermudan price since the geometric weight* $p^{T_j - T_i}$ *decays very slowly in* $T_j - T_i$*. Therefore, when* i *is small the overall increase in correlations of the co-terminal swap rates could be large which in turn leads to a decrease in price and hence negative vegas.*

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0.27	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.60
	2	0.94	0.01	0.01	0.01	0.01	0.01	0.01	0.01	8.86	...
	3	1.46	0.01	0.01	0.01	0.01	0.01	0.00	7.79	...	...
	4	1.82	0.01	0.01	0.01	0.01	0.01	5.71	...	...	...
	5	2.01	0.02	0.02	0.02	0.01	4.02	...	...	...	...
	6	2.08	0.02	0.02	0.01	2.45	...	...	...	...	...
	7	2.05	0.02	0.01	1.56	...	...	...	...	...	...
	8	1.89	0.02	1.00	...	...	...	...	...	...	...
	9	1.68	0.43	...	...	...	...	...	...	...	...
	10	1.57	...	...	...	...	...	...	...	...	...

Table 4.5: The Bermudan swaption's scaled vegas (in 10^4) under the one step covariance model.

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0.08	0.05	0.16	0.30	0.16	0.62	0.70	0.57	2.48	−1.05
	2	0.10	0.46	0.36	0.56	0.86	1.32	2.21	4.79	−1.06	...
	3	0.10	0.22	0.36	0.58	0.93	1.70	3.70	1.42	...	...
	4	0.06	0.16	0.29	0.54	0.95	2.15	3.49	...	...	...
	5	0.05	0.16	0.30	0.55	1.34	3.82	...	...	...	...
	6	0.02	0.10	0.16	0.46	3.89	...	...	...	...	...
	7	0.03	0.01	0.08	3.67	...	...	...	...	...	...
	8	−0.07	−0.24	3.25	...	...	...	...	...	...	...
	9	−0.18	2.25	...	...	...	...	...	...	...	...
	10	1.54	...	...	...	...	...	...	...	...	...

Table 4.6: The Bermudan swaption's scaled vegas (in 10^4) under the weighted covariance model ($\alpha = 0.05$).

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0.10	0.13	0.18	0.25	0.15	0.35	0.30	0.38	0.65	1.86
	2	0.23	0.36	0.46	0.59	0.66	0.82	1.05	1.73	3.83	...
	3	0.22	0.41	0.53	0.67	0.81	1.14	1.87	3.75	...	...
	4	0.23	0.38	0.51	0.69	0.95	1.67	3.45	...	...	...
	5	0.18	0.35	0.53	0.81	1.48	3.00	...	...	...	...
	6	0.17	0.34	0.53	1.05	2.67	...	...	...	...	...
	7	0.13	0.31	0.71	2.68	...	...	...	...	...	...
	8	0.08	0.26	2.62	...	...	...	...	...	...	...
	9	0.08	2.03	...	...	...	...	...	...	...	...
	10	1.55	...	...	...	...	...	...	...	...	...

Table 4.7: The Bermudan swaption's scaled vegas (in 10^4) under the weighted covariance model ($\alpha = 0.3$).

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0.25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	3.59
	2	0.94	0.03	0.01	0.01	0.01	0.01	0.01	0.01	8.87	...
	3	1.44	0.03	0.01	0.01	0.01	0.00	0.00	7.77	...	...
	4	1.78	0.05	0.00	0.00	0.00	0.01	5.69	...	...	...
	5	1.98	0.05	0.00	0.00	0.00	4.03	...	...	...	...
	6	2.03	0.04	0.03	0.00	2.42	...	...	...	...	...
	7	2.03	0.08	0.02	1.58	...	...	...	...	...	...
	8	1.82	0.08	0.99	...	...	...	...	...	...	...
	9	1.66	0.46	...	...	...	...	...	...	...	...
	10	1.57	...	...	...	...	...	...	...	...	...

Table 4.8: The Bermudan swaption's scaled vegas (in 10^4) under the weighted covariance model ($\alpha = 5$).

4.4 The net market vegas for different parameterizations

We recall that the HW process is an example of parametrization by time and it has a certain vega profile with the responses only on the reverse diagonal. The one step and weighted covariance models move the vega response away from the reverse diagonal and this causes their hedging behaviours to be quite different from that of the HW model. However, their vega profiles are still very closely connected as they are different examples of parametrization by time. For the one step and weighted covariance models, we plot the sum of the vegas for each row (expiry) of the swaption matrix. With our initial data, we observe that each row sum is roughly a constant that is independent of α and very close to the co-terminal vega on the same row of the HW model (figure 4.1).

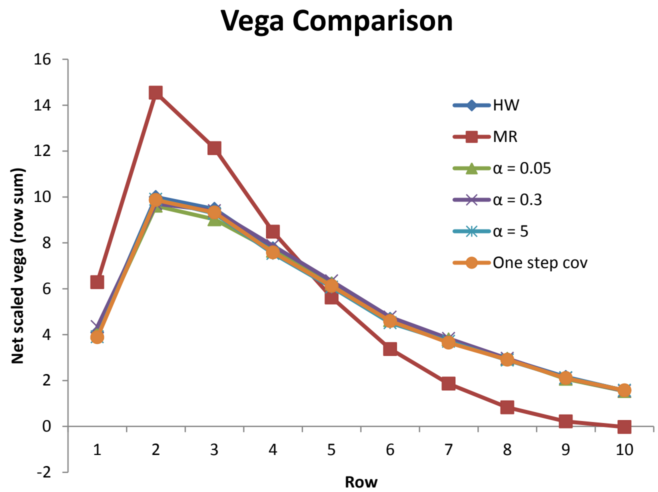

Figure 4.1: The (net) row sum of the scaled vegas (in 10^4) of a 11-years annual Bermudan swaption for different models and parameters.

We state this observation as a result.

Result 1 *For each $i = 1, \ldots, n$ and all $\alpha > 0$, under the assumptions that implied volatilities of the same expiry are not so variant with respect to tenor and the variances $\xi_{T_1}, \ldots, \xi_{T_n}$ are comparable between models the following relation holds true*

$$\sum_{k=1}^{n+1-i} \nu_{i,k}^{\alpha} \approx \nu_{i,n+1-i}^{hw} \tag{21}$$

In order to prove this result, we need the following sub-result that works with the log-transformation of the implied volatilities.

For $k = 1, \ldots, n+1-i$, let $\Sigma_{i,k} := \ln \widetilde{\sigma}_{i,k}$. We define the total derivative of the Bermudan swaption's price with respect to these log-implied volatilities as

$$\hat{\nu}_{i,k} := \frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}\Sigma_{i,k}}.$$

Again, in order to distinguish different models we denote $\hat{\nu}_{i,k}^{\alpha}$ for the one step and weighted covariance models. For the HW model, the equivalent co-terminal term is denoted by $\hat{\nu}_{i,n+1-i}^{\mathrm{hw}}$ and note that it is the only term that matters. We state the sub-result as a lemma.

Lemma 1 *For each $i = 1, \ldots, n$ and all $\alpha > 0$, under the second assumption in result 1 the following relation holds true*

$$\sum_{k=1}^{n+1-i} \hat{\nu}_{i,k}^{\alpha} \approx \hat{\nu}_{i,n+1-i}^{hw} \tag{22}$$

Proof: We will first prove that for the one step and weighted covariance models the row sum $\sum_{k=1}^{n+1-i}\hat{\nu}_{i,k}^{\alpha}$ is independent of α. Note that $\sum_{k=1}^{n+1-i}\hat{\nu}_{i,k}^{\alpha}$ roughly represents the effect of a parallel additive shift of $\Sigma_{i,k}$ for $k=1,\dots,n+1-i$ on the Bermudan price as we can see from the Taylor expansion of the price function $\hat{V}_{T_0}$. This is equivalent to a parallel multiplicative shift of $\widetilde{\sigma}_{i,k}$ for $k=1,\dots,n+1-i$ since

$$\Sigma_{i,k}\longrightarrow\Sigma_{i,k}+\ln\epsilon \iff \widetilde{\sigma}_{i,k}\longrightarrow\epsilon\widetilde{\sigma}_{i,k}.$$

One then can write

$$\sum_{k=1}^{n+1-i}\hat{\nu}_{i,k}^{\alpha} \approx \frac{\hat{V}_{T_0}(\epsilon\widetilde{\sigma}_{i,1},\dots,\epsilon\widetilde{\sigma}_{i,n+1-i})-\hat{V}_{T_0}(\widetilde{\sigma}_{i,1},\dots,\widetilde{\sigma}_{i,n+1-i})}{\ln\epsilon},$$

where $\epsilon>1$ and sufficiently small. It remains to show that the effect of the parallel multiplicative shift of the $\mathrm{i^{th}}$ row of the swaption matrix on the Bermudan price is independent of α.

In the following we use the analysis obtained in appendix A. The main purpose is to assess the effects of the above parallel multiplicative shift on each of the estimated covariances that feed into the one step and weighted covariance models, i.e. $\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i},\ln y_{T_i}^{j,n+1-j})$ for $j=i+1,\dots,n$. This can be seen via the effects on the covariances of the log-LIBORs. From the local fit in appendix A.2, we have the following approximation for the covariances of the log-LIBORs at each exercise date T_i

$$\widetilde{\sigma}_{i,k}^2T_i\approx\sum_{l=i}^{i+k-1}\sum_{l^*=i}^{i+k-1}\zeta_l^{i,k}(0)\zeta_{l^*}^{i,k}(0)\mathrm{Cov}^{\mathrm{ma}}(\ln L_{T_i}^{l},\ln L_{T_i}^{l^*}),\qquad k=1,\dots,n+1-i,$$

where $\{\zeta_l^{i,k}(0)\}_{l=i,\dots,n}$ are constants that only depend on the initial discount curve (see appendix A). Therefore, under a parallel multiplicative shift of the $\mathrm{i^{th}}$ row of the swaption matrix we have that

$$\mathrm{Cov}^{\mathrm{ma}}(\ln L_{T_i}^{l},\ln L_{T_i}^{l^*})\longrightarrow\epsilon^2\mathrm{Cov}^{\mathrm{ma}}(\ln L_{T_i}^{l},\ln L_{T_i}^{l^*}),\qquad l,l^*=i,\dots,n.$$

Since the covariance of the log-swap rates can be approximated by summing up the covariances of the corresponding spanning log-LIBORs (see appendix A.2), we have

$$\begin{aligned}\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i},\ln y_{T_i}^{j,n+1-j}) &\approx \sum_{k=i}^{n}\sum_{l=j}^{n}\zeta_k^{i,n+1-i}(0)\zeta_l^{j,n+1-j}(0)\mathrm{Cov}^{\mathrm{ma}}(\ln L_{T_i}^{k},\ln L_{T_i}^{l})\\ \Rightarrow \mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i},\ln y_{T_i}^{j,n+1-j}) &\longrightarrow \epsilon^2\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i},\ln y_{T_i}^{j,n+1-j}),\qquad j=i+1,\dots,n,\end{aligned}$$

Recall the market quantity $B_i=\sum_{j=i+1}^{n}p^{T_j-T_i}\mathrm{Cov}^{\mathrm{ma}}(\ln y_{T_i}^{i,n+1-i},\ln y_{T_i}^{j,n+1-j})$. For all values of α, we then have that

$$B_i\longrightarrow\epsilon^2B_i.$$

We further recall the construction of $\sqrt{\xi_{T_i}^{\alpha}}$ in equation (14) where we have

$$\sqrt{\xi_{T_i}^{\alpha}}=\frac{B_i}{\widetilde{\sigma}_{i,n+1-i}\sqrt{T_i}\sum_{j=i+1}^{n}p^{T_j-T_i}\sqrt{\frac{1}{\xi_{T_j}^{\alpha}}}\widetilde{\sigma}_{j,n+1-j}\sqrt{T_j}}.$$

Similar to the arguments in section 4.1, the cascade structure in equation (15) implies that $\{\sqrt{\xi^{\alpha}_{T_j}}\}_{j=i+1,\ldots,n}$ are invariant under a parallel multiplicative shift of the i$^{\text{th}}$ row. It then follows from (14) that

$$\sqrt{\xi^{\alpha}_{T_i}} \longrightarrow \epsilon\sqrt{\xi^{\alpha}_{T_i}},$$

and hence further $\sqrt{\frac{1}{\xi^{\alpha}_{T_i}}}\widetilde{\sigma}_{i,n+1-i}\sqrt{T_i}$ is invariant. For $1 \leq s < i$, we have that

$$\sqrt{\xi^{\alpha}_{T_s}} = \frac{B_s}{\widetilde{\sigma}_{s,n+1-s}\sqrt{T_s}\sum_{j=s+1}^{n} p^{T_j-T_s}\sqrt{\frac{1}{\xi^{\alpha}_{T_j}}}\widetilde{\sigma}_{j,n+1-j}\sqrt{T_j}}.$$

When $s = i-1$, it is clear that $\sqrt{\xi^{\alpha}_{T_s}}$ is also invariant because $\sqrt{\frac{1}{\xi^{\alpha}_{T_i}}}\widetilde{\sigma}_{i,n+1-i}\sqrt{T_i}$ is invariant. Inductively, we have that $\{\sqrt{\xi^{\alpha}_{T_s}}\}_{1\leq s<i}$ are all invariant. It is now clear that a parallel multiplicative shift of the i$^{\text{th}}$ row will only shift $\sqrt{\xi^{\alpha}_{T_i}} \longrightarrow \epsilon\sqrt{\xi^{\alpha}_{T_i}}$ regardless of α.

Since $\Sigma_{i,k}$ is just the log of $\widetilde{\sigma}_{i,k}$ for each $k = 1, \ldots, n+1-i$, one can write the analogous formulae for $\hat{\nu}^{\alpha}_{i,k}$ following exactly the same arguments as in (18) and (19). Therefore, for the off reverse diagonal entries: $k = 1, \ldots, n-i$

$$\hat{\nu}^{\alpha}_{i,k} = \sum_{s=1}^{i}\frac{\partial \hat{V}_{T_0}}{\partial\sqrt{\xi^{\alpha}_{T_s}}} \times \frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,k}},$$

and for the reverse diagonal entry $k = n+1-i$

$$\hat{\nu}^{\alpha}_{i,n+1-i} = \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \sum_{s=1}^{i}\frac{\partial \hat{V}_{T_0}}{\partial\sqrt{\xi^{\alpha}_{T_s}}} \times \frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,n+1-i}}.$$

Hence, we have the row sum

$$\begin{aligned}\sum_{k=1}^{n+1-i}\hat{\nu}^{\alpha}_{i,k} &= \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \sum_{k=1}^{n+1-i}\sum_{s=1}^{i}\frac{\partial \hat{V}_{T_0}}{\partial\sqrt{\xi^{\alpha}_{T_s}}} \times \frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,k}} \\ &= \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \sum_{s=1}^{i}\frac{\partial \hat{V}_{T_0}}{\partial\sqrt{\xi^{\alpha}_{T_s}}} \times \left(\sum_{k=1}^{n+1-i}\frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,k}}\right).\end{aligned}$$

Similar to that we discussed earlier, the sum $\sum_{k=1}^{n+1-i}\frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,k}}$ roughly represents the effect of a parallel multiplicative shift of the i$^{\text{th}}$ row of the swaption matrix on $\sqrt{\xi^{\alpha}_{T_s}}$ by looking at the Taylor expansion on $\sqrt{\xi^{\alpha}_{T_s}}$. We previously concluded for the one step and weighted covariance models that this shift will leave $\sqrt{\xi^{\alpha}_{T_j}}$ unchanged for $j \neq i$. It follows that $\sum_{k=1}^{n+1-i}\frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_s}}}{\mathrm{d}\Sigma_{i,k}} \approx 0$ for $s < i$. Hence

$$\sum_{k=1}^{n+1-i}\hat{\nu}^{\alpha}_{i,k} \approx \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \frac{\partial \hat{V}_{T_0}}{\partial\sqrt{\xi^{\alpha}_{T_i}}} \times \sum_{k=1}^{n+1-i}\frac{\mathrm{d}\sqrt{\xi^{\alpha}_{T_i}}}{\mathrm{d}\Sigma_{i,k}}.$$

Since the parallel multiplicative shift of the i^{th} row will alter $\sqrt{\xi^\alpha_{T_i}} \longrightarrow \epsilon\sqrt{\xi^\alpha_{T_i}}$, we then can write that

$$\sum_{k=1}^{n+1-i} \hat{\nu}^\alpha_{i,k} \approx \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi^\alpha_{T_i}}} \times \frac{\epsilon\sqrt{\xi^\alpha_{T_i}} - \sqrt{\xi^\alpha_{T_i}}}{\ln \epsilon}. \tag{23}$$

Because we assume the $\sqrt{\xi^\alpha_{T_i}}$'s are similar for all α under our initial data, we can now conclude that $\sum_{k=1}^{n+1-i} \hat{\nu}^\alpha_{i,k}$ is independent of α.

We now prove the second part of the lemma that connects the weighted covariance process with the HW process. For the HW process, we recall that

$$\xi_{T_i} = \left(\frac{T_{n+1} - T_i}{(1+\alpha_i y_0^{i,n+1-i})(\psi_{T_{n+1}} - \psi_{T_i})} \right)^2 \widetilde{\sigma}^2_{i,n+1-i} T_i,$$

where $\psi_{T_i} = \frac{1}{a}(1 - e^{-aT_i}), a > 0$. We fixed a at 3% so that the ξ_{T_i}'s are similar to the $\xi^\alpha_{T_i}$'s of the weighted covariance process. It is clear that shifting $\Sigma_{i,n+1-i} \longrightarrow \Sigma_{i,n+1-i} + \ln \epsilon$ or equivalently $\widetilde{\sigma}_{i,n+1-i} \longrightarrow \epsilon \widetilde{\sigma}_{i,n+1-i}$ will only shift

$$\sqrt{\xi_{T_i}} \longrightarrow \epsilon \sqrt{\xi_{T_i}}, \tag{24}$$

and leave $\{\sqrt{\xi_{T_j}}\}_{j \neq i}$ remain unchanged. This is exactly the same effect that a parallel multiplicative shift of the i^{th} row has on the weighted covariance process. We now write the analogous formula for $\hat{\nu}^{\mathrm{hw}}_{i,n+1-i}$ following the same argument as in (20)

$$\hat{\nu}^{\mathrm{hw}}_{i,n+1-i} = \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi_{T_i}}} \times \frac{\mathrm{d}\sqrt{\xi_{T_i}}}{\mathrm{d}\Sigma_{i,n+1-i}}.$$

This then follows immediately from (24) that

$$\hat{\nu}^{\mathrm{hw}}_{i,n+1-i} \approx \frac{\partial \hat{V}_{T_0}}{\partial \Sigma_{i,n+1-i}} + \frac{\partial \hat{V}_{T_0}}{\partial \sqrt{\xi_{T_i}}} \times \frac{\epsilon\sqrt{\xi_{T_i}} - \sqrt{\xi_{T_i}}}{\ln \epsilon},$$

which is approximately the same as (23) since ξ_{T_i} and $\xi^\alpha_{T_i}$ are comparable. The proof is now complete.

We are now able to prove result 1.

Proof of result 1: since $\Sigma_{i,k} = \ln \widetilde{\sigma}_{i,k}$, one can write the following for the one step and weighted covariance models

$$\nu^\alpha_{i,k} = \hat{\nu}^\alpha_{i,k} \frac{\mathrm{d}\Sigma_{i,k}}{\mathrm{d}\widetilde{\sigma}_{i,k}} = \hat{\nu}^\alpha_{i,k} \frac{1}{\widetilde{\sigma}_{i,k}}.$$

Hence

$$\sum_{k=1}^{n+1-i} \nu^\alpha_{i,k} = \sum_{k=1}^{n+1-i} \hat{\nu}^\alpha_{i,k} \frac{1}{\widetilde{\sigma}_{i,k}}.$$

For the HW model, the equivalent quantity is $\nu^{\mathrm{hw}}_{i,n+1-i} = \hat{\nu}^{\mathrm{hw}}_{i,n+1-i} \frac{1}{\widetilde{\sigma}_{i,n+1-i}}$. Observe that if the implied volatility is not so variant in tenor k, we will be able to remove $\widetilde{\sigma}^{-1}_{i,k}$ from the sum and replace them by a constant C. This assumption is also supported by the data we work with in this paper (table 4.1). It then follows by lemma 1 that

$$\sum_{k=1}^{n+1-i} \nu_{i,k}^{\alpha} \approx C \sum_{k=1}^{n+1-i} \hat{\nu}_{i,k}^{\alpha} \approx C\hat{\nu}_{i,n+1-i}^{\text{hw}} \approx \nu_{i,n+1-i}^{\text{hw}}.$$

The proof is now complete.

5 A hedging result

In this section, we construct a vega-delta neutral portfolio for a Bermudan swaption under different models. The portfolio will consist of vanilla swaptions and interest rate swaps. We then calculate the gamma profile of the portfolio for each model and compare them accordingly. Here we employ the same example of Bermudan swaption as we considered in section 4 with the data provided in tables 4.1 and 4.2.

We first look at the gamma profiles of the HW and MR hedging portfolios in section 5.1. We then continue our investigation with the one step and weighted covariance models in section 5.2. In all cases, we want to stress that the vegas of the Bermudan are the main factor that affects the gamma profile of the hedging portfolio. For the first two models, we observe a distinct difference between them in terms of the total gamma of vega-delta neutral portfolio. These will be addressed in detail in section 5.1.3. It is also seen in this section that the parallel gamma profiles of these models are qualitatively similar and not ideal in practice. Due to the more realistic specification of the later models, one can actually hope that the gamma profile of the hedging portfolio will be improved. This is a potential advantage of the one step and weighted covariance models and we will analyze it in detail in section 5.2.3.

5.1 A hedging portfolio for the HW and MR models

We now proceed to the construction of the vega-delta neutral portfolios for the HW and MR models.

5.1.1 Vega hedge

We first construct a vega neutral portfolio. Since the Bermudan prices under the HW and MR models are only sensitive to the changes in the co-terminal implied volatilities, we will only need to hedge those risks by trading suitable proportions of the co-terminal vanilla swaptions. At this stage, the portfolio will consist of a Bermudan swaption with today's value $\hat{V}_{T_0}$ and an appropriate proportion N_i^{sption} of the corresponding i$^{\text{th}}$ co-terminal vanilla swaption with today's value $\widetilde{V}_{T_0}^{i,n+1-i}$ for each $i = 1, \ldots, n$. We require the portfolio to be vega neutral, i.e. for each $i = 1, \ldots, n$, N_i^{sption} is chosen such that

$$\frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}} + N_i^{\text{sption}} \frac{\mathrm{d}\widetilde{V}_{T_0}^{i,n+1-i}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}} = \nu_{i,n+1-i} + N_i^{\text{sption}} \frac{\mathrm{d}\widetilde{V}_{T_0}^{i,n+1-i}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}} = 0,$$

where the derivatives can be calculated numerically by the finite difference method and note that $\nu_{i,n+1-i}$ indicates the i$^{\text{th}}$ bucket vega of the Bermudan for the HW and MR models. Hence, we have that

$$N_i^{\text{sption}} = -\nu_{i,n+1-i} / \left(\frac{\mathrm{d}\widetilde{V}_{T_0}^{i,n+1-i}}{\mathrm{d}\widetilde{\sigma}_{i,n+1-i}} \right).$$

We display the vegas of the Bermudan calculated for the HW and MR models together in table 5.1 and the co-terminal vanilla swaptions' vegas (with notional being 1 million) in table 5.2. The calculations of N_i^{sption} then follow directly and we display the results in table 5.3.

i	1	2	3	4	5	6	7	8	9	10
HW	4.09	10.00	9.48	7.76	6.23	4.69	3.74	2.94	2.16	1.56
MR	6.28	14.54	12.12	8.48	5.62	3.37	1.86	0.83	0.22	-0.02

Table 5.1: The Bermudan swaption's scaled vegas (in 10^4) for the HW and MR models.

i	1	2	3	4	5	6	7	8	9	10
Vanilla	0.27	0.35	0.38	0.38	0.36	0.32	0.27	0.21	0.15	0.08

Table 5.2: The co-terminal vanilla swaptions' scaled vegas (in 10^4).

i	1	2	3	4	5	6	7	8	9	10
HW	-15	-28	-25	-20	-17	-15	-14	-14	-15	-21
MR	-23	-41	-32	-22	-16	-11	-7	-4	-1	0

Table 5.3: Vega hedging (N_i^{sption}) for the HW and MR models.

In table 5.3, one observes that N_i^{sption}'s of the MR model are about one and a half times as big in magnitude as those of the HW model for small i. The gap between N_i^{sption}'s of the two models gets smaller as i increases and after a certain i we observe the reverse situation, i.e. N_i^{sption}'s of the MR model become much smaller in magnitude compared with those of the HW model. This is an immediate result from the difference in their vegas which is a direct consequence of the difference between the parameterizations by time and by expiry as we explored in sections 3.2 and 4.2.

5.1.2 Delta hedge

After the vega hedging step, the hedging portfolio is no longer exposed to the vega risks but still exposed to the delta risks, i.e. the risks with respect to the movements of the underlyings. The next step is, therefore, to neutralize the delta risks but still maintain the vega neutrality. We will do so by using the co-initial swaps with today's values $V_{T_0}^i$ for $i = 1, \ldots, n+1$ as the hedging instruments. Note that the corresponding swap rates of the co-initial swaps are $y^{0,i}$ which start today and mature at T_i for $i = 1, \ldots, n+1$. We will work with the co-initial swap rates as the underlyings instead of the pure discount bonds as their prices are directly available in the market.

We first define the deltas of a Bermudan to be

$$\hat{\Delta}_j := \frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}y^{0,j}},$$

for $j = 1, \ldots, n+1$. These derivatives can be calculated by the finite difference method. At this stage, the current vega neutral portfolio has non-zero deltas.

Because the co-initial swaps are not sensitive to any changes in the implied volatilities, adding them to the portfolio will not affect the vega neutrality. Denote the proportions of the co-initial swaps that we wish to acquire by N_i^{swap} for $i = 1, \ldots, n+1$. We now have a new hedging portfolio with today's value $V_{T_0}^{\text{port}}$

$$V_{T_0}^{\text{port}} = \hat{V}_{T_0} + \sum_{i=1}^{n} N_i^{\text{sption}} \widetilde{V}_{T_0}^{i,n+1-i} + \sum_{i=1}^{n+1} N_i^{\text{swap}} V_{T_0}^{i}.$$

The proportions $\{N_i^{\text{swap}}\}_{i=1,\ldots,n+1}$ need to be chosen so that the hedging portfolio becomes delta neutral. The j^{th} delta position of the portfolio is denoted by Δ_j where

$$\Delta_j := \frac{\mathrm{d}V_{T_0}^{\text{port}}}{\mathrm{d}y^{0,j}},$$

for $j = 1, \ldots, n+1$. Therefore, in order to neutralize the delta risks we need to solve for vector $\mathbf{N}^{\text{swap}} = (N_1^{\text{swap}}, \ldots, N_{n+1}^{\text{swap}})$ so that $\Delta_j = 0$ for $j = 1, \ldots, n+1$. This is a straightforward task and it effectively requires a matrix multiplication to get the vector $\mathbf{N}^{\text{swap}}$. The portfolios for the HW and MR models are now vega-delta neutral.

We present the Bermudan's deltas and N_i^{swap}'s in figure 5.1 and table 5.4 respectively. Note that there is a large jump in the last delta of the Bermudan which is not straightforward at first glance. The reason is that shifting the last co-initial swap rate will shift the last LIBOR and result in a parallel shift of all the co-terminal swap rates. On the other hand, the equivalent effect of shifting the other co-initial swap rates is that it will only decrease slightly one of the co-terminal swap rates and leave the rest almost unchanged. As a result, the last delta is positive and large in magnitude while the others are small and negative. These effects can be seen via the one to one correspondence between the co-initial swap rates and the discount bonds.

Remark 5 *In practice, delta is usually quoted as the change in price when the underlying rate moves by* 1 *bp (*0.0001*) so we will scale the "true" delta as calculated above by a factor of* 0.0001*, i.e.* $\hat{\Delta}_j \to 0.0001\hat{\Delta}_j$*. For example, a delta of around* -3000 *in figure 5.1 means that the Bermudan price (with notional* 100 *million) will decrease by around* 3000 *if the corresponding co-initial swap rate increases by* 1 *bp.*

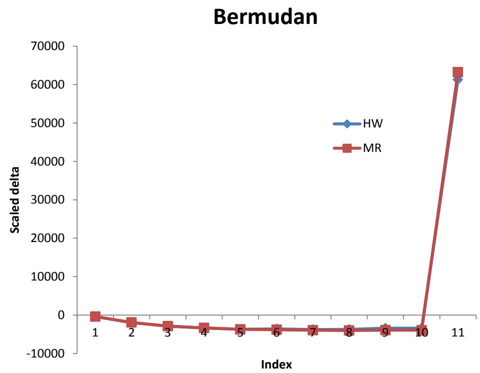

Figure 5.1: The scaled deltas of the Bermudan under the HW and MR models.

i	1	2	3	4	5	6	7	8	9	10	11
HW	-1.4	-2.0	-1.3	-0.9	-0.8	-1.2	-1.6	-2.6	-4.2	-8.2	21.8
MR	-4.4	-7.6	-4.7	-2.0	-0.0	1.4	2.5	3.5	4.2	4.8	-0.9

Table 5.4: Delta hedging (N_i^{swap}) for the HW and MR models (correspond to swaps with notional $N = 1$ million).

5.1.3 The gammas of the HW and MR hedging portfolios

Recall that each of the deltas is itself a function of the co-initial swap rates so they will move whenever these rates move. This change of deltas requires us to re-balance the portfolio since the previous portfolio is no longer delta-neutral. It is then important to consider the gammas of the portfolio as a measure of the sensitivity of the deltas. As the re-balancing cost could sometimes be high, it is desirable from a practical point of view that the gammas of the portfolio should have as small magnitudes as possible which implies that there is very little need for re-balancing the portfolio. For each $i = 1, \ldots, n+1$, we carry out a parallel shift of the co-initial swap rates with $\epsilon = 1$ bp (0.0001) and re-evaluate the delta to calculate the gamma. For the Bermudan, we then end up with the gamma vector $\hat{\mathbf{\Gamma}}$ where each of the co-ordinates is

$$\hat{\Gamma}_i := \frac{\hat{\Delta}_i(y^{0,1}+\epsilon, \ldots, y^{0,n+1}+\epsilon) - \hat{\Delta}_i(y^{0,1}, \ldots, y^{0,n+1})}{\epsilon}, \qquad i = 1, \ldots, n+1. \tag{25}$$

For the vega-delta neutral portfolio, we have the corresponding gamma vector $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_{n+1})$. The quantity in (25) is referred to as the parallel gamma and tells the trader how much the delta moves when the market moves. It is a proxy for the row sums of the full gamma matrix

$$\hat{\Gamma}_i \approx \sum_{j=1}^{n+1} \frac{\mathrm{d}^2 \hat{V}_{T_0}}{\mathrm{d}y^{0,i}\mathrm{d}y^{0,j}}.$$

Remark 6 *Again, we note that although the "true" gammas are the second order derivatives of the price with respect to the underlyings, it is market convention to quote them as the change in the scaled deltas (by a factor of* 0.0001*) when the underlying rate(s) increases by* 1 *bp. For example, one will quote the parallel gamma* $\hat{\Gamma}_i$ *as calculated in* (25) *as*

$$0.0001\hat{\Delta}_i(y^{0,1}+\epsilon,\dots,y^{0,n+1}+\epsilon) - 0.0001\hat{\Delta}_i(y^{0,1},\dots,y^{0,n+1}).$$

Therefore, the gammas we display later are the "true" gammas scaled by a factor of 10^{-8}*.*

An aggregated gamma quantity of interest is the total gamma. This is the sum of all entries in the gamma matrix $\Gamma = \sum_{i=1}^{n+1}\Gamma_i$ and it measures the total gamma exposure of the vega-delta neutral portfolio. We approximate this quantity by adding up all co-ordinates of the gamma vector since each co-ordinate is a proxy for the row sum of the gamma matrix. In table 5.5, we display the scaled total gamma for each component of the HW and MR portfolios. The result shows that the magnitude of the total gamma of the MR portfolio is around twice as big as that of the HW portfolio. This difference mainly comes from the big gap between the total gamma contributions of the co-terminal swaptions of the two portfolios. On the other hand, the contributions from the co-initial swaps are seen to be much smaller in magnitude so we will not discuss them in detail.

	Bermudan	vanilla swaptions	co-initial swaps	portfolio
HW	196	-286	2	-88
MR	192	-369	-7	-184

Table 5.5: Scaled total gammas of the HW and MR portfolios.

In order to understand the difference in the total gammas of the two portfolios, we plot the total gamma contribution from each individual co-terminal swaption in figure 5.2. Note that the total gamma contribution of the vanilla swaptions (-286 for HW and -369 for MR after scaling) is basically $\sum_{i=1}^{n} N_i^{\text{sption}} \sum_{j=1}^{n+1} \widetilde{\Gamma}_j^{i,n+1-i}$. The plot in figure 5.2 displays $N_i^{\text{sption}} \sum_{j=1}^{n+1} \widetilde{\Gamma}_j^{i,n+1-i}$ after scaling for each i. The difference in the negative peaks is clearly caused by the difference in N_i^{sption}'s as we observed in table 5.3. Recall that the difference in N_i^{sption}'s is a consequence of the difference in the vegas of the Bermudan for the two models which is characterized by the difference between the parametrizations by time and by expiry. From this analysis, one then concludes that the HW portfolio has a better total gamma than the MR portfolio. It indicates that in a wider context a parametrization by time process will potentially lead to a better total gamma profile of the hedging portfolio than a parametrization by expiry process.

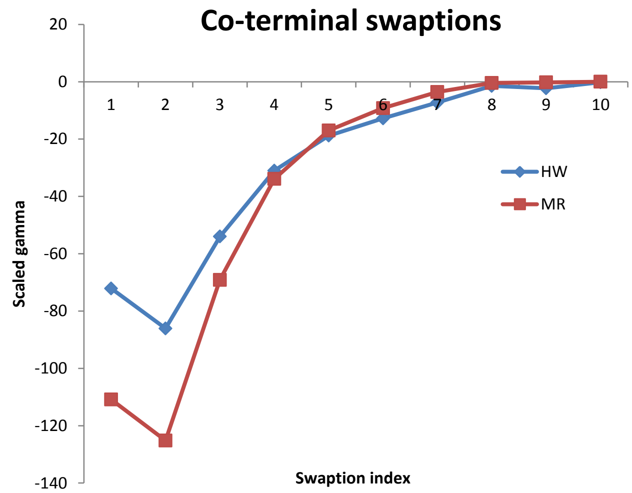

Figure 5.2: Scaled total gamma contributions from the co-terminal vanilla swaptions.

We now look at the gamma vectors of the HW and MR portfolios. Figure 5.3 displays the scaled gamma vectors of the portfolios before the hedge (just the Bermudan) and after the hedge (together with the co-terminal swaptions and the co-initial swaps). The results show that the gamma vectors of the Bermudan swaption for both models are very similar. After the hedge, the portfolios of the two models still have qualitatively very similar gamma profiles. Note that the large jump in the last gamma in the left plot is a direct consequence of the jump in the last delta in figure 5.1. From the right plot in figure 5.3, we see that for both models as all co-initial swap rates increase the deltas Δ_i's will increase for $i < n+1$ and decrease for $i = n+1$. We further observe that the rates of the increase are much lower than the rate of the decrease. Clearly, we do not have a good hedge of the last gamma for both the HW and MR models.

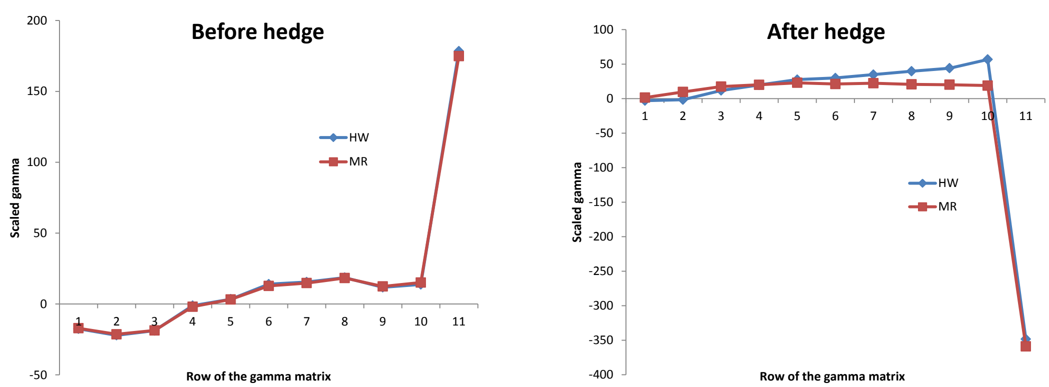

Figure 5.3: Scaled gamma vectors of the HW and MR portfolios before and after the hedge.

To gain more insight into the gamma behaviours, we will look at the gamma contribution from each hedging component. We further denote the gamma vector of the co-terminal swaption with today's value $\widetilde{V}_{T_0}^{i,n+1-i}$ by $\widetilde{\mathbf{\Gamma}}^{\mathbf{i,n+1-i}} = (\widetilde{\Gamma}_1^{i,n+1-i}, \ldots, \widetilde{\Gamma}_{n+1}^{i,n+1-i})$. Similarly, we denote the gamma vector of the co-initial swap with today's value $V_{T_0}^i$ by $\mathbf{\Gamma^i} = (\Gamma_1^i, \ldots, \Gamma_{n+1}^i)$. Recall that the portfolio consists of

$$V_{T_0}^{\text{port}} = \hat{V}_{T_0} + \sum_{i=1}^{n} N_i^{\text{sption}} \widetilde{V}_{T_0}^{i,n+1-i} + \sum_{i=1}^{n+1} N_i^{\text{swap}} V_{T_0}^i,$$

and hence the portfolio's parallel gamma can be written as

$$\Gamma_j = \underbrace{\hat{\Gamma}_j}_{\text{Bermudan}} + \sum_{i=1}^{n} N_i^{\text{sption}} \underbrace{\widetilde{\Gamma}_j^{i,n+1-i}}_{\text{Co-terminal swaptions}} + \sum_{i=1}^{n+1} N_i^{\text{swap}} \underbrace{\Gamma_j^i}_{\text{Co-initial swaps}},$$

for $j = 1, \ldots, n+1$. Figure 5.4 represents the scaled gamma contribution vector from each hedging component of the portfolio. The left and right plots display $\sum_{i=1}^{n} N_i^{\text{sption}} \widetilde{\Gamma}_j^{i,n+1-i}$ and $\sum_{i=1}^{n+1} N_i^{\text{swap}} \Gamma_j^i$ after scaling respectively for each $j = 1, \ldots, n+1$. Note that the index i for both sums indicates the corresponding co-terminal swaption and the co-initial swap. While the co-initial swaps do not have very large gammas, the contribution from the co-terminal swaptions clearly determines the gamma behaviour of the whole portfolio. We observe in the left plot of figure 5.4 that for both models the last gamma contribution $\sum_{i=1}^{n} N_i^{\text{sption}} \widetilde{\Gamma}_{n+1}^{i,n+1-i}$ is extremely negative and it pulls the last gamma Γ_{n+1} of the portfolio down to be also very negative. This seems to be a common problem for both models. The reason is that both the HW and MR models assign the vega responses to only the reverse diagonal of the swaption matrix. This results in large values in magnitude of N_i^{sption}'s which then lead to very negative values of $\sum_{i=1}^{n} N_i^{\text{sption}} \widetilde{\Gamma}_{n+1}^{i,n+1-i}$. In section 5.2.3, we will examine for the one step and weighted covariance models how moving the vega responses away from the reverse diagonal and spreading them over the swaption matrix can influence (improve) the gamma profile of the portfolio.

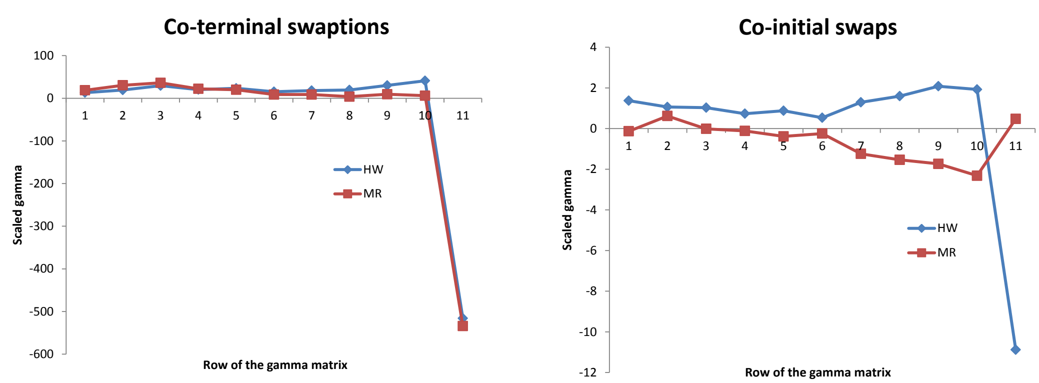

Figure 5.4: Scaled gamma contribution vectors of the co-terminal swaptions and the co-initial swaps for the HW and MR portfolios.

5.2 A hedging portfolio for the one step and weighted covariance models

This section discusses the vega-delta neutral portfolios for the one step and weighted covariance models. Similar to our treatment of the HW and MR models, we again use the vanilla swaptions and the co-initial swaps to vega-delta hedge the Bermudan.

5.2.1 Vega hedge

In section 4.4, we discussed the vegas of a Bermudan computed for the one step and weighted covariance models. Since Bermudan prices under these models respond to the changes of certain implied volatilities in the swaption matrix, we will need to hedge those risks by trading suitable proportions of the appropriate vanilla swaptions. At this stage, the portfolio will consist of a Bermudan swaption with today's value $\hat{V}_{T_0}$ and an appropriate proportion $N_{i,j}^{\text{sption}}$ of the corresponding vanilla swaption with today's value $\widetilde{V}_{T_0}^{i,j}$ (with notional being 1 million) for each $i = 1, \ldots, n$ and $j = 1, \ldots, n+1-i$. We require the portfolio to be vega neutral, i.e. for each $i = 1, \ldots, n$ and $j = 1, \ldots, n+1-i$, $N_{i,j}^{\text{sption}}$ is chosen such that

$$\frac{\mathrm{d}\hat{V}_{T_0}}{\mathrm{d}\widetilde{\sigma}_{i,j}} + N_{i,j}^{\text{sption}} \frac{\mathrm{d}\widetilde{V}_{T_0}^{i,j}}{\mathrm{d}\widetilde{\sigma}_{i,j}} = \nu_{i,j}^{\alpha} + N_{i,j}^{\text{sption}} \frac{\mathrm{d}\widetilde{V}_{T_0}^{i,j}}{\mathrm{d}\widetilde{\sigma}_{i,j}} = 0,$$

where the derivatives can be calculated numerically by the finite difference method. The vegas of vanilla swaptions are displayed in table 5.6. We recall that $\nu_{i,j}^{\alpha}$ is the (i,j) entry in the Bermudan's vega matrix of the one step and weighted covariance models. Hence, we have that

$$N_{i,j}^{\text{sption}} = -\nu_{i,j}^{\alpha} / \left(\frac{\mathrm{d}\widetilde{V}_{T_0}^{i,j}}{\mathrm{d}\widetilde{\sigma}_{i,j}} \right).$$

We now present the results from the vega-hedging step in the following tables.

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	0.03	0.05	0.08	0.10	0.13	0.16	0.19	0.21	0.24	0.27
	2	0.04	0.08	0.12	0.16	0.20	0.24	0.28	0.32	0.35	...
	3	0.05	0.10	0.15	0.20	0.25	0.29	0.34	0.38	...	...
	4	0.06	0.12	0.17	0.23	0.28	0.33	0.38	...	...	...
	5	0.06	0.13	0.19	0.25	0.30	0.36	...	...	...	...
	6	0.07	0.13	0.20	0.26	0.32	...	...	...	...	...
	7	0.07	0.14	0.21	0.27	...	...	...	...	...	...
	8	0.07	0.14	0.21	...	...	...	...	...	...	...
	9	0.07	0.15	...	...	...	...	...	...	...	...
	10	0.08	...	...	...	...	...	...	...	...	...

Table 5.6: The vanilla swaptions' scaled vegas (in 10^4).

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	-10.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-0.0	-13.4
	2	-22.7	-0.1	-0.1	-0.1	-0.1	-0.0	-0.0	-0.0	-25.2	...
	3	-28.6	-0.1	-0.1	-0.1	-0.1	-0.0	-0.0	-20.5	...	...
	4	-31.1	-0.1	-0.1	-0.1	-0.1	-0.0	-15.1	...	...	...
	5	-31.5	-0.2	-0.1	-0.1	-0.0	-11.3	...	...	...	...
	6	-30.5	-0.2	-0.1	-0.0	-7.7	...	...	...	...	...
	7	-28.7	-0.2	-0.1	-5.8	...	...	...	...	...	...
	8	-25.6	-0.1	-4.7	...	...	...	...	...	...	...
	9	-22.4	-2.9	...	...	...	...	...	...	...	...
	10	-20.9	...	...	...	...	...	...	...	...	...

Table 5.7: Vega hedging ($N_{i,j}^{\text{sption}}$) for the one step covariance model.

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	-3.0	-1.0	-2.1	-2.8	-1.2	-3.9	-3.8	-2.7	-10.3	3.9
	2	-2.4	-5.7	-3.0	-3.5	-4.3	-5.6	-8.0	-15.2	3.0	...
	3	-1.9	-2.2	-2.4	-2.9	-3.8	-5.8	-11.0	-3.7	...	...
	4	-1.1	-1.4	-1.7	-2.4	-3.4	-6.5	-9.2	...	...	...
	5	-0.8	-1.3	-1.6	-2.3	-4.4	-10.7	...	...	...	...
	6	-0.3	-0.8	-0.8	-1.8	-12.2	...	...	...	...	...
	7	-0.4	-0.1	-0.4	-13.6	...	...	...	...	...	...
	8	0.9	1.7	-15.4	...	...	...	...	...	...	...
	9	2.4	-15.4	...	...	...	...	...	...	...	...
	10	-20.4	...	...	...	...	...	...	...	...	...

Table 5.8: Vega hedging ($N_{i,j}^{\text{sption}}$) for the weighted covariance model ($\alpha = 0.05$).

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	-3.8	-2.4	-2.3	-2.4	-1.2	-2.2	-1.6	-1.8	-2.7	-6.9
	2	-5.6	-4.4	-3.8	-3.7	-3.3	-3.4	-3.8	-5.5	-10.9	...
	3	-4.3	-4.1	-3.6	-3.4	-3.3	-3.9	-5.6	-9.9	...	...
	4	-4.0	-3.3	-3.0	-3.0	-3.4	-5.1	-9.1	...	...	...
	5	-2.8	-2.8	-2.8	-3.3	-4.9	-8.4	...	...	...	...
	6	-2.5	-2.5	-2.7	-4.0	-8.4	...	...	...	...	...
	7	-1.9	-2.2	-3.4	-9.9	...	...	...	...	...	...
	8	-1.1	-1.8	-12.3	...	...	...	...	...	...	...
	9	-1.0	-13.9	...	...	...	...	...	...	...	...
	10	-20.5	...	...	...	...	...	...	...	...	...

Table 5.9: Vega hedging ($N_{i,j}^{\text{sption}}$) for the weighted covariance model ($\alpha = 0.3$).

Tenor		1	2	3	4	5	6	7	8	9	10
Expiry	1	-9.2	-0.3	-0.2	-0.1	-0.1	-0.0	-0.0	-0.0	-0.0	-13.3
	2	-22.9	-0.4	-0.1	-0.1	-0.0	-0.0	-0.0	-0.0	-25.2	...
	3	-28.2	-0.3	-0.1	-0.1	-0.0	-0.0	-0.0	-20.4	...	...
	4	-30.4	-0.5	-0.0	-0.0	-0.0	-0.0	-15.0	...	...	...
	5	-30.9	-0.4	-0.0	-0.0	-0.0	-11.3	...	...	...	...
	6	-29.7	-0.3	-0.2	0.0	-7.6	...	...	...	...	...
	7	-28.4	-0.6	-0.1	-5.8	...	...	...	...	...	...
	8	-24.7	-0.6	-4.7	...	...	...	...	...	...	...
	9	-22.1	-3.2	...	...	...	...	...	...	...	...
	10	-20.9	...	...	...	...	...	...	...	...	...

Table 5.10: Vega hedging ($N_{i,j}^{\text{sption}}$) for the weighted covariance model ($\alpha = 5$).

5.2.2 Delta hedge

Similar to the HW and MR models, we use the co-initial swaps with today's values $V_{T_0}^i$ for $i = 1, \ldots, n+1$ to delta hedge the vega neutral portfolio. To be specific, we construct the hedging portfolio with today's value $V_{T_0}^{\text{port}}$

$$V_{T_0}^{\text{port}} = \hat{V}_{T_0} + \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} N_{i,j}^{\text{sption}} \widetilde{V}_{T_0}^{i,j} + \sum_{i=1}^{n+1} N_i^{\text{swap}} V_{T_0}^i.$$

In order to neutralize the deltas, one then needs to solve for vector $\mathbf{N}^{\text{swap}} = (N_1^{\text{swap}}, \ldots, N_{n+1}^{\text{swap}})$ so that $\Delta_j = 0$ where each Δ_j is the overall j^{th} delta position of the portfolio for $j = 1, \ldots, n+1$. This step again requires us to effectively do a matrix multiplication to obtain the vector $\mathbf{N}^{\text{swap}}$.

Since the deltas of the Bermudan under the one step and weighted covariance models are very similar to those under the HW and MR models, there is no extra interest in plotting them. We display the vectors $\mathbf{N}^{\text{swap}}$ for the one step and weighted covariance models in table 5.11.

i	1	2	3	4	5	6	7	8	9	10	11
$\alpha = 0.05$	-3.2	-6.3	-3.5	0.1	0.9	1.8	3.8	6.0	5.8	11.5	-19.1
$\alpha = 0.3$	-4.0	-6.0	-3.6	-1.4	1.3	2.1	3.9	5.4	5.7	8.6	-13.8
$\alpha = 5$	-3.6	-6.2	-2.3	-0.1	1.7	2.9	3.2	4.4	4.3	5.5	-8.3
One step cov	-3.7	-5.9	-2.6	-0.3	1.6	2.6	3.6	4.3	4.6	5.4	-8.3

Table 5.11: Delta hedging (N_i^{swap}) for the one step and the weighted covariance models (correspond to swaps with notional $N = 1$ million).

5.2.3 The gammas of the one step and weighted covariance hedging portfolios

In this section, we carry out a similar comparison as in section 5.1.3 but for the HW and the one step and weighted covariance portfolios. We first report the total gamma of each portfolio produced by the one step and weighted covariance models in table 5.12. Recall that the total gamma is approximated by summing up all co-ordinates of the parallel gamma vector. One can see that the results are very similar and comparable between models. Furthermore, they slightly deviate from that of the HW model and are still much smaller in magnitude compared with that of the MR model (see table 5.5). This again supports our findings in section 5.1.3 that a parametrization by time process potentially leads to a better vega-delta neutral portfolio than a parametrization by expiry process in terms of the total gamma.

Note that the overall contributions from the vanilla swaptions in all four portfolios are quite close to the overall contributions from the co-terminal swaptions in the HW case. In order to calculate these contributions for the one step and weighted covariance models, one first calculates the total gamma for each vanilla swaption involved in the vega hedge and then multiplies with the corresponding proportion of holding $N_{i,j}^{\text{sption}}$ in the portfolio. We then sum these products up to obtain the overall contribution. It turns out that as we sum them up for each row of the swaption matrix, we effectively get the same plot as figure 5.2 of the HW model. This observation can be linked back to the vega "row sum" observation in result 1. The Bermudan's vega $\nu_{i,j}^{\alpha}$ of the one step and weighted covariance models is directly connected to the proportion $N_{i,j}^{\text{sption}}$ in the vega hedge. Intuitively, the similarity between the vega row sum $\sum_{j=1}^{n+1-i} \nu_{i,j}^{\alpha}$ and the HW's i$^{\text{th}}$ bucket vega $\nu_{i,n+1-i}^{\text{hw}}$ for each $i = 1, \ldots, n$ should also carry over to the row sum† of the total gamma to some extent due to the direct connection of the proportions of vanilla swaptions in their portfolios, i.e. $\{N_{i,j}^{\text{sption}}\}_{j=1,\ldots,n+1-i}$ of the one step and weighted covariance models and N_{i}^{sption} of the HW model.

	Bermudan	vanilla swaptions	co-initial swaps	portfolio
$\alpha = 0.05$	199	-292	-8	-101
$\alpha = 0.3$	196	-299	-7	-110
$\alpha = 5$	195	-282	-6	-93
One step cov	196	-282	-6	-93

Table 5.12: Scaled total gammas of the one step and weighted covariance portfolios.

We now consider the gamma vectors which are calculated by the analogue of (25) for the relevant instruments. The results are displayed in figure 5.5. Again, we observe that the Bermudan's gamma vectors are very close across models. The after-hedge gamma vectors, on the other hand, seem to be quite variant. On average, Γ_i's of the one step and weighted covariance portfolios are seen to be lower in magnitude than those of the HW portfolio, especially the last gamma. This is a big improvement of the one step and weighted covariance models over the HW model.

†The "row sum" in this discussion corresponds to the row of the swaption matrix, NOT the row of the gamma matrix.

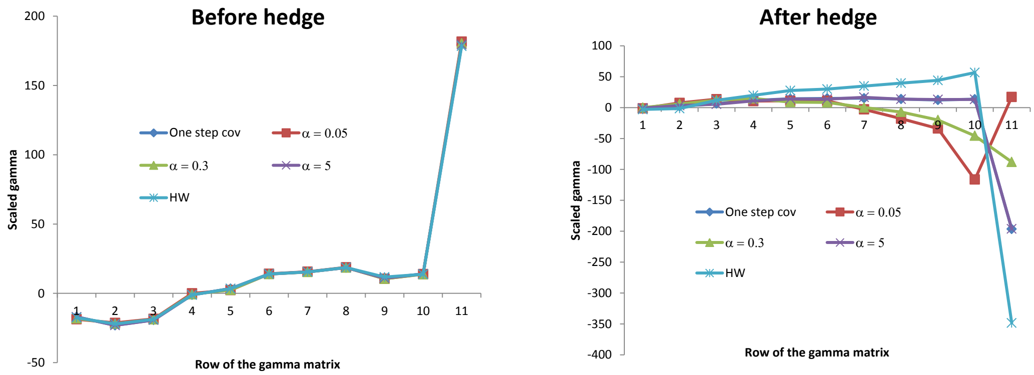

Figure 5.5: Scaled gamma vectors of the HW and the one step and weighted covariance portfolios before and after the hedge.

We again look at the gamma contribution vector from each hedging component of the portfolio in figure 5.6. As expected for all models, the magnitude of each co-ordinate of the vanilla swaptions' gamma vector dominates that of the co-initial swaps and clearly determines the gamma behaviour of the whole portfolio. Hence, we will focus on the gamma behaviour of the vanilla swaptions only. We observe that on average the contributions from the vanilla swaptions of the one step and weighted covariance portfolios seem to be much lower in magnitude than those from the co-terminal swaptions of the HW portfolio. Observe that as we move away from the reverse diagonal entries and assign more weight to other entries of the swaption matrix, the magnitude of each co-ordinate of the gamma vector can be significantly reduced.

We further observe a trend in the vanilla swaptions' gamma contributions as the parameter α of the weighted covariance model varies. When $\alpha = 0.05$, the gamma contribution is seen to be more evenly spread over all co-ordinates of the gamma vector compared with other values of α. As α increases, there are less gamma contributions from the earlier co-ordinates ($i < n+1$) but the last gamma contribution gets bigger in magnitude (more negative). Note that the case $\alpha = 5$ is very similar to the one step covariance case (almost coincide as we see in the plots). This is because the two models have very similar vega profiles and N_{ij}^{sption}'s from the vega hedge. Finally, an important point that we want to stress here is that with the one step and weighted covariance models the swaption's holder has the flexibility to control the gamma vector of the portfolio just by simply tuning the geometric weight parameter α.

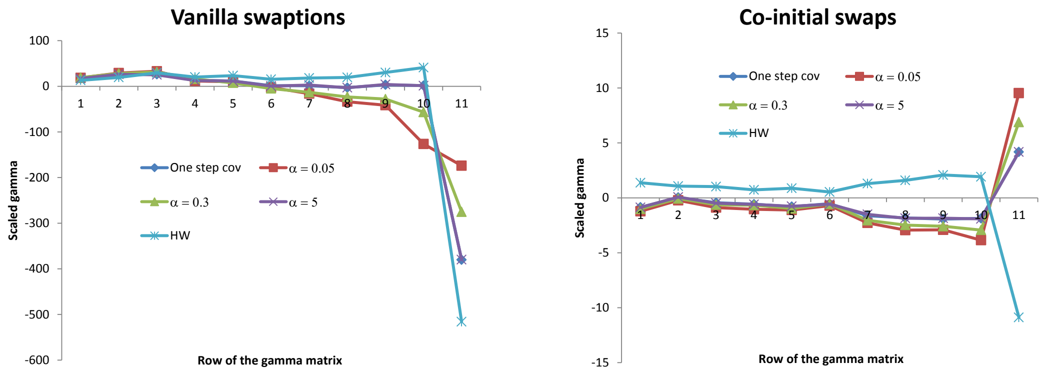

Figure 5.6: Scaled gamma contribution vectors of the vanilla swaptions and the co-initial swaps for the HW and the one step and weighted covariance portfolios.

6 Conclusions

This paper has developed a new framework for the choice of driving process for the one-dimensional SMF model. Our approach is motivated by the problem of pricing and hedging Bermudan swaptions. We retain the computational benefit of a single-factor model but attempt to incorporate the information from a multi-factor world. It turns out that the choice of driving process x has a strong impact on the hedging behaviour of the model. In terms of the existing choices, when we construct a vega-delta neutral portfolio for a Bermudan swaption the HW model gives a much lower total gamma in magnitude compared with the MR model. The reason was found to be the fundamental difference between their imposed parametrizations by time and by expiry which leads to the difference in their vega profiles. We analyzed this issue in detail and concluded that the former outperforms the later. The HW model, however, still lacks some flexibility in terms of the control over the parallel gamma vector of the vega-delta neutral portfolio. The main reason for this weakness was found to be the fact that the HW model only assigns the vega responses to the reverse diagonal of the swaption matrix.

In this paper, we introduce the one step and weighted covariance models which are different examples of parametrization by time. We observed that the new models give a very similar quality of hedge to the HW model in terms of the total gamma of vega-delta neutral portfolio. Additionally, they have an extra flexibility of the exponentially decaying weights that helps with the control over response to changes in the swaption matrix. For a certain choice of weights, the weighted covariance model spreads the vega responses over the swaption matrix and consequently reduces the magnitudes of all co-ordinates of the parallel gamma vectors of the vega-delta neutral portfolio. This is an advantage in practice.

We believe that the driving process x plays a fundamental role in evaluating any other product and application. Furthermore, the underlying parametrization that x imposes should be one of the first criteria to consider for practitioners. It is promising that a parametrization by time process can be used in a wider context.

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A Estimating the market implied covariance/correlation structure

In this appendix, we show how we estimate the covariances of the log of the co-terminal forward swap rates at their setting dates from the market. The estimation is carried out in two stages using a full rank multi-factor LIBOR Market model (LMM). We first describe how to approximate the correlations of the log-LIBORs at each exercise date by a global fit to the swaption matrix. With the knowledge of these correlations, we deduce the corresponding covariances of the log-LIBORs by performing a local fit to each row of the swaption matrix. The final stage of the approximation is to use the covariances of the log-LIBORs at each exercise date to determine the target quantities, $\mathrm{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$ for $i = 1, \ldots, n-1$ and $j > i$. This can be done by using the relationship between the swap rates and the LIBORs.

A.1 Approximating the terminal correlations, a global fit approach

We first introduce the n-factor LMM under the terminal measure $\mathbb{S}^{n+1}$. Suppose we are given n deterministic instantaneous volatility functions $\sigma_i(t), t \leq T_i$ for each $i = 1, \ldots, n$. We further introduce the instantaneous correlation $\rho_{i,j} \in [-1, 1]$ for each pair of Brownian factors W_i and W_j, i.e. $\mathrm{d}W_t^i \mathrm{d}W_t^j = \rho_{ij}\mathrm{d}t$. Under the terminal measure $\mathbb{S}^{n+1}$, the n-factor LMM reads

1. $i = n, t \leq T_n$:

$$\mathrm{d}L_t^n = \sigma_n(t) L_t^n \mathrm{d}W_t^n,$$

2. $i < n, t \le T_i$:

$$\mathrm{d}L_t^i = -\left(\sum_{j=i+1}^{n} \frac{\alpha_j L_t^j}{1+\alpha_j L_t^j}\sigma_i(t)\sigma_j(t)\rho_{ij}\right) L_t^i \mathrm{d}t + \sigma_i(t) L_t^i \mathrm{d}W_t^i. \tag{26}$$

The formal solution to the SDE (26) is

$$L_t^i = L_0^i \times \exp\left(\int_0^t \sum_{j=i+1}^{n} \frac{\alpha_j L_s^j}{1+\alpha_j L_s^j}\sigma_i(s)\sigma_j(s)\rho_{ij}\mathrm{d}s - \int_0^t \frac{\sigma_i^2(s)}{2}\mathrm{d}s + \int_0^t \sigma_i(t)\mathrm{d}W_t^i\right).$$

As the drift terms are stochastic, one usually seeks for a fast numerical scheme to approximate the solution. In the current literature, by "freezing the drift" at time zero we obtain a very crude drift approximation which could encourage quite a significant arbitrage

$$L_t^i \approx L_0^i \times \exp\left(\sum_{j=i+1}^{n} \frac{\alpha_j L_0^j}{1+\alpha_j L_0^j}\int_0^t \sigma_i(s)\sigma_j(s)\rho_{ij}\mathrm{d}s - \int_0^t \frac{\sigma_i^2(s)}{2}\mathrm{d}s + \int_0^t \sigma_i(t)\mathrm{d}W_t^i\right). \tag{27}$$

However, it is particularly useful when calibrating the model to the "terminal correlation" because it allows for an analytically closed form formula. Here, by "terminal correlation" we mean the correlation between the log-LIBORs at the same setting date, e.g. $\mathrm{Corr}(\ln L_{T_i}^k, \ln L_{T_i}^l)$ for $k, l \ge i$. Similar versions for other setting dates hold as well, see for example (Brigo & Mercurio, 2001) or (Rebonato, 2004). By (27), we get

$$\mathrm{Corr}(\ln L_{T_i}^k, \ln L_{T_i}^l) \approx \frac{\int_0^{T_i} \sigma_k(t)\sigma_l(t)\rho_{kl}\mathrm{d}t}{\sqrt{\int_0^{T_i} \sigma_k^2(t)\mathrm{d}t}\sqrt{\int_0^{T_i} \sigma_l^2(t)\mathrm{d}t}} \qquad k, l \ge i. \tag{28}$$

The above quantities can only be approximated given the parameters for the instantaneous volatilities and the instantaneous correlations.

Instantaneous correlation: we use the following simple and financially appealing form

$$\rho_{ij} = \rho_{ij}(\beta) = \exp(-\beta|T_i - T_j|),$$

where $\beta > 0$. For more details of this choice, readers are referred to (Rebonato, 2002, 2004) and (Brigo & Mercurio, 2001).

Instantaneous volatility: we choose the "humped shape" function, originally proposed by (Rebonato, 2002, 2004)

$$\sigma_i(t) = \sigma_i(t; a, b, c, d) = [a + b(T_i - t)]e^{-c(T_i - t)} + d,$$

where $a, b, c, d \in \mathbb{R}$ are the four parameters to be chosen appropriately.

A global fit: we now aim to recover all the parameters of the n-factor LMM model in order to approximate the terminal correlations in (28). This can be done by performing a global fit to the swaption matrix which contains implied volatilities of different expiries and tenors.

Note that each swap rate can be written as

$$
\begin{aligned}
y_t^{i,j} &= \sum_{k=i}^{i+j-1} w_k^{i,j}(t) L_t^k, \\
w_k^{i,j}(t) &= \frac{\alpha_k D_{tT_{k+1}}}{\sum_{l=i}^{i+j-1} \alpha_l D_{tT_{l+1}}},
\end{aligned}
$$

By Itô's lemma, under the terminal measure we have that

$$
\begin{aligned}
\mathrm{d}y_t^{i,j} &= \ldots \mathrm{d}t + \left(\sum_{k=i}^{i+j-1} \zeta_k^{i,j}(t) \sigma_k(t) \mathrm{d}W_t^k \right) y_t^{i,j}, \\
\zeta_k^{i,j}(t) &= \frac{\widetilde{w}_k^{i,j}(t) L_t^k}{y_t^{i,j}}, \\
\widetilde{w}_k^{i,j}(t) &= w_k^{i,j}(t) + \sum_{l=i,l\neq k}^{i+j-1} \frac{\partial w_l^{i,j}(t)}{\partial L_t^k} L_t^l.
\end{aligned} \tag{29}
$$

Hence, if the corresponding swaption is to be valued using the Black's formula with implied volatility $\widetilde{\sigma}_{i,j}$, we would want to get

$$
\widetilde{\sigma}_{i,j}^2 T_i = \sum_{k=i}^{i+j-1} \sum_{l=i}^{i+j-1} \int_0^{T_i} \zeta_k^{i,j}(t) \zeta_l^{i,j}(t) \sigma_k(t) \sigma_l(t) \rho_{kl} \mathrm{d}t,
$$

by ignoring the drift in 29 and assuming that the terms $\zeta_k^{i,j}(t)$ are all deterministic. Empirically, as shown in (Rebonato, 2002), (Rebonato, 2004) and (Brigo & Mercurio, 2001), one can actually obtain that each $\zeta_k^{i,j}(t)$ is approximately equal to its value today, i.e. $\zeta_k^{i,j}(0)$. The above equation now becomes of a much simpler form

$$
\widetilde{\sigma}_{i,j}^2 T_i \approx \sum_{k=i}^{i+j-1} \sum_{l=i}^{i+j-1} \zeta_k^{i,j}(0) \zeta_l^{i,j}(0) \int_0^{T_i} \sigma_k(t) \sigma_l(t) \rho_{kl} \mathrm{d}t. \tag{30}
$$

This allows us to a carry a fast yet accurate enough approximation scheme for both the global and the local fits as we shall see later.

We can now use a least squares fit method to do a global fit to the swaption matrix. For a particular choice of parameters $\{\beta, a, b, c, d\}$, we define the model volatilities $\{\sigma_{i,j}(\beta; a, b, c, d)\}$ to satisfy

$$
\sigma_{i,j}^2(\beta; a, b, c, d) T_i := \sum_{k=i}^{i+j-1} \sum_{l=i}^{i+j-1} \zeta_k^{i,j}(0) \zeta_l^{i,j}(0) \int_0^{T_i} \sigma_k(t; a, b, c, d) \sigma_l(t; a, b, c, d) \rho_{kl}(\beta) \mathrm{d}t.
$$

One then defines

$$
\chi^2 := \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} \left[\sigma_{i,j}(\beta; a, b, c, d) - \widetilde{\sigma}_{i,j} \right]^2,
$$

and looks for the optimal set of parameters $\{\beta, a, b, c, d\}$ that minimizes χ^2. At the end of this stage, we will have recovered all the parameters of the n-factor LMM and hence the terminal correlations in (28) can be estimated.

Remark 7 *Note that in this step, we do not take the value* $\int_0^{T_i} \sigma_k(t;a,b,c,d)\sigma_l(t;a,b,c,d)\rho_{kl}(\beta)\mathrm{d}t$ *as an approximation for* $Cov(\ln L_{T_i}^k, \ln L_{T_i}^l)$ *(the global approach does not reflect enough accuracy). The local step presented next will give a better approximation for these covariances.*

We assume the correlations between the log-LIBORs obtained from the global fit are not affected by any changes in the market data. This is because the changes in these correlations are recorded to be small and do not have a big impact on the approximations from the local fit in A.2. In our check, we find that the effect is numerically insignificant and this suggests that historical data can be used for the global fit. Hence, we keep the parameters of the instantaneous volatility and the instantaneous correlation functions the same at all time.

A.2 Approximating the covariances, a local fit approach

Recall that from the global fit we can approximate the correlations $\mathrm{Corr}(\ln L_{T_i}^k, \ln L_{T_i}^l)$ for $k, l \geq i$ at any exercise date T_i. In order to deduce the corresponding covariances of the log-LIBORs at T_i, we use the implied volatilities on the i^{th} row of the swaption matrix. We employ the approximation in (30) but use $\mathrm{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^l)$ instead of $\int_0^{T_i} \sigma_k(t)\sigma_l(t)\rho_{kl}\mathrm{d}t$, i.e.

$$\widetilde{\sigma}_{i,j}^2 T_i \approx \sum_{k=i}^{i+j-1} \sum_{l=i}^{i+j-1} \zeta_k^{i,j}(0)\zeta_l^{i,j}(0)\mathrm{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^l). \tag{31}$$

The following approximation steps are described to solve for $\mathrm{Var}(\ln L_{T_i}^j)$ where j runs from i to n. Since we keep all the correlations fixed, solving for the variances automatically implies the covariances. Note that we ignore the effect of changing the measure as it is small and irrelevant for the discussion.

- **Step 1**: we start from $\widetilde{\sigma}_{i,1}$ of the i^{th} caplet. Black's formula implies that $\widetilde{\sigma}_{i,1}^2 T_i \approx \mathbf{Var}(\ln \mathbf{L_{T_i}^i})$.

- **Step 2**: next, we consider $\widetilde{\sigma}_{i,2}$. Also, by Black's formula and the approximation in (31)

$$\begin{aligned}
\widetilde{\sigma}_{i,2}^2 T_i &\approx (\zeta_i^{i,2}(0))^2 \mathrm{Var}(\ln L_{T_i}^i) \\
&\quad + (\zeta_{i+1}^{i,2}(0))^2 \mathbf{Var}(\ln \mathbf{L_{T_i}^{i+1}}) + 2\zeta_i^{i,2}(0)\zeta_{i+1}^{i,2}(0)\mathrm{Cov}(\ln L_{T_i}^i, \ln L_{T_i}^{i+1}) \\
&= (\zeta_i^{i,2}(0))^2 \mathrm{Var}(\ln L_{T_i}^i) + (\zeta_{i+1}^{i,2}(0))^2 \mathbf{Var}(\ln \mathbf{L_{T_i}^{i+1}}) \\
&\quad + 2\zeta_i^{i,2}(0)\zeta_{i+1}^{i,2}(0)\mathrm{Corr}(\ln L_{T_i}^i, \ln L_{T_i}^{i+1})\sqrt{\mathrm{Var}(\ln L_{T_i}^i)}\sqrt{\mathbf{Var}(\ln \mathbf{L_{T_i}^{i+1}})}
\end{aligned} \tag{32}$$

  It is straightforward to solve this equation for the unknown $\mathbf{Var}(\ln \mathbf{L_{T_i}^{i+1}})$. Hence, $\mathrm{Cov}(\ln L_{T_i}^i, \ln L_{T_i}^{i+1})$ can be recovered.

- **Step j+1-i**: notice that each time we move from $\widetilde{\sigma}_{i,j-i}$ to $\widetilde{\sigma}_{i,j+1-i}$, there is one more unknown to solve, i.e. $\mathbf{Var}(\ln \mathbf{L_{T_i}^j})$. With the knowledge of the terminal correlation, we can recover

$\text{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^j)$ for $k = 1, \ldots, j-1$. This is clear by the following relation

$$
\begin{aligned}
\widetilde{\sigma}_{i,j+1-i}^2 T_i &\approx \sum_{k=i}^{j}\sum_{l=i}^{j} \zeta_k^{i,j+1-i}(0)\zeta_l^{i,j+1-i}(0)\text{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^l) \\
&= \cdots + (\zeta_j^{i,j+1-i}(0))^2 \mathbf{Var}(\ln \mathbf{L}_{\mathbf{T_i}}^{\mathbf{j}}) \qquad (33)\\
&\quad +2\sum_{k=i}^{j-1} \zeta_k^{i,j+1-i}(0)\zeta_j^{i,j+1-i}(0)\text{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^j) \\
&= \cdots + (\zeta_j^{i,j+1-i}(0))^2 \mathbf{Var}(\ln \mathbf{L}_{\mathbf{T_i}}^{\mathbf{j}}) \qquad (34)\\
&\quad +2\sum_{k=i}^{j-1} \zeta_k^{i,j+1-i}(0)\zeta_j^{i,j+1-i}(0)\text{Corr}(\ln L_{T_i}^k, \ln L_{T_i}^j)\sqrt{\text{Var}(\ln L_{T_i}^k)}\sqrt{\mathbf{Var}(\ln \mathbf{L}_{\mathbf{T_i}}^{\mathbf{j}})}
\end{aligned}
$$

where the $\ldots$ terms and $\text{Var}(\ln L_{T_i}^k)$, for $k = 1, \ldots, j$ are known from the previous steps.

- **Step n+1-i**: At the end of this step, we will have recovered the variances and covariances of all the alive log-LIBORs at T_i.

For $j > i$, by using the approximation $\zeta_k^{i,j}(t) \approx \zeta_k^{i,j}(0)$ in (29) we have that

$$
\mathrm{d}\ln y_t^{i,j} \approx \ldots \mathrm{d}t + \left(\sum_{k=i}^{i+j-1} \zeta_k^{i,j}(0)\sigma_k(t)\mathrm{d}W_t^k\right). \qquad (35)
$$

Hence, our target market quantity can be written as

$$
\text{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j}) \approx \text{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{j,n+1-j}), \qquad j > i
$$

which follows from the independence of increment in (35). Again, since we use $\text{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^l)$ instead of $\int_0^{T_i} \sigma_k(t)\sigma_l(t)\rho_{kl}\mathrm{d}t$, it follows from (35) that

$$
\text{Cov}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_i}^{j,n+1-j}) \approx \sum_{k=i}^{n}\sum_{l=j}^{n} \zeta_k^{i,n+1-i}(0)\zeta_l^{j,n+1-j}(0)\text{Cov}(\ln L_{T_i}^k, \ln L_{T_i}^l). \qquad (36)
$$

When setting the various models, we will denote the covariances of the log of the swap rates estimated from the market by this two step procedure as $\text{Cov}^{\text{ma}}(\ln y_{T_i}^{i,n+1-i}, \ln y_{T_j}^{j,n+1-j})$.