Representations of the First Hitting Time Density of an Ornstein-Uhlenbeck Process

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Three expressions are provided for the first hitting time density of an Ornstein-Uhlenbeck process to a fixed level. The first one hinges on an eigenvalue expansion involving zeros of the parabolic cylinder functions. The second one is an integral representation involving some special functions whereas the third is given in terms of a functional of a 3-dimensional Bessel bridge. These expressions are used to compute numerically the density in some examples.

1. Introduction

In this paper, we gather different expressions for the density function of the first hitting time to a fixed level by an Ornstein-Uhlenbeck process, abbreviated as OU-process. This density is used in different areas of mathematical finance. Indeed, it is connected to some pricing formulas of interest rate path dependent options when the dynamics of the underlying asset is assumed to be a mean reverting OU-process. For this, we refer to Leblanc & Scaillet [16] and the references therein. The knowledge of the sought density is also relevant in credit risk modelling, see e.g. Jeanblanc & Rutkowski [10]. It is also required in other fields of applied mathematics. For instance in biology, see Smith [22], the hitting time is used for modelling the time between firings of a nerve cell.

Recently, Leblanc & Scaillet [16] and Leblanc, Renault & Scaillet [15] showed that the density can be expressed as the Laplace transform of a functional of a 3-dimensional Bessel bridge. However, the authors used therein an erroneous spatial homogeneity property for the 3-dimensional Bessel bridge, a mistake that has been noticed by several authors, including Göing-Jaeschke & Yor [6]. The feature of this representation is of probabilistic nature and the details are given in Section 5. We provide two other explicit expressions obtained by different techniques which feature is of analytic nature. The first expression is a series expansion involving the eigenvalues of a Sturm-Liouville boundary value problem associated with the Laplace transform of the first hitting time, see Keilson & Ross [11]. An analytic continuation argument is used to compute the Fourier transform of the first hitting time which gives an integral representation of the density. As discussed above – in specific contexts in mathematical finance – there is a need to perform numerical computations of the density. The three representations suggest new methods to approximate the density. We point out that the OU-process is used here as a case study. Indeed, it is possible to adapt readily the methodologies described below for a large class of one dimensional diffusions.

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The paper is organised as follows. In the next section the OU-process is reviewed and basic properties of the first hitting time are presented. In Section 3, 4 and 5 the series, the integral, and the Bessel bridge representations of the density are derived respectively. Section 6 is devoted to numerical computations of the density. Finally, for completeness, some properties of Hermite and parabolic cylinder functions are recalled in Section 7.

2. Preliminaries on Ornstein-Uhlenbeck processes

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. The associated OU-process $(U_t)_{t\geq 0}$, with parameter $\lambda \in \mathbb{R}$, is defined to be the unique solution to the stochastic differential equation

$$dU_t = dB_t - \lambda U_t dt, U_0 = x \in \mathbb{R}. \tag{2.1}$$

Denote by $\mathbf{P}_x^{(\lambda)}$ the law of (U_t) when $U_0 = x \in \mathbb{R}$ and hence $\mathbf{P}_x^{(0)} := \mathbf{P}_x$ is the law of (B_t) started at x. Thanks to Girsanov theorem the absolute-continuity relationship

$$d\mathbf{P}^{(\lambda)}_{x|\mathcal{F}_t} = e^{-\frac{\lambda}{2}\left(B_t^2 - x^2 - t\right) - \frac{\lambda^2}{2}\int_0^t B_s^2 ds} d\mathbf{P}_{x|\mathcal{F}_t} \tag{2.2}$$

holds for any $t > 0$. Moreover, the process $(U_t)_{t\geq 0}$ is a strong Markov process with infinitesimal generator, denoted by $\mathcal{G}$, given on $\mathcal{C}_b(\mathbb{R})$ by

$$\mathcal{G}f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) - \lambda x \frac{\partial f}{\partial x}(x) \ , x \in \mathbb{R}. \tag{2.3}$$

The linear stochastic differential (2.1), when integrated, yields the realization

$$U_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s} dB_s\right)$$

for $t \geq 0$. By the Dambis, Dubins-Schwartz theorem, see Revuz & Yor [19, p.181], there is a Brownian motion (W_t) such that $\int_0^t e^{\lambda s} dB_s = W_{\tau(t)}$, for any $t \geq 0$, where $\tau(t) = (2\lambda)^{-1}(e^{2\lambda t} - 1)$. Hence, the representation $U_t = e^{-\lambda t}\big(x + W_{\tau(t)}\big)$ holds and this is known as Doob's transform. For a fixed real a, introduce $\sigma_a = \inf\{s > 0 : U_s = a\}$. Its law is absolutely continuous with respect to the Lebesgue measure. We set $\mathbf{P}_x^{(\lambda)}(\sigma_a \in dt) = p^{(\lambda)}_{x\to a}(t)dt$, $t > 0$, and assume that $\lambda > 0$ so that (U_t) is positively recurrent and, therefore σ_a is finite. We focus here on the situation when (U_t) starts below the hitting barrier and notice that we recover the other case by replacing a and x with $-a$ and $-x$ in the density (since $(-U_t)$ is also an OU-process). For the Laplace transform $u_a^\alpha(x) := \mathbf{E}_x^{(\lambda)}[e^{-\alpha\sigma_a}]$, $\alpha > 0$, we recall the following well-known result, see Siegert [21] or Breiman [3].

Proposition 2.1. *For $x < a$, the Laplace transform of σ_a is given by*

$$u_a^\alpha(x) = \frac{H_{-\alpha/\lambda}(-x\sqrt{\lambda})}{H_{-\alpha/\lambda}(-a\sqrt{\lambda})} = \frac{e^{\lambda x^2/2} D_{-\alpha/\lambda}(-x\sqrt{2\lambda})}{e^{\lambda a^2/2} D_{-\alpha/\lambda}(-a\sqrt{2\lambda})} \tag{2.4}$$

where $H_\nu(\cdot)$ and $D_\nu(\cdot)$ are the Hermite and parabolic cylinder functions, respectively.

Proof. Thanks to the general theory of linear diffusion, we refer to Itô & McKean [9], $u_a^\alpha(\cdot)$ is the unique solution of the following Sturm-Liouville boundary value problem

$$\begin{aligned} &\mathcal{G}u(x) = \alpha u(x) \text{ for } x < a \\ &u(x)_{|x=a} = 1 \text{ and } \lim_{x\to-\infty} u(x) = 0. \end{aligned} \tag{2.5}$$

This is a singular boundary value problem since the interval is not bounded. We refer to [9, p.130] where it is shown that the solution to the above problem takes the form $u_a^\alpha(x) = \Psi_\alpha(x)/\Psi_\alpha(a)$, where $\Psi_\alpha(\cdot)$ is, up to some multiplicative constant, the unique increasing positive solution of the equation $\mathcal{G}\Psi = \alpha\Psi$. By the definition of Hermite functions, see Section 7, we easily see that $\Psi_\alpha(x) = H_{-\alpha/\lambda}(x\sqrt{\lambda})$ leading to (2.4) as required. □

Remark 2.2. By the scaling property of (B_t) or Proposition 2.1 we see that $\mathbf{E}_x^{(\lambda)}\left[e^{-\alpha\sigma_a}\right] = \mathbf{E}_{x\sqrt{\lambda}}^{(1)}\left[e^{-\alpha/\lambda\sigma_{a\sqrt{\lambda}}}\right]$. Hence

$$p_{x\to a}^{(\lambda)}(t) = \lambda p_{x\sqrt{\lambda}\to a\sqrt{\lambda}}^{(1)}(\lambda t) \tag{2.6}$$

for $t > 0$, and therefore the study of $p_{x\to a}^{(\lambda)}(\cdot)$ may be reduced to the case $\lambda = 1$.

Remark 2.3. For the special case $a = 0$ there is a simple expression for $p_{x\to 0}^{(\lambda)}(\cdot)$. Indeed, we shall first recall that, for the Brownian motion recovered by letting $\lambda \to 0$, we have

$$p_{x\to a}^{(0)}(t) = \frac{|a-x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-x)^2}{2t}\right), \tag{2.7}$$

for $t > 0$. Now, with $\hat{\sigma}_a = \inf\{s \geq 0 : W_s + x = a\sqrt{1+2\lambda s}\}$, Doob's transform implies the identity $\hat{\sigma}_a = \tau(\sigma_a)$ a.s. as was noticed by Breiman [3]. We deduce that $p_{x\to 0}^{(\lambda)}(t) = \tau'(t)p_{x\to 0}^{(0)}(\tau(t))$. Thus

$$p_{x\to 0}^{(\lambda)}(t) = \frac{|x|}{\sqrt{2\pi}} \left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2} \exp\left(-\frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}\right) \tag{2.8}$$

for $t > 0$, which is also found in Pitman & Yor [18].

We close this section with a couple of remarks concerning extension of the results to the cases of transient OU-processes and mean reverting OU-processes.

Remark 2.4. Recall that if λ is negative then the process (U_t) is transient. However, from (2.2) and the chain rule, we deduce that

$$d\mathbf{P}_{x|\mathcal{F}_t}^{(\lambda)} = \exp\left(\lambda(U_t^2 - x^2 - t)\right) d\mathbf{P}_{x|\mathcal{F}_t}^{(-\lambda)}$$

for $t > 0$, as found in [2]. This combined with the optional stopping theorem yields

$$p_{x\to a}^{(\lambda)}(t) = \exp\left(\lambda(a^2 - x^2 - t)\right) p_{x\to a}^{(-\lambda)}(t)$$

for any $t > 0$.

Remark 2.5. If we replace $(B_t)_{t\geq 0}$ by $(B_t + \mu t)_{t\geq 0}$ in (2.1), for some real μ, then the resulting process is a mean reverting one for which a realization is given by

$$U_t^{(\mu)} = \frac{\mu}{\lambda} + e^{-\lambda t}\left(x - \frac{\mu}{\lambda} + \int_0^t e^{\lambda s} dB_s\right)$$

for $t > 0$. The corresponding first hitting time density, denoted by $p_{x\to a}^{(\lambda,\mu)}(\cdot)$, is easily seen to be related to that with $\mu = 0$ via the formula $p_{x\to a}^{(\lambda,\mu)}(t) = p_{x-\frac{\mu}{\lambda}\to a-\frac{\mu}{\lambda}}^{(\lambda)}(t)$ for any $t > 0$.

3. The series representation

This section is devoted to inverting the Laplace transform of the distribution of σ_a by means of the Cauchy Residue Theorem. Let $D_\nu(\cdot)$ be the parabolic cylinder function with index $\nu \in \mathbb{R}$, see Section 7. For b given and fixed, denote by $(\nu_{j,b})_{j\geq 1}$ the ordered sequence of positive zeros of $\nu \mapsto D_\nu(b)$. We are ready to state the following result.

Theorem 3.1. *For any couple of real numbers $x < a$ we have the series expansion*

$$p^{(\lambda)}_{x\to a}(t) = -\lambda e^{\lambda(x^2-a^2)/2} \sum_{j=1}^{\infty} \frac{D_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})}{D'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})} e^{-\lambda \nu_{j,-a\sqrt{2\lambda}} t} \tag{3.1}$$

where $D'_{\nu_{j,b}}(b) = \frac{\partial D_\nu(b)}{\partial \nu}|_{\nu=\nu_{j,b}}$. For any $t_* > 0$, the series converges uniformly for $t > t_*$.

Proof. The substitution $v(x) = e^{-x^2/4} u(x/\sqrt{2\lambda})$ transforms (2.5) into the Weber equation $v'' - \big(\frac{\alpha}{\lambda} - \frac{1}{2} + q(x)\big)v = 0$ where $q(x) = x^2/4$. A fundamental solution of the latter equation is given by $x \mapsto D_{-\alpha/\lambda}(-x)$. Since $x \mapsto q(x)$ is real-valued, continuous and $q(x) \to \infty$ as $x \to \infty$, the Weber operator has a pure point spectrum, we refer to Hille [8, Theorem 10.3.4]. Moreover, the eigenvalues are simple, positive and bounded from below, see Ricciardi & Sato [20] for more details about the distribution of the spectrum. As a consequence, the Laplace transform (2.4) is meromorphic as a function of the parameter α, whose poles are negative simple and are given by the sequence $\{\alpha_j = -\lambda \nu_{j,-a\sqrt{2\lambda}}\}_{j=1,\ldots}$. The residue of $\alpha \mapsto u^\alpha_a(x)$ at $\alpha = \alpha_j, j \geq 1$ is easily found to equal

$$\mathrm{Res}_{\alpha=\alpha_j}\, u^\alpha_a(x) = -\lambda e^{\lambda(x^2-a^2)/2} D_{-\alpha_j/\lambda}(-x\sqrt{2\lambda}) / D'_{-\alpha_j/\lambda}(-a\sqrt{2\lambda}).$$

To check that the conditions of [7, Theorem 10.7c] are satisfied, we make use of the asymptotic properties of parabolic cylinder functions recalled in Section 7. The Heaviside expansion theorem in [7] gives the expression of the density where the parameters are given by the eigenvalues of the associated Sturm-Liouville expansion. The uniform convergence of the series on $[t_*, \infty)$, for any $t_* > 0$, follows from the asymptotic formulas (7.7) and (7.8). $\square$

The following local limit result is essentially due to the fact that the series of formula (3.1) is uniformly convergent.

Corollary 3.2. *Let the situation be as in Theorem 3.1, then*

$$\lim_{T\to\infty} e^{\lambda \nu_{1,-a\sqrt{2\lambda}} T} \mathbf{P}^{(\lambda)}_x (\sigma_a > T) = \frac{e^{\lambda(x^2-a^2)/2}}{\nu_{1,-a\sqrt{2\lambda}}} \frac{D_{\nu_{1,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})}{D'_{\nu_{1,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})}.$$

Remark 3.3. The above representation appeared without a rigorous justification in many references. For instance, we found it in Keilson & Ross [11] and Ricciardi & Sato [20] where the authors study the zeros of the Hermite functions. A similar expression is given by Novikov, Frishling & Kordzakhia [4] for the density of the first passage time of the Brownian motion to the square root boundary, connected to the distribution we are focusing on by Doob's transform.

Remark 3.4. The distribution of σ_a is infinitely divisible and may be viewed as an infinite convolution of elementary mixtures of exponential distributions. In [13], Kent establishes a link between the canonical measure of the first hitting time of a fixed level by a one dimensional diffusion and the spectral measure of its infinitesimal generator. When the left end point of the diffusion is not natural, the same author gives the series expansion based on the spectral decomposition, see [12]. However, in our case, the left end point is natural therefore such methodology cannot be applied directly.

4. The integral representation

In this section we compute the cosine transform of the distribution of σ_a. Then the density $p^{(\lambda)}_{x\to a}(\cdot)$ is computed out from the cosine transform and its inverse defined on $L^1(\mathbb{R}^+)$ via

$$p^{(\lambda)}_{x\to a}(t) = \frac{2}{\pi}\int_0^\infty \cos(\alpha t)\mathbf{E}^{(\lambda)}_x\big[\cos(\alpha\sigma_a)\big]d\alpha,\ t>0.$$

Theorem 4.1. *Fix $x<a$ and $t>0$ we have*

$$p^{(\lambda)}_{x\to a}(t) = \frac{2\lambda}{\pi}\int_0^\infty \cos(\alpha\lambda t)\frac{Hr_{-\alpha}(-\sqrt{\lambda}a)Hr_{-\alpha}(-\sqrt{\lambda}x)+Hi_{-\alpha}(-\sqrt{\lambda}x)Hi_{-\alpha}(-\sqrt{\lambda}a)}{Hr^2_{-\alpha}(-\sqrt{\lambda}a)+Hi^2_{-\alpha}(-\sqrt{\lambda}a)}d\alpha \tag{4.1}$$

where $Hr_{-\alpha}(\cdot)$ and $Hi_{-\alpha}(\cdot)$ are specified by formulae (7.4) and (7.5) respectively.

Proof. We shall only treat the case $\lambda=1$ which can be completed by using (2.6). The Laplace transform $\nu\mapsto \mathbf{E}^{(1)}_x[e^{-\nu\sigma_a}]$ is analytic on the domain $\{\nu\in\mathbb{C}|\operatorname{Re}(\nu)\geq 0\}$. Recall that $\nu_{1,-a\sqrt{2}}$ denotes the smallest positive zero of the function $\nu\mapsto D_\nu(-a\sqrt{2})$ and that the ratio of parabolic cylinder functions is analytic on $\{\nu\in\mathbb{C}|\operatorname{Re}(\nu)>-\nu_{1,-a\sqrt{2}}\}$. By analytical continuation, the Laplace transform is also analytical on $\{\nu\in\mathbb{C}|\operatorname{Re}(\nu)>-\nu_{1,-a\sqrt{2}}\}$ and hence $\mathbf{E}^{(1)}_x[e^{-\nu\sigma_a}] = H_{-\nu}(-x)/H_{-\nu}(-a)$ on $\{\nu\in\mathbb{C}|\operatorname{Re}(\nu)\geq 0\}$ due to Proposition 2.1. We have that

$$\begin{aligned}\mathbf{E}^{(1)}_x\big[\cos(\alpha\sigma_a)\big] &= \operatorname{Re}\left(\frac{H_{i\alpha}(-x)}{H_{i\alpha}(-a)}\right)\\ &= \frac{Hr_{-\alpha}(-a)Hr_{-\alpha}(-x)+Hi_{-\alpha}(-x)Hi_{-\alpha}(-a)}{Hr^2_{-\alpha}(-a)+Hi^2_{-\alpha}(-a)}.\end{aligned}$$

The statement follows from the injectivity of the cosine transform. □

5. The Bessel bridge representation

As mentioned in the introduction computing explicitly $p^{(\lambda)}_{x\to a}(t)$ amounts to characterising the distribution of a quadratic functional of a 3-dimensional Bessel bridge denoted by $(R_t)_{t\geq 0}$. In order to explain the connection let us recall that, informally, the process $(R_s)_{s\leq t}$ conditionally on $R_0=x$ and $R_t=y$, denoted by $(r_s)_{s\leq t}$, is the so-called 3-dimensional Bessel bridge over the interval $[0,t]$ between x and y. Formally, (r_s) is seen to be the unique strong solution of the stochastic differential equation

$$dr_s = \left(\frac{y-r_s}{t-s}+\frac{1}{r_s}\right)ds + dB_s,\ r_0=x,\ s<t,$$

which is its decomposition as a semi-martingale in its own filtration. Now, we are ready to quote the following result from [6] and provide its detailed proof.

Theorem 5.1. *For $x<a$ and $t>0$ we have*

$$p^{(\lambda)}_{x\to a}(t) = e^{-\lambda(a^2-x^2-t)/2}p^{(0)}_{x\to a}(t)\mathbf{E}_{0\to a-x}\left[\exp\left(-\frac{\lambda^2}{2}\int_0^t (r_s-a)^2ds\right)\right] \tag{5.1}$$

where (r_s) is a 3-dimensional Bessel bridge over the interval $[0,t]$ between 0 and $a-x$ and $\mathbf{P}_{0\to a-x}$ stands for its law. The density $p^{(0)}_{x\to a}(\cdot)$ is given in (2.7).

Proof. Relation (2.2) combined with Doob's optional stopping theorem yields $p^{(\lambda)}_{x\to a}(t) = e^{-\lambda(a^2-x^2-t)/2}p^{(0)}_{x\to a}(t)b(t)$ where we set

$$b(t) = \mathbf{E}_x\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t B_s^2 ds\Big)\Big|\sigma_a^{(0)} = t\Big]$$

and $\sigma_a^{(0)} = \inf\{s > 0 : B_s = a\}$. Next, we need to recall Williams' time reversal formula for which we set $L_a = \sup\{s \geq 0 : R_s = a\}$. Corollary 4.4, in [19, p.498], implies that the processes $(y - B_{\sigma_y^{(0)}-u})_{u\leq\sigma_y^{(0)}}$ and $(R_u)_{u\leq L_y}$ are equivalent. It remains to use successively the spatial homogeneity, the symmetry of (B_t), Williams' time reversal identity and the transience property of (R_t) in order to write

$$\begin{aligned} b(t) &= \mathbf{E}_{a-x}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t (B_s-a)^2 ds\Big)\Big|\sigma_0^{(0)} = t\Big] \\ &= \mathbf{E}_0\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t (R_s-a)^2 ds\Big)\Big|L_{a-x} = t\Big] \\ &= \mathbf{E}_0\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^t (R_s-a)^2 ds\Big)\Big|R_t = a-x\Big]. \end{aligned}$$

This completes the proof. □

6. Numerical Illustrations

Two standard techniques for approximating the density of the first hitting time of diffusions commonly used, are the following: the numerical approach to the solution of the partial differential equation associated to the density (analytic method) and direct Monte Carlo simulation (probabilistic method). The three representations of the hitting time density of OU-process suggest alternative ways to approximate the density. Below, we provide a short description of the approximation and illustrate them with two examples.

6.1. The first approximation is to use the series expansion (3.1). The infinite series is truncated after the first N terms, that is,

$$f_S(t) = -\lambda e^{\lambda(x^2-a^2)/2}\sum_{j=1}^N b_j \exp\big(-\lambda a_j t\big),\ t > 0,$$

where $a_j = \nu_{j,-a\sqrt{2\lambda}}$ and $b_j = D_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})/D'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})$. For t small, $f_S(t)$ is negative or decreasing. Let t_0 be the point where $f_S(t_0) = 0$ or $f'_S(t_0) = 0$. Hence, the approximation of $p^{(\lambda)}_{x\to a}(\cdot)$ is given by $f_S(t)$ for $t \geq t_0$ and 0 for $0 < t < t_0$. The parabolic cylinder function $D_\nu(x)$ can be estimated by the series expansion given by (7.6) and (7.1). From this, numerical values of $\nu_{j,-a\sqrt{2\lambda}}$, $D_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})$ and $D'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})$ can be estimated where the last term is computed by the differential quotient. A problem is to choose suitable N for a prescribed truncation error. Since we are approximating a density, there are many ways to measure the quality of the chosen truncation parameter N. We give an average error $\bar{e}$ based on large-n asymptotics that is independent of the argument t and is easy to compute. Integrating the absolute value of the N^{th} term of the series and using the asymptotic formulas (7.7) and (7.8) yield

$$\int_0^\infty |\lambda e^{\lambda(x^2-a^2)/2} b_N \exp\big(-\lambda a_N t\big)|dt = e^{\lambda(x^2-a^2)/2}|b_N|/a_N \sim \pi^{-1}e^{\lambda(x^2-a^2)/2}N^{-1}.$$

The average error is defined to by $\bar{e} = \pi^{-1}e^{\lambda(x^2-a^2)/2}N^{-1}$. When $\nu_{j,-a\sqrt{2\lambda}}$, $j = 1,\dots,N$, are estimated, it is easy to numerically compute the expectation of a bounded function of the hitting time (e.g. prices of interest rate options presented in [16]) using the approximation. Then $\bar{e}$ gives a measure how precise the expectation is estimated. In the examples below we chose $N = 100$ and in Example 1 $\bar{e}$ is equal to 0.005.

6.2. It is not a good method to approximate the integral in (4.1) by the corresponding Riemann sum. Instead, we make use of the trapezoidal rule. The approximation using the integral representation is then given by

$$f_I(t) = \frac{e^{A/2}}{2t}\frac{H_{-A/(2\lambda t)}(-x\sqrt{\lambda})}{H_{-A/(2\lambda t)}(-a\sqrt{\lambda})} + \frac{e^{A/2}}{t}\sum_{k=1}^{N}(-1)^k \operatorname{Re}\left(\frac{H_{-A/(2\lambda t)-k\pi i/(\lambda t)}(-x\sqrt{\lambda})}{H_{-A/(2\lambda t)-k\pi i/(\lambda t)}(-a\sqrt{\lambda})}\right)$$

where $A > 0$. It follows from the computations in Section 4 that the Laplace transform is given by $\mathbf{E}_x^{(\lambda)}[e^{-\nu\sigma_a}] = H_{-\nu/\lambda}(-x\sqrt{\lambda})/H_{-\nu/\lambda}(-a\sqrt{\lambda})$. Also, for this approach, the question remains about a good choice for A and N. The numerical computations of the integral leads to the discretisation error and the truncation error (both depends on the argument t). A bound for the discretisation error is Ce^{-A} where C is constant that dominates the density. In the examples below $A = 18.1$ so the discretisation error is of order (10^{-7}). There is no simple bound for the truncation error. One can choose N when the value of the last term is small. We set $N = 500$ in the examples which is a conservative choice. In practice, one can determine A and N based on trial and error. We refer tp Abate & Whitt [1], for precise statements and more details on this approximation method.

6.3. For the Bessel bridge approach, it is needed to resort to some simulation techniques to compute the functional of the 3-dimensional Bessel bridges in the expression (5.1). With the notation $\mathbf{E}\big[G\big(\int_0^t h(r_s)ds\big)\big]$ where $G(x) = e^{-x}$ is a bounded function and $h(x) = \lambda^2(x-a)^2/2$ is a regular function, the three steps to follow are:

(1) First, we compute the integral by considering the corresponding Riemann sum, that is, $\mathbf{E}\big[G\big(\int_0^t h(r_s)\,ds\big)\big] \simeq \mathbf{E}\big[G\big(\sum_{k=1}^n h(r_{kt/n})\big)\big]$.
(2) We approach (r_t) with another process $(\bar{r}_t)$ by means of the Euler scheme which gives $\mathbf{E}\big[G\big(\sum_{k=1}^n h(r_{kt/n})\big)\big] \simeq \mathbf{E}\big[G\big(\sum_{k=1}^n h(\bar{r}_{kt/n})\big)\big]$. The same step of discretisation is chosen for the Euler scheme and the Riemann sum.
(3) Finally, to estimate the expectation we use Monte Carlo method by simulating a large number M of independent paths of the process $(\bar{r}_t)$ and hence $\mathbf{E}\big[G\big(\sum_{k=1}^n h(\bar{r}_{kt/n})\big)\big] \simeq \frac{1}{M}\sum_{i=1}^M G\big(\sum_{k=1}^n h(\bar{r}^{(i)}_{kt/n})\big)$.

Putting these steps together, the approximation formula for (5.1) is given by

$$f_B(t) = e^{-\lambda(a^2-x^2-t)/2}\frac{|a-x|}{\sqrt{2\pi t^3}}\exp\left(-\frac{(a-x)^2}{2t}\right)\frac{1}{M}\sum_{i=1}^{M} G\left(\sum_{k=1}^{n} h(\bar{r}^{(i)}_{kt/n})\right).$$

6.4. The two first approaches are analytic methods and very easy to implement using programs like Maple or Mathematica,where it is possible to use built-in functions. However, these require the knowledge of the Laplace transform of the first hitting time which can be computed only for some specific continuous Markov processes. The Bessel bridge approach is a probabilistic method. Its main advantage compared to the direct Monte Carlo one is that it overcomes the problem of detecting the time at which the approximated process crosses the boundary. We refer to [5] for an explanation of the difficulties encountered with the direct Monte Carlo method. We also emphasise that this algorithm estimates directly the density whereas the direct Monte Carlo provides an approximation of the distribution function. This method can readily be used to treat similar problems for continuous Markov processes which laws are absolutely continuous with respect to the Wiener measure.

6.5. In order to test the performance of the three methodologies, we carried out two numerical examples. In both examples we have used the following approximation parameters. For the series we used $N = 100$ of the series expansion of $f_S(\cdot)$. For the integral representation we have chosen $A = 18.1$ and took $N = 500$ terms in the series of $f_I(\cdot)$. In the series of $f_B(\cdot)$ for the Bessel bridge method, we have simulated $M = 10^5$ paths of the Bessel bridge with $n = 1000$ time steps on the interval $[0, 4]$. In both examples the parameter of the OU-process is $\lambda = 1$, which is sufficient by (2.6).

Example 1: We examine the example $a = 0$, which is the only case where the density is known in closed form, indeed given by (2.8). The OU-process is starting from $x = -1$. The numerical approximations of the density $p^{(1)}_{(-1)\to 0}(t)$ are collected in Table 1. The table shows that the analytical approaches are accurate up to 10^{-5} digits whereas the simulation approach is accurate up to 10^{-3} digits. Note that for the series method $t_0 = 0.044$ and hence for $t = 0.04$ the approximated value for the density is set to be 0 as described in Subsection 6.1. In fact, $f_S(0.04) = -0.0019$.

t	0.04	0.08	0.1	0.25	0.5	0.75
Explicit	0.000310	0.057549	0.144538	0.762172	0.760954	0.584084
Series	0	0.057540	0.144538	0.762172	0.760954	0.584084
Integral	0.000310	0.057540	0.144538	0.762172	0.760954	0.584084
Bessel bridge	0.000310	0.057538	0.144534	0.762074	0.760946	0.584362
t	1	1.5	2	2.5	3	4
Explicit	0.441483	0.257945	0.154101	0.092934	0.056248	0.020670
Series	0.441483	0.257945	0.154101	0.092934	0.056248	0.020670
Integral	0.441483	0.257945	0.154122	0.092612	0.055968	0.020670
Bessel bridge	0.441648	0.258012	0.154107	0.092841	0.056203	0.020596

Table 1. Different values of the density $p^{(1)}_{(-1)\to 0}(t)$, that is, the parameters are $a = 0$, $x = -1$ and $\lambda = 1$.

Example 2: In this example the OU-process starts from $x = 0$. We computed the density $p^{(1)}_{0\to a}(t)$ for a equals 0.50, 0.75 and 1.00. In Figure 1, the results of the three densities are presented. In this example there is no check of the numerical values since there is no closed form formulas. But from Figure 1, one see that three methods give numerical values that are very close and can hardly be distinguished.

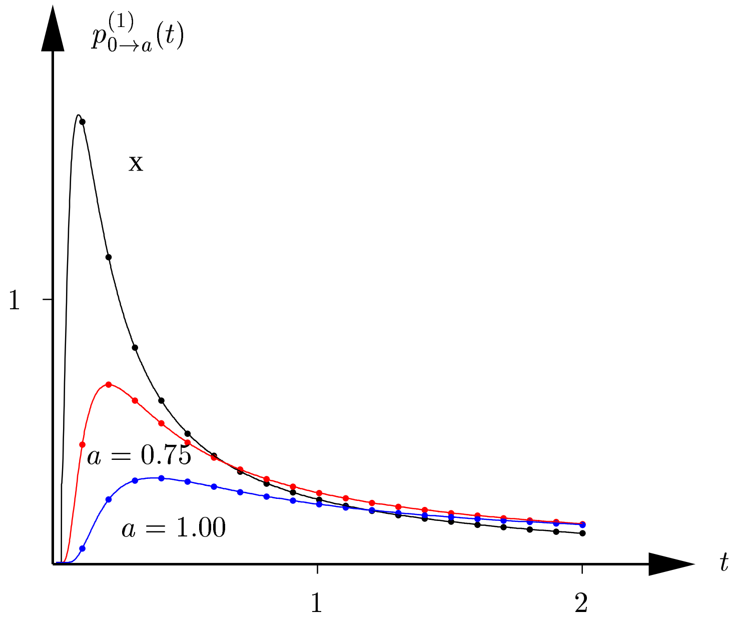

Figure 1. Drawings of the density $t \mapsto p^{(1)}_{0\to a}(t)$ for a equals 0.50, -0.75 and 1.00 when $\lambda = 1$ and $x = 0$. Solid line: series method, bullet: Bessel bridge method and cross: integral method.

7. Hermite functions and their complex decomposition

The special functions used in previous sections are recalled in this section and most of the results can be found in Lebedev [14, Chapter 10]. The Hermite function $H_\nu(z)$ is defined by the following series representation

$$H_\nu(z) = \frac{1}{2\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(\frac{m-\nu}{2}\right)(2z)^m, |z| < \infty, \tag{7.1}$$

and satisfy the recurrence relations

$$\frac{\partial H_\nu}{\partial z}(z) = 2\nu H_{\nu-1}(z) \text{ and } H_{\nu+1}(z) = 2zH_\nu(z) - 2\nu H_{\nu-1}(z).$$

$H_\nu(z)$ is an entire function of both variable z and parameter ν. The couple $H_\nu(\pm\cdot)$ form a fundamental solution to the ordinary Hermite equation

$$\mathcal{G}u + \nu u = 0, \tag{7.2}$$

where $\mathcal{G}$ is the infinitesimal operator given in (2.3). The Hermite function, see [14, p.297], has the integral representation

$$H_\nu(z) = \frac{2^{\nu+1}}{\Gamma((1-\nu)/2)} \int_0^\infty e^{-s^2} s^{-\nu}(s^2+z^2)^{\nu/2} ds, \tag{7.3}$$

for $\mathrm{Re}(\nu) < 1$ and $|\arg z| < \pi/2$. We have that $H_\nu(0) = 2^\nu \Gamma(1/2)/\Gamma((1-\nu)/2)$. With the notation

$$\frac{H_{i\alpha}(z)}{H_{i\alpha}(0)} = Hr_\alpha(z) + iHi_\alpha(z),$$

thanks to the representation (7.3) we easily get that

$$Hr_\alpha(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}\cos\left(\frac{\alpha}{2}\log\left(1+\left(\frac{z}{s}\right)^2\right)\right)ds \tag{7.4}$$

$$Hi_\alpha(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}\sin\left(\frac{\alpha}{2}\log\left(1+\left(\frac{z}{s}\right)^2\right)\right)ds. \tag{7.5}$$

Replacing ν by $i\alpha$ in (7.2) and equalising the real and imaginary parts yield the system

$$\mathcal{G}Hr_\alpha - \alpha Hi_\alpha = 0 \text{ and } \mathcal{G}Hi_\alpha + \alpha Hr_\alpha = 0,$$

with boundary conditions $Hr_\alpha(0) = 1$ and $Hi_\alpha(0) = 0$. The Weber equation $u'' + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right)u = 0$ has as a particular solution the parabolic cylinder function

$$D_\nu(z) = 2^{-\nu/2}e^{-z^2/4}H_\nu(z/\sqrt{2}), z \in \mathbb{R}. \tag{7.6}$$

Finally, as for $\nu \to \infty$, we have the asymptotic formula

$$D_\nu(z) \sim \sqrt{2}(\nu+1/2)^{\nu/2}e^{-(\nu+1/2)/2}\cos\left(z\sqrt{\nu+1/2} - \pi\nu/2\right)\left(1+O(\nu^{-1/2})\right), z \in \mathbb{R}.$$

We deduce from the formula the following large-n asymptotics

$$\nu_{n,-a\sqrt{2\lambda}} \sim 2n - 1 + 4\frac{\lambda a^2}{\pi^2} - 2\frac{\sqrt{\lambda}a}{\pi}\sqrt{4n-1+4\frac{\lambda a^2}{\pi^2}} \tag{7.7}$$

and

$$\begin{aligned} \frac{D_{\nu_{n,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})}{D'_{\nu_{n,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})} \quad &\sim \quad (-1)^n 2\frac{\sqrt{2\nu_{n,-a\sqrt{2\lambda}}+1}}{\pi\sqrt{2\nu_{n,-a\sqrt{2\lambda}}+1}+2a\sqrt{\lambda}} \\ &\times \quad \cos\left(x\sqrt{\lambda(2\nu_{n,-a\sqrt{2\lambda}}+1)} + \frac{\pi\nu_{n,-a\sqrt{2\lambda}}}{2}\right) \end{aligned} \tag{7.8}$$

where $\nu_{n,b}$ is the $n'th$ positive zero of $\nu \mapsto D_\nu(b)$ for a fixed b.

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