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ON A FLUCTUATION IDENTITY FOR RANDOM WALKS AND LÉVY PROCESSES

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Abstract

In this paper, some identities in laws involving ladder processes for random walks and Lévy processes are extended and unified.

1. *Fluctuation theory for random walks*

An essential component of fluctuation theory in discrete time is the study of the bivariate renewal process formed by the coordinates of the successive maxima of a random walk S and the corresponding times. This process is called the *strictly* or *weakly ascending ladder process*, depending on whether or not we count the times where the maximum is reached with the same value; the descending ladder processes are the ascending ladder processes of the dual random walk, $-S$. In the classical era, a breakthrough was achieved by Spitzer, Baxter and others, who introduced Wiener–Hopf techniques and established several fundamental identities that relate the distributions of the ascending and descending ladder processes to that of the underlying random walk. These results were originally formulated in a generating function guise, but since then alternative, probabilistic versions have been established, and these have led to a refined understanding of the underlying identities. In particular, a reformulation of one of these identities by Alili and Doney [**2**] has led to new results for random walks in [**3**], [**9**], [**12**] and [**13**], and for Lévy processes in [**1**] and [**6**]. In this paper, we establish a further refinement of Alili and Doney's identity, for both random walks and Lévy processes, and give some applications; in particular, we extend the uniform law given in [**13**].

The formal mathematical setting is fixed as follows. Let $S = \{S_n : n \geqslant 0\}$ be a given random walk. That is, $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, where $\{X_i : i \geqslant 0\}$ is a sequence of independent and identically distributed random variables. The strict ascending ladder process $(T, H) = \{(T_k, H_k) : k \geqslant 0\}$ is defined as follows:

$$T_0 = 0, \qquad T_k = \inf\{n > T_{k-1} : S_n > S_{T_{k-1}}\}, \qquad k \geqslant 1, \tag{1.1}$$

and the heights are given by:

$$H_k = S_{T_k}, \qquad k \geqslant 0. \tag{1.2}$$

The coordinates of this process are the record heights of the random walk, coupled with the corresponding indices. It is clear from the Markov property that this is a bivariate renewal process. The fact that the law of the random walk S is determined by the law of the processes (T, H) and (T^*, H^*), where (T^*, H^*) is the *weak* descending ladder process of the dual walk $S^* = -S$, is expressed by

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the well-known Wiener–Hopf factorization. This is in turn implicit in the following identities, originally due to Sparre-Andersen [**14**]:

$$\left(1 - E\left[r^{T_1} e^{-\mu H_1}\right]\right)^{-1} = \exp\left(\sum_{m=1}^{\infty} \frac{r^m}{m} E\left[e^{-\mu S_m} \mathbb{I}_{\{S_m > 0\}}\right]\right) \tag{1.3}$$

and

$$\left(1 - E\left[r^{T_1^*} e^{-\mu H_1^*}\right]\right)^{-1} = \exp\left(\sum_{m=1}^{\infty} \frac{r^m}{m} E\left[e^{-\mu S_m^*} \mathbb{I}_{\{S_m^* \geqslant 0\}}\right]\right), \tag{1.4}$$

which hold for any pair (r, μ) of positive reals. These are the so-called *Wiener–Hopf factors*. Spitzer [**15**] and Baxter [**4**] showed later that relations (1.3) and (1.4) are consequences of a combinatorial lemma. The aim of the next paragraph is to gather together the different reformulations of the above identities that have appeared in the literature.

1.1. *Variations of Wiener–Hopf factorization*

A probabilistic statement that is equivalent to (1.3) is implicit in [**10**, p. 424, equation 9.3] of Feller. It states that for any $n \geqslant 1$ and $x > 0$, we have

$$\frac{1}{n} P(S_n \in dx) = \sum_{k=1}^{n} \frac{1}{k} P(T_k = n,\, H_k \in dx). \tag{1.5}$$

In [**2**], it was shown that for $k, n \geqslant 1$ and $x > 0$, we have

$$\frac{1}{n} P(H_{k-1} < S_n \leqslant H_k,\, S_n \in dx) = \frac{1}{k} P(T_k = n,\, H_k \in dx), \tag{1.6}$$

which is a refinement of relation (1.5). Amongst the results that have been derived from this is a generalized uniform law, discovered by Marchal and given in [**13**], which we will now recall. We write G_n for the first hitting time of the overall maximum by the random walk S up to time n; that is,

$$G_n = \inf\left\{k : S_k = \sup_{0 \leqslant j \leqslant n} S_j\right\}, \qquad n \geqslant 1.$$

The special case $k = 1$ of (1.6) states that

$$P(H_1 \geqslant S_n \in dx) = nP(T_1 = n,\, H_1 \in dx), \qquad x > 0,$$

and Marchal's result disintegrates this by showing that for any $i = 1, \ldots, n$ and $x > 0$,

$$\begin{aligned} P(H_1 \geqslant S_n \in dx,\, G_n = i) &= P(T_1 = n,\, H_1 \in dx) \\ &= \frac{1}{n} P(H_1 \geqslant S_n \in dx). \end{aligned} \tag{1.7}$$

This says that *under the probability* $P(\,\cdot \mid S_n = x,\, 0 < S_n \leqslant H_1)$, $x > 0$, *the time* G_n *of the maximum is uniformly distributed over* $\{1, \ldots, n\}$.

It is natural to ask whether a more general uniform law can be obtained when identity (1.6) is considered for any integer k. More generally, can we unify the identities (1.5), (1.6) and (1.7)? The answer to this question is provided in the following section; see Theorems 1 and 2.

1.2. *On Feller's combinatorial lemma*

We start this section by establishing a refinement of Feller's well-known combinatorial lemma; see [**10**, p. 412], and also [**7**] for the skip-free case. Fix $n \geqslant 1$ and let the sequence $s = \{s_i : 0 \leqslant i \leqslant n\}$ be any deterministic chain such that $s_0 = 0$ and $s_n > 0$. The associated ladder process (T, H) is defined as in (1.1) and (1.2). We define the local time at the maximum L_n to be

$$L_n = \sup\{k : T_k \leqslant n\}. \tag{1.8}$$

By definition, the cyclic rearrangements $\theta_1(s), \dots, \theta_{n-1}(s)$ of s are the shifted chains defined for $j = 0, 1, \dots, n-1$ by

$$\theta_j(s)_i = \begin{cases} s_{i+j} - s_j, & \text{if } i \leqslant n-j, \\ s_{j+i-n} + s_n - s_j, & \text{if } n-j \leqslant i \leqslant n, \end{cases} \qquad i = 0, 1, \dots, n.$$

LEMMA 1. *There is a unique integer k in $\{1, \dots, L_n\}$ such that*

$$H_{L_n - k} \leqslant H_{L_n} - s_n < H_{L_n - k + 1}. \tag{1.9}$$

Moreover, k satisfies the following properties.

(i) *The number of cyclical rearrangements of (s_i) such that n is a ladder time, is k.*

(ii) *The number k is preserved by any cyclical rearrangement of (s_i).*

Proof. First, observe that because $s_n > 0$ we have $0 \leqslant H_{L_n} - s_n < H_{L_n}$. Since $k \mapsto H_{L_n - k}$ decreases, there exists a unique $k \in \{1, \dots, L_n\}$ satisfying relation (1.9). Now, we may draw a picture in order to see easily that the only cyclical rearrangements of (s_i) such that n is a ladder time are those that are obtained by splitting the chain (s_i) at a ladder time that is greater than the first passage time above $\sup_{0 \leqslant i \leqslant n} s_i - s_n$ and less than or equal to the first hitting time of $\sup_{0 \leqslant i \leqslant n} s_i$. Note that $\sup_{0 \leqslant i \leqslant n} s_i = H_{L_n}$. From the definition of k, there are exactly k such ladder times. Statement (ii) is a straightforward consequence of (i). □

Now let $S = \{S_n : n \geqslant 0\}$, $S_0 = 0$, be a real-valued random walk. As before, let ν_k be a random variable that is independent of S and uniformly distributed on $\{0, 1, \dots, k-1\}$. Set

$$G_n^k = T_{L_n - \nu_k}; \tag{1.10}$$

then here is our main result.

THEOREM 1. *For all k and n such that $1 \leqslant k \leqslant n$, and for all $i = 1, \dots, n$, we have*

$$P(H_{L_n - k} \leqslant H_{L_n} - S_n < H_{L_n - k + 1},\, S_n \in dx, G_n^k = i) = \frac{1}{k} P(T_k = n, H_k \in dx). \tag{1.11}$$

Proof. Fix a path S such that $H_{L_n - k} \leqslant H_{L_n} - S_n < H_{L_n - k + 1}$. According to Lemma 1(ii), each of the n cyclical rearrangements of S satisfies the same inequalities. Next, according to Lemma 1(i), to each of the n cyclical rearrangements of S, say S', there correspond exactly k cyclical rearrangements such that n is a ladder time; these cyclical rearrangements are $\theta_{T_{L_n - j}}(S')$, $j = 0, \dots, k-1$.

Conversely, the n cyclical rearrangements of each path $\theta_{T_{L_n-j}}(S')$ (where $j = 0,\dots,k-1$ are fixed) are the same as those of the initial chain S. Finally, observe that the value S_n is preserved by any cyclical rearrangement, and that $\{H_k \in dx\} = \{S_n \in dx\}$ on the set $\{T_k = n\}$. $\square$

REMARK. In fact, Theorem 1 is a consequence of the following path transformation, which may also be proved via Lemma 1. Let u_n be a random variable that is independent of S and is uniformly distributed on $\{0,1,\dots,n-1\}$. Conditionally on $\{T_k = n\}$, the chain $\theta_{u_n}(S)$ has the same law as the chain S conditionally on $\{H_{L_n-k} \leqslant H_{L_n} - S_n < H_{L_n-k+1}\}$. Conversely, conditionally on $\{H_{L_n-k} \leqslant H_{L_n} - S_n < H_{L_n-k+1}\}$, the chain $\theta_{G_n^k}(S)$ has the same law as the chain S, conditionally on $\{T_k = n\}$.

The following time-reversal property of the ladder process will enable us to state Theorem 1 in another way; see Theorem 2.

LEMMA 2. *For any integer $n \geqslant 1$, the identity in the law stating that*

$$\{(T_k, H_k) : k \leqslant L_n\} \stackrel{(d)}{=} \{(T_{L_n} - T_{L_n-k}, H_{L_n} - H_{L_n-k}) : k \leqslant L_n\}$$

holds conditionally on S_n.

Proof. First we introduce the past maximum process of S: $\bar{S}_j = \sup_{i \leqslant j} S_i$, then for $k = 0,1,\dots,L_n$, we call $e^{(k)}$ the kth excursion of the reflected process $\bar{S} - S$:

$$e_i^{(k)} = (\bar{S} - S)_{T_k+i}, \qquad 0 \leqslant i < T_{k+1} - T_k,$$

and we denote by $\tilde{e}_i^{(L_n)}$ the excursion that is in progress at time n:

$$\tilde{e}_i^{(L_n)} = (\bar{S} - S)_{T_{L_n}+i}, \qquad 0 \leqslant i \leqslant n - T_{L_n}.$$

By construction, the reflected process $\{(\bar{S} - S)_k : 0 \leqslant k \leqslant n\}$ is the concatenation of $e^{(0)}, e^{(1)}, \dots, e^{(L_n-1)}$ and $\tilde{e}^{(L_n)}$. Let us denote it by

$$\{(\bar{S} - S)_k : 0 \leqslant k \leqslant n\} = e^{(0)} \circ e^{(1)} \circ \dots \circ e^{(L_n-1)} \circ \tilde{e}^{(L_n)}.$$

Now, recall that $\bar{S} - S$ is a Markov process. It follows from the Markov property that if σ is a geometrically distributed random variable that is independent of S, then conditionally on L_σ, the excursions $e^{(0)}, e^{(1)}, \dots, e^{(L_\sigma-1)}$ are i.i.d. and independent of $\tilde{e}^{(L_\sigma)}$. Hence the process

$$e^{(0)} \circ e^{(1)} \circ \dots \circ e^{(L_\sigma-1)} \circ \tilde{e}^{(L_\sigma)},$$

has the same law as the process

$$e^{(L_\sigma-1)} \circ e^{(L_\sigma-2)} \circ \dots \circ e^{(0)} \circ \tilde{e}^{(L_\sigma)},$$

and, conditioning on $\sigma = n$, we obtain:

$$e^{(0)} \circ e^{(1)} \circ \dots \circ e^{(L_n-1)} \circ \tilde{e}^{(L_n)} \stackrel{(d)}{=} e^{(L_n-1)} \circ e^{(L_n-2)} \circ \dots \circ e^{(0)} \circ \tilde{e}^{(L_n)}.$$

Then our lemma is a consequence of the above identity. $\square$

As a consequence, Theorem 1 may be reformulated in the following way.

THEOREM 2. *For any integers* $1 \leqslant k \leqslant n$ *and for all* $i = 1, \dots, n$, *we have*

$$P(H_{k-1} < S_n \leqslant H_k, S_n \in dx, T_{L_n} - T_{\nu_k} = i) = \frac{1}{k} P(T_k = n, H_k \in dx). \qquad (1.12)$$

REMARKS.

1. Note that all the results of both this section and the next one are also valid for any chain S whose increments are cyclically exchangeable (that is, $\theta_j(S)$ is distributed as S for each $j = 0, 1, \dots, n-1$).

2. By summing (1.12) in i, we obtain the Alili–Doney identity (1.6), and Marchal's identity (1.7) is a consequence of (1.12) for $k = 1$.

1.3. *Some consequences*

In the following result, the two first assertions are consequences of Theorem 1, whereas the last statement follows from Theorem 2.

COROLLARY 1. *For any integers* $1 \leqslant k \leqslant n$, *the following statements hold.*

(i) *Conditionally on the event* $\{H_{L_n - k} \leqslant S_n < H_{L_n - k+1}, S_n = x\}$, *the random variable* G_n^k *(defined in* (1.10)*) is uniformly distributed on* $\{1, \dots, n\}$.

(ii) *Summing* (1.11) *over* i *yields*

$$kP(H_{L_n - k} \leqslant H_{L_n} - S_n < H_{L_n - k+1}, S_n \in dx) = nP(T_k = n, H_k \in dx).$$

(iii) *Conditionally on* $\{H_{k-1} < S_n \leqslant H_k, S_n \in dx\}$, *the random variable* $T_{L_n} - T_{\nu_k}$ *is uniformly distributed on* $\{1, \dots, n\}$.

Next, we will establish a different generalization of Marchal's uniform law (1.7). In order to simplify the formulas in what follows, let us introduce the time $N_y = \inf\{j \geqslant 0, H_j \geqslant y\}$, for $y > 0$. Then observe that $\{N_{S_n} = k\} = \{H_{k-1} < S_n \leqslant H_k\}$, for any k. Since $T_0 \equiv 0$, we see that (1.7) says that, conditional on $N_{S_n} = 1$, G_n is uniformly distributed over $\{T_0 + 1, \dots, n\}$, and it is natural to ask whether a similar result holds over $\{T_{k-1} + 1, \dots, n\}$ conditional on $N_{S_n} = k$, for any $1 \leqslant k \leqslant n$. For this, we need the following consequence of (1.7).

LEMMA 3. *For any fixed integer* $0 < k \leqslant n - 1$, *and for all pairs of integers* $i \in \{k-1, k, \dots, n-1\}$ *and* $j \in \{1, \dots, n\}$ *such that* $i + j \leqslant n$, *we have*

$$P(T_k = n,\, H_k \in dx,\, T_{k-1} = i) = P(N_{S_n} = k,\, S_n \in dx,\, G_n = j + i,\, T_{k-1} = i). \qquad (1.13)$$

Proof. It is enough to apply the strong Markov property at the stopping time T_{k-1} in the right-hand side of (1.13), and then to use Marchal's identity (1.7) and the fact that (T, H) is a renewal process. □

REMARK. We can also prove the above result by a path transform. Indeed, for a fixed pair (i, j) fulfilling the above condition, we can establish a one-to-one correspondence between paths satisfying $\{T_k = n,\, H_k = x,\, T_{k-1} = i\}$ and those satisfying $\{N_{S_n} = k,\, S_n = x,\, G_n = j + i,\, T_{k-1} = i\}$. The path transform consists in leaving unchanged the section of the paths from 0 to i, and applying Marchal's path transform to the remaining parts of the paths.

The required uniform law follows immediately.

PROPOSITION 1. *Conditionally on $\{S_n = x,\ N_{S_n} = k\}$, the time G_n is uniformly distributed on $\{T_{k-1}+1,\ T_{k-1}+2, \dots, n\}$. That is, for a fixed $i \in \{k-1, k, \dots, n-1\}$, we have*

$$P(G_n = j \mid N_{S_n} = k,\ S_n = x,\ T_{k-1} = i) = \frac{1}{n-i} \tag{1.14}$$

for $i < j < n$.

We also point out that specializing (1.13) to the case $j = n - i$ gives

$$P(T_{k-1} = i \mid N_{S_n} = k, S_n = x) = (n-i)\frac{k}{n} P(T_{k-1} = i \mid T_k = n, H_k = x),$$

and hence

$$E\left[\frac{1}{n - T_{k-1}} \mid N_{S_n} = k, S_n = x\right] = \frac{k}{n}. \tag{1.15}$$

This makes an interesting contrast to

$$E\big[n - T_{k-1} \mid T_k = n, H_k = x\big] = \frac{n}{k}, \tag{1.16}$$

which can also be proved by elementary methods.

2. *Fluctuation identities for Lévy processes*

Let $(X_t,\ t \geqslant 0)$, be a real Lévy process started at 0. We assume that 0 is regular for $(0, \infty)$; that is, $P(\inf\{t : X_t > 0\} = 0) = 1$. As a consequence, 0 is a regular state (regular for itself) for the strong Markov process $Y_t = \sup_{s \leqslant t} X_s - X_t$, called the *reflected process at the maximum*; that is, $P(\inf\{t > 0 : Y_t = 0\} = 0) = 1$. Then the process Y admits a local time L at 0, which is defined in the sense of Markov processes as in [**5**, Chapter IV]. Let τ be the right continuous inverse of L: $\tau_t = \inf\{s : L_s > t\}$, and set $\Gamma_t = X(\tau_t)$. Then the process (τ, Γ) is the continuous-time analogue of the discrete ladder process (T, H). More precisely, it is a bivariate subordinator whose law is characterized as follows. For any couple (α, β) of positive reals, we have

$$E\left(e^{-\alpha\tau_1 - \beta\Gamma_1}\right) = e^{-\kappa(\alpha,\beta)},$$

where the characteristic exponent κ of (τ, Γ) is given by

$$\kappa(\alpha, \beta) = k \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-\alpha t - \beta x}) t^{-1} P(X_t \in dx)\right). \tag{2.1}$$

The analogue of (1.5) for Lévy processes was discovered by Bertoin and Doney, and is described in [**6**]. Now, define the first passage-time process of X by

$$\sigma_x = \inf\{t : X_t > x\}, \qquad x \geqslant 0.$$

In [**1**], the identity between measures on $(0, \infty)^3$,

$$t^{-1} P(L_{\sigma_x} \in du,\ X_t \in dx)\, dt = u^{-1} P(\tau_u \in dt,\ \Gamma_u \in dx)\, du, \tag{2.2}$$

was proved in the same way that, in discrete time, (1.6) was deduced from (1.3); see [**2**]. It is then natural to adapt our identities (1.11) and (1.12) to Lévy processes. In this case, the proof has to be different. Indeed, under our regularity assumption, the process X has infinitely many ladder times, so that there is no analogue for the combinatorial Lemma 1. Let us first state the following continuous-time equivalent to Lemma 2.

LEMMA 4. *For any $t \geqslant 0$, the identity in the law stating that*

$$\{(\tau_u, \Gamma_u) : u \leqslant L_t\} \stackrel{(d)}{=} \{(\tau_{L_t} - \tau_{L_t - u}, \Gamma_{L_t} - \Gamma_{L_t - u}) : u \leqslant L_t\}$$

holds conditionally on X_t.

The proof of this lemma uses classical arguments of excursion theory, and follows the same scheme as in Lemma 2, so we omit it here.

Then the idea of the proof of our result in continuous time is to discretize the process X on the time interval $[0, 1]$, in order to obtain a sequence of random walks that converges almost surely towards X, and such that the corresponding sequence of ladder processes converges almost surely towards (τ, Γ). Set $M_t = \sup_{v \leqslant t} X_v$.

THEOREM 3. *Let ν_u be a uniformly distributed random variable on $[0, u]$ that is independent of X. Then we have the following identity between measures, with $(s, t, u, x) \in [0, t] \times (0, \infty)^3$:*

$$P(L_t - L_{\sigma_{M_t - x}} \in du,\, X_t \in dx,\, \tau_{L_t - \nu_u} \in ds)\, dt = u^{-1} P(\tau_u \in dt,\, \Gamma_u \in dx)\, du\, ds \tag{2.3}$$

and

$$P(L_{\sigma_x} \in du,\, X_t \in dx,\, \tau_{L_t} - \tau_{\nu_u} \in ds)\, dt = u^{-1} P(\tau_u \in dt,\, \Gamma_u \in dx)\, du\, ds \tag{2.4}$$

Proof. We first give the idea of the proof of (2.4). Note that from the proof of [**5**, p. 175, Theorem VI.19], $x \mapsto L(\sigma_x)$ is the right continuous inverse of the process Γ; that is

$$L_{\sigma_x} = \inf\{t : \Gamma_t > x\}, \qquad x \geqslant 0.$$

Now, for $n \geqslant 1$, consider the random walk $X_k^{(n)} \stackrel{(\mathrm{def})}{=} X_{k/n}$, $k \geqslant 0$, and define its ladder process $(T^{(n)}, H^{(n)})$ as in (1.1) and (1.2), and the 'local time' $L^{(n)}$ as in (1.8). Identity (2.4) is then a consequence of Theorem 2 combined with the convergence, as $n \to \infty$, of the sequence of three-dimensional processes

$$\left\{ \left(X_{[nt]}^{(n)}, n^{-1} T_{[nt]}^{(n)}, H_{[nt]}^{(n)} \right) : t \geqslant 0 \right\}$$

towards (X, τ, Γ), in the sense of finite-dimensional distributions. Then, identity (2.3) follows from identity (2.4) and Lemma 2. □

The resulting uniform laws are then the same as in Corollary 1 for random walks.

COROLLARY 2. *For any fixed $t, u, x > 0$, the following statements hold.*

(i) *Conditionally on $L_t - L_{\sigma_{M_t - x}} = u$ and $X_t = x$, the random variable $\tau_{L_t - \nu_u}$ is uniformly distributed on $[0, t]$.*

(ii) *Conditionally on $L_{\sigma_x} = u$ and $X_t = x$, the random variable $\tau_{L_t} - \tau_{\nu_u}$ is uniformly distributed on $[0, t]$.*

REMARKS.

1. Note that τ_{L_t} is the unique time at which X reaches its overall maximum on $[0, t]$; that is, $\tau_{L_t} = \inf\{v \leqslant t : X_v = \sup_{0 \leqslant l \leqslant t} X_l\}$. In the above statement, if $x = 0$,

then necessarily $u = 0$ and Corollary 2 is reduced to the well-known uniform law giving the time of the maximum for (cyclically) exchangeable bridges from 0 to 0; see [**11**] and [**8**].

2. When X is Brownian motion, the uniform laws of Corollary 2 are consequences of the path transformation stated in [**7**, Theorem 7].

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