

MARTIN BOUNDARIES ASSOCIATED WITH A KILLED RANDOM WALK

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ABSTRACT. – We start by studying the connection between the full Martin boundary associated with a space time version of a random walk which is killed on entering the negative half-line, and that associated with the bivariate renewal process of weak increasing ladder heights and times in the random walk. We show that although the corresponding spatial boundaries are isomorphic, the space time boundaries are not. The rest of the paper is devoted to determining these boundaries explicitly in the special case that the moment generating function of the step distribution exists in a non-empty interval. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Dans un premier temps, nous étudions la correspondance entre la frontière espace-temps de Martin totale associée à une marche aléatoire à valeurs réelles tuée lorsqu'elle franchit la demi-droite réelle négative, et celle associée au processus de renouvellement bivarié formé par les échelles et les temps de croissance. Nous montrons que les frontières spatiales correspondantes sont isomorphes, alors que les frontières espace-temps ne le sont pas. Le reste du travail est consacré à déterminer explicitement ces frontières dans le cas particulier où la fonction génératrice correspondant à la distribution des pas existe dans un intervalle non vide. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Let $S = (S_n, n \geq 0)$ be an aperiodic random walk on the integers, with mass function $p(\cdot)$; thus $S_n = S_0 + \sum_1^n X_i$ for $n \geq 1$, where the X 's are independent and identically distributed with $\mathbb{P}(X_i = x) = p(x)$ for $x \in \mathbb{Z}$ and $i = 1, 2, \dots$. Writing $\tau = \inf\{n > 0, S_n < 0\}$ we define the Markov chain S^* as S killed at time τ ; thus S^* has state space $\mathbb{Z}^* = \{0, 1, \dots\}$ and substochastic transition function

$$p(x, y) = p(y - x), \quad x, y \in \mathbb{Z}^*. \quad (1.1)$$

In this paper we investigate the Martin boundaries of two bivariate Markov chains associated with S . The first is $\{(S_n^*, n), n \geq 0\}$, a space time version of S^* , which we denote by \underline{S}^* . This has state space $\mathcal{E} = Z^* \times Z^*$ and substochastic transition function

$$p^*(\underline{x}, \underline{y}) = p(y - x)\delta(l + 1, m), \tag{1.2}$$

where $\delta(i, j)$ is the Kronecker delta function, $\underline{x} = (x, l)$, and $\underline{y} = (y, m)$. (We will adopt the convention that $\underline{x} = (x, l)$, $\underline{y} = (y, m)$ and $\underline{z} = (z, n)$ without further notice in the sequel.) The second is $\{(H_n^+, T_n^+), n \geq 0\}$, the bivariate process of weak increasing ladder heights and times in S , which we denote by $\underline{L}^+ = (\underline{L}_n^+, n \geq 0)$. This is another (possibly substochastic) transient Markov chain on \mathcal{E} , with transition function

$$q^+(\underline{x}, \underline{y}) = \mathbb{P}(H_1^+ = y - x, T_1^+ = m - l). \tag{1.3}$$

Our original aim was to exhibit the minimal Martin boundary M_0^* of \underline{S}^* , or equivalently to find the totality of minimal regular functions for \underline{S}^* . However, this lead us inevitably to study M_0^+ , the corresponding object for \underline{L}^+ , and also M^* and M^+ , the respective *full* Martin boundaries. (Recall that this is equivalent to finding all possible limits of the appropriate relativised Green’s functions, k^* and k^+ .) The point is that there is an important relationship between the Green’s functions g^* and g^+ of \underline{S}^* and \underline{L}^+ (see Lemma 2.1) which enables us to define a linear mapping Φ such that the relativised Green’s functions k^* and k^+ satisfy

$$k^*(\cdot, \underline{y}) = \Phi(k^+(\cdot, \underline{y})). \tag{1.4}$$

Now in [5], the corresponding relation between the relativised Green’s functions of the univariate processes S^* and $H^+ = (H_n^+, n \geq 0)$ was studied. In that situation it was shown that the analogue of (1.4) is preserved *in the limit*, and hence that both the minimal and full Martin boundaries of S^* and H are isomorphic to each other. In our bivariate situation things are not so simple, and although $\Phi(h^+)$ is regular (respectively super-regular) for \underline{S}^* whenever h^+ is regular (super-regular) for \underline{L}^+ , we are *not* able to show that every such function h^* is of the form $\Phi(h^+)$. In fact we exhibit examples of sequences $\underline{y}^{(r)}$ such that both $h^*(\cdot) = \lim_{r \rightarrow \infty} k^*(\cdot, \underline{y}^{(r)})$ and $h^+(\cdot) = \lim_{r \rightarrow \infty} k^+(\cdot, \underline{y}^{(r)})$ exist and $h^* \neq \Phi(h^+)$. Thus in general M^* is not isomorphic to M^+ , so we have no guarantee that M_0^* is isomorphic to M_0^+ . This means that our only way of finding M_0^* is to find the full Martin boundary M^* , and then identify the points corresponding to minimal regular functions. To do this in full generality is clearly an impossible task (the full Martin boundary of S^* is not yet known in all cases), but we are able to achieve it in some important cases. In these cases it does in fact turn out that M_0^* is isomorphic to M_0^+ .

To describe these situations, we need to introduce some notation for the exponential family associated to \mathbb{P} . Write $M(s) = \mathbb{E}(s^{X_1})$, and put

$$\alpha = \inf\{s > 0, M(s) < \infty\}, \quad \beta = \sup\{s > 0, M(s) < \infty\}.$$

The function $\mu(s) = sM'(s)/M(s)$ is a continuous and increasing function on (α, β) , and its inverse will be denoted by $s(\cdot)$. Write

$$\mu_- = \mu(\alpha) \geq -\infty, \quad \mu_+ = \mu(\beta) \leq +\infty,$$

and I for the interval (μ_-, μ_+) together with μ_- if $M(\alpha)$ is finite and μ_+ if $M(\beta)$ is finite. The exponential family $\mathbb{P}^{(\mu)}$ is defined for $\mu \in I$ through its probability mass function

$$p^{(\mu)}(x) = s(\mu)^x p(x)/M(s(\mu)),$$

and it is easily checked that $\mathbb{E}^{(\mu)}(X_1) = \mu$. Thus $\mathbb{P} \equiv \mathbb{P}^{(\mu_0)}$, where $\mu_0 = \sum xp(x)$, and $s(\mu_0) = 1$.

Recall that in [7], the minimal Martin boundary M_0 of the unrestricted space-time process $\underline{S} = \{(S_n, n), n \geq 0\}$ was shown to contain a point μ with a corresponding regular function

$$h^{(\mu)}(\underline{x}) = s(\mu)^x t(\mu)^l \quad \text{where } t(\mu) = 1/M(s(\mu)) \tag{1.5}$$

for each $\mu \in I$; also $\mathbb{P}^{(\mu)}$ is the harmonic transform of \mathbb{P} by $h^{(\mu)}$, and $h^{(\mu)}$ is the limit of the relativised Green's function as $\underline{y}^{(r)} \rightarrow \infty$ with $\theta^{(r)} := y^{(r)}/m^{(r)} \rightarrow \mu$. Moreover if $b < \infty$, where

$$b := \max\{x: p(x) > 0\} \tag{1.6}$$

we have $\mu_+ = b$, but $M(\beta) = \infty$, so $\mu_+ \notin I$. However, there is a point μ_+ in M_0 corresponding to a regular function which is semi-degenerate, i.e. it is supported by a straight line. The harmonic transform of \mathbb{P} by this function corresponds to a degenerate random walk with step length b .

These results suggest that both M_0^* and M_0^+ should be isomorphic to $I^+ := I \cap [0, \infty)$ when $b = \infty$, and to $I^+ \cup \{b\}$ when $b < \infty$. In Theorem 3.1 we show that this is correct for M_0^+ , but we cannot rule out the possibility that M_0^* contains additional points.

We then restrict our selves to the case that

$$E(X_1) = 0, \quad \text{Var}(X_1) = \sigma_0^2 < \infty, \quad \text{and} \quad 0 < \mu_+ < \infty.$$

Furthermore, we assume *either* that

$$b < \infty, \quad \text{so that } I^+ = [0, b),$$

which we call Case A, or that

$$b = \infty, \quad M(\mu_+) < \infty, \quad \mathbb{E}^{(\mu_+)}(X_1 - \mu_+)^2 = \hat{\sigma}^2 < \infty,$$

and

$$p^+(r) := p^{(\mu_+)}(r) \text{ is regularly varying at } \infty \text{ with index } -\kappa, \quad 3 < \kappa < \infty.$$

We refer to this latter situation as Case B; of course in this case $I^+ = [0, \mu_+]$.

In Cases A and B we are able (see Theorems 6.1 and 6.2) to find all possible values of $\lim_{r \rightarrow \infty} k^*(\cdot, \underline{y}^{(r)})$ and $\lim_{r \rightarrow \infty} k^+(\cdot, \underline{y}^{(r)})$ as $\underline{y}^{(r)} \rightarrow \infty$ in any manner through the reduced state space \mathcal{E}_0 , which consists of all states in \mathcal{E} which are accessible from $\underline{0}$. The obvious cases are when

$$\theta^{(r)} := y^{(r)} / m^{(r)} \rightarrow \mu \in \text{int } I^+,$$

but we also have to consider the cases when the limit is 0, $\mu_+, \mu \in (\mu_+, \infty)$, or ∞ . The case $\theta^{(r)} \rightarrow 0$ is particularly delicate, and several subcases have to be considered, depending at what rate the convergence takes place. Thus for example, $\lim_{r \rightarrow \infty} k^+(\cdot, \underline{y}^{(r)})$ is different if $\theta^{(r)} \rightarrow 0$ with $y^{(r)}$ fixed than it is if $\theta^{(r)} \rightarrow 0$ with both $m^{(r)} \rightarrow \infty$ and $y^{(r)} \rightarrow \infty$. A similar phenomenon occurs in the case $\theta^{(r)} \rightarrow \infty$, which can occur with $m^{(r)}$ fixed or with $m^{(r)} \rightarrow \infty$.

The key to all these results is to obtain good estimates for the Green’s functions g^+ and g^* . We find that we have to adopt different methods in different cases. In some cases we can analyse g^* directly by adapting the method used in [2], which incidentally is the paper which stimulated the present work. In other cases g^* is analysed indirectly by first studying g^+ , and then using the relation (1.4). Finding good estimates of g^+ , without making superfluous assumptions, is quite tricky (see, for example, Theorem 3.7 of [9]), and we rely heavily on a recently discovered identity from [1].

These results specify the full Martin boundaries M^+ and M^* in Cases A and B, and then we can read off M_0^+ and M_0^* . We are not able to say much about M_0^* when $\mu_+ < 0$ or $\mu_+ = 0$. In the first of these cases M_0^+ is empty, and in the second it has one point, corresponding to the identity function. The obvious conjecture is that M_0^* is isomorphic to M_0^+ , but we have not been able to establish this, except in some very special cases. The possibility of describing M^* seems even more remote.

The paper is organised as follows. In Section 2 we study the mapping Φ . In Section 3 M_0^+ is determined in all cases. Section 4 is devoted to some asymptotic estimates for $\mathbb{P}(S_m = y)$, which are applied to give results for the Green’s functions in Section 5, and in Section 6 we determine the Martin boundaries in Cases A and B.

2. The mapping from M^+ to M^*

To avoid trivial cases we will assume henceforth that

$$\exists x_1 > 0, x_2 < 0 \quad \text{with } p(x_1) > 0, p(x_2) > 0. \tag{2.1}$$

Our only other assumption in this section is that S is aperiodic. Note first that the Green’s function $g^*(\underline{x}, \underline{y}) = \sum_{n=0}^{\infty} p_n^*(\underline{x}, \underline{y})$, where p_n^* denotes the n -step transition function for \underline{S}^* , can be written as

$$g^*(\underline{x}, \underline{y}) = \mathbb{P}_x(S_{m-l} = y, \tau > m - l), \tag{2.2}$$

where $\mathbb{P}_x(\cdot)$ stands for $\mathbb{P}(\cdot \mid S_0 = x)$. Since this is finite, \underline{S}^* is of course transient, and following [7] we will take $\underline{0} = (0, 0)$ as a reference state and treat \underline{S}^* as a substochastic

Markov chain on the reduced state space

$$\mathcal{E}_0 = \{ \underline{y} \in Z^* \times Z^* : g^*(\underline{0}, \underline{y}) > 0 \}.$$

This is permissible, because all states that can be reached from states in \mathcal{E}_0 are in \mathcal{E}_0 . Then for $\underline{x}, \underline{y} \in \mathcal{E}_0$ we define the relativised Green’s function

$$k^*(\underline{x}, \underline{y}) = g^*(\underline{x}, \underline{y}) / g^*(\underline{0}, \underline{y}).$$

By definition a function $f : \mathcal{E}_0 \rightarrow \mathbb{R}^+$ is regular (super-regular) for \underline{S}^* if and only if

$$P^* f = (\leq) f \quad \text{on } \mathcal{E}_0, \tag{2.3}$$

where

$$(P^* f)(\underline{x}) = \sum_{\underline{y} \in \mathcal{E}_0} p^*(\underline{x}, \underline{y}) f(\underline{y}). \tag{2.4}$$

For each fixed \underline{y} , the function $k^*(\cdot, \underline{y})$ is super-regular, and the full Martin boundary M^* of \underline{S}^* is in 1:1 correspondence with the set of all possible distinct limits of $k^*(\cdot, \underline{y})$, as $\underline{y} \rightarrow \infty$ in any manner such that $\underline{y} \in \mathcal{E}_0$. The minimal Martin boundary M_0^* consists of the subset of M^* which corresponds to functions which are regular and minimal. We recall that a non-negative regular function is minimal if, whenever \tilde{h} is non-negative regular function with $\tilde{h} \leq h$, then $\tilde{h} = ch$ for some positive constant c . (Throughout this paper, c will denote a generic positive constant, whose value may change from line to line.)

Next, we introduce some notation for the ladder processes associated with S . First, the ladder time processes are defined by $T_0^+ = T_0^- = 0$, and

$$T_{k+1}^+ = \inf\{r > T_k^+ : S_r \geq S_{T_k^+}\}, \quad k = 0, 1, \dots,$$

$$T_{k+1}^- = \inf\{r > T_k^- : S_r < S_{T_k^-}\}, \quad k = 0, 1, \dots,$$

where $\inf\{\emptyset\} = \infty$. (Note that T_1^- coincides with τ .) Then the ladder processes $\underline{L}^\pm = \{\underline{L}_n^\pm, n \geq 0\}$ are defined by

$$\underline{L}_n^\pm = (H_n^\pm, T_n^\pm) \quad \text{on } n : T_n^\pm < \infty,$$

where the ladder heights are given by $H_n^\pm = \pm S(T_n^\pm)$. Thus \underline{L}^\pm are (possibly substochastic) transient Markov chains on \mathcal{E} with transition functions

$$q^\pm(\underline{x}, \underline{y}) = \mathbb{P}(H_1^\pm = y - x, T_1^\pm = m - l) = q^\pm(y - x, m - l)$$

and Green’s functions

$$g^\pm(\underline{x}, \underline{y}) = \sum_{r=0}^\infty q^{\pm(r)}(\underline{x}, \underline{y}).$$

Because of spatial and temporal homogeneity, we can also write $g^+(\underline{x}, \underline{y}) = v(\underline{y} - \underline{x})$ and $g^-(\underline{x}, \underline{y}) = u(\underline{y} - \underline{x})$, where

$$v(\underline{x}) = \sum_{r=0}^{\infty} \mathbb{P}(\underline{L}_r^+ = \underline{x}) \quad \text{and} \quad u(\underline{x}) = \sum_{r=0}^{\infty} \mathbb{P}(\underline{L}_r^- = \underline{x}).$$

In treating the Martin boundary of \underline{L}^+ we again take $\underline{0}$ as a reference state and note that the set of states which can be reached by \underline{L}^+ starting from $\underline{0}$ coincides with \mathcal{E}_0 . (This is an application of the duality lemma, see [8, p. 395].) So for $\underline{x}, \underline{y} \in \mathcal{E}_0$ we define the relativised Green’s function

$$k^+(\underline{x}, \underline{y}) = g^+(\underline{x}, \underline{y})/g^+(\underline{0}, \underline{y}),$$

and write M^+, M_0^+ respectively for the full and the minimal Martin boundaries of \underline{L}^+ . Of course regular, super-regular and minimal functions for \underline{L}^+ are defined as for \underline{S}^* , with P^* replaced by Q^+ , defined by

$$(Q^+ f)(\underline{x}) = \sum_{\underline{y} \in \mathcal{E}_0} q^+(\underline{x}, \underline{y}) f(\underline{y}).$$

The key to our analysis is the following bivariate extension of a result due to Spitzer (see [12, p. 209]) which expresses the Green’s function of \underline{S}^* in terms of the Green’s functions of \underline{L}^+ and \underline{L}^- .

LEMMA 2.1. – For $\underline{x}, \underline{y} \in \mathcal{E}_0$ we have

$$g^*(\underline{x}, \underline{y}) = \sum_{\underline{z} \in \mathcal{E}_0} u(x - z, n - l) g^+(\underline{z}, \underline{y}), \tag{2.5}$$

where we note that the sum extends over \underline{z} such that $0 \leq z \leq x \wedge y$ and $l \leq n \leq m$.

Proof. – Since $g^*(\underline{x}, \underline{y}) = g^*((x, 0), (y, m - l))$ it is enough to prove (2.5) for $l = 0$, and in this case we have $g^*((x, 0), (y, m)) = \mathbb{P}_x(\Lambda)$ where

$$\Lambda = \{S_m = y, \tau > m\}.$$

We decompose Λ according to J_m , the minimum value attained by S by time m , and σ_m the time at which J_m is first attained. This gives

$$\begin{aligned} \mathbb{P}_x(\Lambda) &= \sum_{z=0}^{x \wedge y} \sum_{n=0}^m \mathbb{P}_x(J_m = z, \sigma_m = n, S_m = y, \tau > m) \\ &= \sum_{z=0}^{x \wedge y} \sum_{n=0}^m \mathbb{P}_0(J_n = z - x, \sigma_n = n) \mathbb{P}_0(S_{m-n} = y - z, \tau > m - n) \\ &= \sum_{z=0}^{x \wedge y} \sum_{n=0}^m u(x - z, n) v(y - z, m - n), \end{aligned}$$

where we have used the Markov property, temporal and spatial homogeneity, and duality. This is the same as (2.5) with $l = 0$. \square

COROLLARY 2.2. – *The relativised Green's functions are also connected by*

$$k^*(\underline{x}, \underline{y}) = \sum_{\underline{z} \in \mathcal{E}_0} u(x - z, n - l) k^+(\underline{z}, \underline{y}). \quad (2.6)$$

Proof. – Just note that putting $\underline{x} = \underline{0}$ in (2.5) gives

$$g^*(\underline{0}, \underline{y}) = \sum_0^m u(0, n) g^+((0, n), \underline{y}) = g^+(\underline{0}, \underline{y}),$$

because $u(0, n) = \delta(0, n)$. Then dividing both sides of (2.5) by $g^+(\underline{0}, \underline{y})$ gives (2.6). \square

We can restate this result as saying that, for fixed \underline{y} , $k^*(\cdot, \underline{y})$ is the image under Φ of $k^+(\cdot, \underline{y})$; where Φ maps a real-valued function $h(\underline{x})$, $\underline{x} \in \mathcal{E}_0$ into another such function h^* defined by

$$h^*(\underline{x}) = \sum_{\underline{z} \in \mathcal{E}_0} u(x - z, n - l) h(\underline{z}). \quad (2.7)$$

Note that in (2.7), in general the summation is not finite, as it is in the special case of (2.5), but extends over $0 \leq z \leq x$, $n \geq l$. This is technically the main difference between our situation and the spatial case discussed in [5].

To analyse the mapping Φ , we need the following elementary result.

LEMMA 2.3. – (i) *For $r \geq 0$, $y \geq 1$ we have*

$$u(y, r + 1) = \sum_{w=0}^{\infty} p(w - y) u(\omega, r). \quad (2.8)$$

(ii) *For $r \geq 0$, $y \geq 0$ we have*

$$q^+(y, r + 1) = \sum_{w=0}^{\infty} u(\omega, r) p(w + y). \quad (2.9)$$

Proof. – (i) Since $u(w, 0) = \delta(w, 0)$ for $r = 0$ the right hand side of (2.8) reduces to $p(-y)$ and so does the left hand side. For $r > 0$ note that $u(0, r) = 0$ and decompose according to the value of S_1 .

(ii) For $r = 0$, Eq. (2.9) reduces to

$$\mathbb{P}_0(S_1 = y) = \sum_0^{\infty} u(w, 0) p(w + y) = p(y),$$

and for $r \geq 1$ the right hand side of (2.9) coincides with $\sum_1^{\infty} u(w, r) p(w + y)$. But in this case the duality lemma gives

$$\begin{aligned}
 q^+(y, r + 1) &= \mathbb{P}_0(S_t < 0, 1 \leq t \leq r, S_{r+1} = y) \\
 &= \sum_1^\infty \mathbb{P}_0(S_t < 0, 1 \leq t \leq r - 1, S_r = -w) p(w + y) \\
 &= \sum_1^\infty \mathbb{P}_0(S_t > -w, 1 \leq t \leq r - 1, S_r = -w) p(w + y) \\
 &= \sum_1^\infty u(w, r) p(w + y),
 \end{aligned}$$

which ends the proof. \square

We can now formulate the main result of this section.

PROPOSITION 2.4. – *If a function h is non-negative and super-regular (regular) for \underline{L}^+ then $h^* = \Phi(h)$ is non-negative and super-regular (regular) for \underline{S}^* .*

Proof. – If h is non-negative and $h^* = \Phi(h)$ it is clear that h^* is non-negative and we have

$$\begin{aligned}
 (P^*h^*)(\underline{x}) &= \sum_{y \geq 0} p(y - x) h^*(y, l + 1) \\
 &= \sum_{y \geq 0} p(y - x) \sum_{z=0}^y \sum_{n=l+1}^\infty u(y - z, n - l - 1) h(\underline{z}) \\
 &= \sum_{n \geq l+1} \left(\sum_{0 \leq z \leq x} + \sum_{z > x} \right) \sum_{y \geq z} p(y - x) u(y - z, n - l - 1) h(\underline{z}) \\
 &= \sum_{n \geq l+1} \left(\sum_{0 \leq z \leq x} u(x - z, n - l) h(\underline{z}) + \sum_{z > x} q^+(z - x, n - l) h(\underline{z}) \right) \\
 &= h^*(\underline{x}) - h(\underline{x}) + \sum_{n \geq l} \sum_{z \geq x} q^+(n - l, z - x) h(\underline{z}) \\
 &= h^*(\underline{x}) - h(\underline{x}) + (Q^+h)(\underline{x}),
 \end{aligned}$$

where we have used the results of Lemma 2. The result now follows. \square

3. The minimal boundaries

Our aim now is to specify the minimal regular functions for \underline{L}^+ , so that we can read off the corresponding functions for \underline{S}^* from Proposition 2.4. In [7] it is shown that for any 2-dimensional random walk, every minimal regular function is either non-degenerate, in the sense that it is strictly positive on the whole of the reduced state space, or semi-degenerate in the sense that its support is a straight line. Furthermore in the second case the reduced state space lies completely on one side of the line, and in both cases the function is a multiple of a function of exponential form. Thus a non-degenerate minimal regular function for \underline{L}^+ with $h(\underline{0}) = 1$ has the form $h(\underline{x}) = s^x t^l$, where necessarily

$$(Q^+h)(\underline{0}) = \sum \sum s^x t^l q^+(x, l) = \mathbb{E}_0[s^{H_1^+} t^{T_1^+}] = h(\underline{0}) = 1. \tag{3.1}$$

Hence the identity function is a minimal regular function if and only if T_1^+ is a proper random variable, and this is known to be the case when S either drifts to infinity or oscillates under \mathbb{P} . In particular, if $\mu_0 = \mathbb{E}[X_1]$ exists, this happens if and only if $\mu_0 \geq 0$. To elucidate what other solutions (3.1) can have, recall that the Wiener–Hopf factorization can be written as

$$1 - tM(s) = \{1 - \mathbb{E}[s^{H_1^+} t^{T_1^+}]\} \{1 - \mathbb{E}[s^{-H_1^-} t^{T_1^-}]\}, \tag{3.2}$$

whenever either side is finite. So any root of (3.1) has to be of the form $t = 1/M(s)$ for some $0 < s < \infty$ with $M(s)$ finite, and this can happen with $s \neq 1$ only when the exponential family associated to \mathbb{P} is non-trivial. Recall the notation for this family given in the introduction, and note that for $\mu \in I$

$$\begin{aligned} \mathbb{P}^{(\mu)}(T_1^+ < \infty) &= \sum \sum \mathbb{P}^{(\mu)}(H_1^+ = x, T_1^+ = l) \\ &= \sum \sum s(\mu)^x M(s(\mu))^{-l} \mathbb{P}(H_1^+ = x, T_1^+ = l) \\ &= \mathbb{E}[s(\mu)^{H_1^+} t(\mu)^{T_1^+}], \end{aligned}$$

where we have put $t(\mu) = 1/M(s(\mu))$. Thus (3.1) holds with $s = s(\mu)$, $t = t(\mu)$, if and only if T_1^+ is proper under $\mathbb{P}^{(\mu)}$, or equivalently if and only if $\mu \geq 0$. So we conclude that the only non-degenerate minimal regular functions for \underline{L}^+ are given by

$$h_\mu(\underline{x}) = s(\mu)^x t(\mu)^l \quad \text{for } \mu \in I^+ \equiv I \cap [0, \infty). \tag{3.3}$$

In particular, there are no non-degenerate minimal regular functions if $\mu_+ < 0$; for example if $\mu_0 < 0$ and $\alpha = \beta = 1$.

As for semi-degenerate minimal functions, the only case when \mathcal{E}_0 is bounded above by a line is when $b = \sup\{x: p(x) > 0\}$ is finite, and then the bounding line is $l^+ = \{\underline{x}; x = lb, l \geq 0\}$. Moreover it is easy to see that $q^+(lb, l) = p(b)\delta(1, l)$, and that

$$\hat{h}_b(\underline{x}) = \begin{cases} p(b)^{-l}, & x = lb, l = 0, 1, \dots, \\ 0, & \text{otherwise} \end{cases}$$

defines a minimal regular function whose support is l^+ .

Of course \mathcal{E}_0 is bounded below by the line $l^- = \{\underline{x}, x = 0, l \geq 0\}$ so that the function taking values t^l at $(0, l)$ and zero off l^- would also be a minimal regular function if $\sum t^l q^+(0, l) = 1$. Now

$$q^+(0, l) = \mathbb{P}\{S_1 < 0, S_2 < 0, \dots, S_{l-1} < 0, S_l = 0\},$$

and it is known (see [8, p. 424]) that

$$1 - \sum_1^\infty t^l q^+(0, l) = \exp - \sum_1^\infty \frac{t^n}{n} \mathbb{P}(S_n = 0).$$

Since the lefthandside is a strictly decreasing function of t , if there is a t for which it vanishes, it must be equal to the radius of convergence of the power series

$\sum_1^\infty t^n n^{-1} \mathbb{P}(S_n = 0)$. However it follows from Theorem 5 of [10] that this coincides with $1/M(\hat{s})$, where $\alpha \leq \hat{s} \leq \beta$,

$$M(\hat{s}) = \inf_{\alpha \leq s \leq \beta} M(s),$$

and $\hat{s} > 0$. Thus $\hat{s} = s(\hat{\mu})$ for some $\hat{\mu}$, and

$$\sum_1^\infty \hat{t}^l q((0, l)) = \mathbb{P}^{\hat{\mu}}(T_1 < \infty, H_1 = 0),$$

which is clearly less than one, in view of the fact that $\mathbb{P}^{\hat{\mu}}(S_1 = x_1) > 0$.

Remark 1. – In [10], Kesten assumes that the random walk is strongly aperiodic, rather than just aperiodic. But a perusal of the proof of his Theorem 5 shows that to get the result we require, which is that

$$\lim_{n \rightarrow \infty} (\mathbb{P}(S_n = 0))^{1/n} = M(\hat{s}),$$

the assumption that S is aperiodic and condition (2.1) suffices. Note also that the above result implies that $\hat{s} = s(\mu_-)$ whenever $\mu_- > 0$, so that in all cases I^+ contains its left-hand endpoint, $0 \vee \mu_-$. Of course I^+ is not always closed on the right.

We have thus established

THEOREM 3.1. – *If $b = \infty$ the minimal regular functions for \underline{L}^+ are precisely those given in (3.3). If $b < \infty$ there is the additional minimal regular function \hat{h}_b .*

By computing the images under Φ of these functions, we deduce the following result, most of which is contained in [2];

THEOREM 3.2. – *Let $U^{(\mu)}(x) = \sum_0^x u^{(\mu)}(y)$ be the renewal function in the strict decreasing ladder heights process under $P^{(\mu)}$, so that $u^{(\mu)}(x) = \sum_{r \geq 0} \mathbb{P}^{(\mu)}(H_r^- = x)$. Then*

$$h_\mu^*(\underline{x}) = s(\mu)^x M(s(\mu))^{-l} U^{(\mu)}(x), \quad \mu \in I^+, \tag{3.4}$$

are minimal regular functions for \underline{S}^ . If $b < \infty$ there is, in addition, a semi-degenerate minimal regular function*

$$\hat{h}_b(\underline{x}) = \begin{cases} p(b)^{-l}, & x = lb, l = 0, 1, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. – Remembering that $H_r^- = -S(T_r^-)$ we have, in the obvious notation

$$\begin{aligned} u^{(\mu)}(\underline{y}) &= \sum_0^\infty \mathbb{P}^{(\mu)}(H_r^- = y, T_r^- = m) \\ &= \sum_0^\infty \mathbb{P}^{(\mu)}(S_m = -y, S_u > -y, \text{ for } u < m) \end{aligned}$$

$$\begin{aligned}
&= \sum_0^\infty s(\mu)^{-y} t(\mu)^m \mathbb{P}(H_r^- = y, T_r^- = m) \\
&= s(\mu)^{-y} t(\mu)^m u(\underline{y}).
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi(h_\mu)(\underline{x}) &= \sum_{n \geq l} \sum_{z=0}^x u(x-z, n-l) s(\mu)^z t(\mu)^n \\
&= s(\mu)^x t(\mu)^l \sum_{m \geq 0} \sum_{y=0}^x u(y, m) s(\mu)^{-y} t(\mu)^m \\
&= s(\mu)^x t(\mu)^l U^{(\mu)}(x) = h_\mu^*(\underline{x}).
\end{aligned}$$

As for $\hat{h}_b(\underline{x})$, note that since the line l^+ bounds \mathcal{E}_0 above, if $\underline{z} \in \mathcal{E}_0 \setminus l^+$ then $z < bn$. Thus in

$$\Phi(\hat{h}_b)(\underline{x}) = \sum_{n \geq l} \sum_{nb \leq x} u(x-nb, n-l) p(b)^n$$

we see that the sum is 0 unless $\underline{x} \in l^+$, and then it reduces to $u(0, 0) p(b)^{-l} = p(b)^{-l}$. \square

Remark 2. – This result has a clear probabilistic significance. The h_μ^* -transform of \underline{S}^* corresponds to a conditioning which makes $S_n^*/n \rightarrow \mu$ a.s., and (3.4) shows that we can think of this as a 2-stage process. First we make the exponential transform to get a version of S with mean μ , and then we h -transform a killed version of this, using the unique minimal regular function $U^{(\mu)}(\cdot)$. Indeed it has been shown in [2] that, when I^+ is a non-degenerate interval and $\mu \in \text{int}(I^+)$, the law of S conditional upon $\tau > n$ and either $S_n \geq \mu n$ or $S_n \leq \mu n$, depending on whether $\mu \leq \mu_0$ or $\mu > \mu_0$, converges (in the sense of finite-dimensional distributions) to $\mathbb{P}^{(\mu, +)}$, the harmonic transform of $\mathbb{P}^{(\mu)}$.

4. Some local and ratio limit results

In this section we give a collection of estimates for $\mathbb{P}(S_m = y)$, which are valid as $\underline{y} \rightarrow \infty$, with $\underline{y} \in \mathcal{E}_0$, in different ways. From now on we will assume that S is aperiodic, and that

$$E(X_1) = 0, \quad \text{Var}(X_1) = \sigma_0^2 < \infty, \quad \text{and} \quad 0 < \mu_+ < \infty. \quad (4.1)$$

Furthermore, we assume either that

$$b < \infty, \quad \text{so that } I^+ = [0, b), \quad (4.2)$$

which we call Case A, or that, writing \mathbb{P}^+ , \mathbb{E}^+ , and p^+ for $\mathbb{P}^{(\mu_+)}$, $\mathbb{E}^{(\mu_+)}$, and $p^{(\mu_+)}$

$$b = \infty, \quad M(\mu_+) < \infty, \quad \mathbb{E}^{(\mu_+)}(X_1 - \mu_+)^2 = \hat{\sigma}^2 < \infty, \quad (4.3)$$

and

$$p^+ \text{ is regularly varying at } \infty \text{ with index } -\kappa, \quad 3 < \kappa < \infty. \quad (4.4)$$

We refer to this latter situation as Case B; of course in this case $I^+ = [0, \mu_+]$.

Next, we put $b_0 = \mu_+$ in Case B, and fix $b_0 \in (0, b)$ in Case A. Then define

$$H(\mu) = \log t(\mu) + \mu \log s(\mu), \quad 0 \leq \mu \leq b_0, \tag{4.5}$$

where we recall that $t(\mu) = 1/M(s(\mu))$. We then see, from the definition of $s(\mu)$, that

$$H'(\mu) = \frac{s'(\mu)}{s(\mu)} \left\{ \mu - \frac{s(\mu)M'(s(\mu))}{M(s(\mu))} \right\} + \log s(\mu) = \log s(\mu). \tag{4.6}$$

Furthermore one can check that

$$H''(\mu) = \frac{s'(\mu)}{s(\mu)} = \frac{1}{s(\mu)\mathbb{E}^{(\mu)}\{(X_1 - \mu)^2\}}. \tag{4.7}$$

It follows that H is a twice differentiable function, that both H and H' are monotone increasing, and that $H(0+) = H'(0+) = 0$. Moreover if we put $\theta = y/m$ then for $0 \leq \theta \leq b_0$ we have

$$\mathbb{P}(S_m = y) = \{M(s(\theta))\}^m s(\theta)^{-y} \mathbb{P}^{(\theta)}(S_m = y) = e^{-mH(\theta)} \mathbb{P}^{(\theta)}(S_m = y). \tag{4.8}$$

Thus we will get a good estimate for $\mathbb{P}(S_m = y)$ by approximating $H(\theta)$ by a Taylor expansion, if we can estimate $\mathbb{P}^{(\theta)}(S_m = y)$. It is clear that, in Case A and Case B, $\sigma_\mu^2 := \mathbb{E}^{(\mu)}(X_1 - \mu)^2$ is a continuous function, bounded away from 0 and ∞ , for $0 \leq \mu \leq b_0$. Furthermore, the same holds for $v(\mu) := \mathbb{E}^{(\mu)}|X_1 - \mu|^3$ on $[\delta, b_0 - \delta]$ for each $\delta > 0$, but not on $[0, b_0]$, unless we assume additionally that $v(0)$ and $v(b_0)$ are finite. We will not make this additional assumption, but still claim;

LEMMA 4.1. – *In Case A and Case B it holds that, uniformly for $\theta = y/m \in [0, b_0]$,*

$$\mathbb{P}^{(\theta)}(S_m = y) = \frac{1}{\sqrt{2\pi m\sigma_\theta}} + o\left(\frac{1}{\sqrt{m}}\right) \quad \text{as } m \rightarrow \infty. \tag{4.9}$$

Proof. – If θ is bounded away from 0 and b_0 this follows from Theorem 6 of [11], and we only give the proof for the case $\theta \rightarrow 0$, as the case $\theta \rightarrow b_0$ is similar. If y/\sqrt{m} is bounded, then (4.9) follows from the classical local limit theorem. To see this, note that $\sigma_\theta \rightarrow \sigma_0$ as $\theta \rightarrow 0$, and, using a Taylor expansion and the remarks following (4.7)

$$mH(\theta) = m \left\{ H(0) + \theta H'(\theta) + \frac{1}{2}\theta^2 H''(\theta) + o(\theta^2) \right\} = \frac{m\theta^2}{2\sigma_0^2} + o(1),$$

and since $m\theta^2$ is bounded

$$\begin{aligned} \sigma_\theta \sqrt{2\pi m} \mathbb{P}^{(\theta)}(S_m = y) &= \sigma_\theta e^{mH(\theta)} \sqrt{2\pi m} \mathbb{P}(S_m = y) \\ &= \sigma_\theta e^{mH(\theta)} \left\{ \frac{e^{-\frac{y^2}{2m\sigma_0^2}}}{\sigma_0} + o(1) \right\} \rightarrow 1. \end{aligned}$$

In the case that $\theta \rightarrow 0$ and $y/\sqrt{m} \rightarrow \infty$ we use a Berry–Esseen type local limit theorem from [4]. Specifically, if we apply Lemma 3 therefrom with W_1 having the $\mathbb{P}^{(\theta)}$ distribution of X_1 , (4.9) will follow provided that, as $m \rightarrow \infty$,

$$\frac{\nu(\theta)}{\sqrt{m}} \rightarrow 0 \quad \text{and} \quad \sqrt{m} \int_{c/\nu(\theta)}^{\pi} e^{-m(1-|\psi(t)|)} dt \rightarrow 0 \tag{4.10}$$

where $\psi(t) = \mathbb{E}^{(\theta)}(e^{itX_1})$. Since $M(s)$ exists for some $s > 1$, we know that $\mathbb{E}\{(X_1^+)^3\} < \infty$, and so

$$\begin{aligned} \frac{\nu(\theta)}{\sqrt{m}} &= \frac{\sum_{r=1}^{\infty} r^3 s(\theta)^{-r} p(-r)}{\sqrt{m} M(s(\theta))} + o(1) \sim \frac{1}{\sqrt{m}} \sum_{r=1}^{\infty} r^2 p(-r) \cdot r \left(1 + \frac{\theta}{\sigma_0^2}\right)^{-r} \\ &\leq \frac{\sigma_0^2}{\sqrt{m}\theta} \sum_{r=1}^{\infty} r^2 p(-r) \leq \frac{c\sqrt{m}}{y}. \end{aligned}$$

This establishes the first condition in (4.10), and the second follows from the first and the easily checked fact that $1 - |\psi(t)| \geq ct^2$ for all sufficiently small t and θ . \square

The following consequence of estimate (4.9) will be important;

COROLLARY 4.2. – *Suppose $(\lambda_m, m \geq 1)$ is a sequence of positive integers such that $\lambda_m = o(\sqrt{m})$ as $m \rightarrow \infty$. Then with $\theta = y/m$*

$$\frac{\mathbb{P}^{(\theta)}(S_{m-n} = y + z)}{\mathbb{P}^{(\theta)}(S_m = y)} \rightarrow 1 \quad \text{as } m \rightarrow \infty,$$

uniformly for

$$0 \leq \theta \leq b_0, \quad 0 \leq z \leq \lambda_m, \quad \text{and} \quad 0 \leq n \leq \lambda_m.$$

Proof. – Suppose first that $\hat{\theta} := \frac{y+z}{m-n} \leq b_0$, and take m large enough that $\lambda_m < m$. Then using (4.6) twice we get

$$\begin{aligned} \mathbb{P}^{(\theta)}(S_{m-n} = y + z) &= t(\theta)^{m-n} s(\theta)^{y+z} \mathbb{P}(S_{m-n} = y + z) \\ &= \left\{ \frac{t(\theta)}{t(\hat{\theta})} \right\}^{m-n} \left\{ \frac{s(\theta)}{s(\hat{\theta})} \right\}^{y+z} \mathbb{P}^{(\hat{\theta})}(S_{m-n} = y + z), \end{aligned} \tag{4.11}$$

where we recall that $t(\theta) = 1/M(s(\theta))$. Now

$$0 \leq \hat{\theta} - \theta = \frac{zm + ny}{m(m-n)} \leq \frac{z + nb_0}{m} \leq (1 + b_0)\lambda_m \rightarrow 0, \tag{4.12}$$

and $\sigma(\cdot)$ is uniformly continuous on $[0, b_0]$, so it follows from (4.9) that

$$\mathbb{P}^{(\hat{\theta})}(S_{m-n} = y + z) / \mathbb{P}^{(\theta)}(S_m = y) \rightarrow 1 \quad \text{uniformly as } m \rightarrow \infty.$$

Also putting $y + z = (m - n)\hat{\theta}$ we get, after some manipulation involving (4.6),

$$\left\{ \frac{t(\theta)}{t(\hat{\theta})} \right\}^{m-n} \left\{ \frac{s(\theta)}{s(\hat{\theta})} \right\}^{y+z} = \exp -(m - n) \{ H(\hat{\theta}) - H(\theta) - (\hat{\theta} - \theta)H'(\theta) \}. \tag{4.13}$$

Now it follows from (4.7) that H'' is bounded on $[0, b_0]$, so we see, using the bound (4.12), that

$$(m - n)|H(\hat{\theta}) - H(\theta) - (\hat{\theta} - \theta)s(\theta)| \leq cm(\lambda_m)^2,$$

and the result follows. If $\hat{\theta} > b_0$, a similar argument shows that

$$\frac{\mathbb{P}^{(\theta)}(S_{m-n} = y + z)}{\mathbb{P}^{(b_0)}(S_{m-n} = y + z)} \rightarrow 1 \quad \text{and} \quad \frac{\mathbb{P}^{(\theta)}(S_m = y)}{\mathbb{P}^{(b_0)}(S_m = y)} \rightarrow 1 \quad \text{uniformly as } m \rightarrow \infty.$$

Since the classical local limit theorem shows that

$$\frac{\mathbb{P}^{(b_0)}(S_{m-n} = y + z)}{\mathbb{P}^{(b_0)}(S_m = y)} \rightarrow 1 \quad \text{uniformly as } m \rightarrow \infty,$$

the result follows. \square

In Case B, we also have to consider the situation that $\underline{y} \in \mathcal{E}_0$ and $\theta > \mu_+$. If $y \in (m\mu_+, m\mu_+ + \Delta\sqrt{m})$, where Δ is fixed, the classical local limit theorem gives a good estimate for $\mathbb{P}^{(\mu_+)}(S_m = y)$, but to deal with other cases we need to exploit the assumption (4.4). Specifically, recalling that \mathbb{P}^+ , p^+ , and σ_+ stand for $\mathbb{P}^{(\mu_+)}$, $p^{(\mu_+)}$, and σ_{μ_+} and writing ϕ for the standard Normal density function, the following is an immediate consequence of Theorem 2 of [6];

PROPOSITION 4.3. – *In Case B we have, uniformly in y such that $z := \frac{y - m\mu_+}{\sqrt{m}} \rightarrow \infty$,*

$$\mathbb{P}^+(S_m = y) = \frac{1}{\sqrt{m}\sigma_+} \phi\left(\frac{z}{\sigma_+}\right) \{1 + o(1)\} + mp^+([y - m\mu_+])\{1 + o(1)\} \quad \text{as } m \rightarrow \infty. \tag{4.14}$$

The final piece of information we need about $\mathbb{P}(S_m = y)$ is a ratio limit theorem in Case A, which may have other applications.

PROPOSITION 4.4. – *Let S be any integer-valued, aperiodic random walk which for some $b > 0$ has $p(b) > 0$, $p(x) = 0$ for all $x > b$. Write $l^+ = \{(mb, m), m \geq 0\}$ for the upper boundary of \mathcal{E}_0 . Then for fixed $\underline{x} = (x, l) \in \mathcal{E}_0$ the convergence*

$$\frac{\mathbb{P}(S_{m-l} = y - x)}{\mathbb{P}(S_m = y)} \rightarrow \hat{h}_b(\underline{x}) = \begin{cases} p(b)^{-l} & \text{if } \underline{x} \in l^+, \\ 0 & \text{if } \underline{x} \notin l^+, \end{cases} \tag{4.15}$$

holds uniformly as $y/m \rightarrow b$ with $\underline{y} = (y, m) \in \mathcal{E}_0$.

Proof. – We consider first the case that $\underline{x} = (lb, l) \in l^+$, when with $\tilde{S}_r = br - S_r, r \geq 0$, we can rewrite (4.15) as

$$\frac{\mathbb{P}(\tilde{S}_{m-l} = \tilde{y})}{\mathbb{P}(\tilde{S}_m = \tilde{y})} \rightarrow p(b)^{-l}, \tag{4.16}$$

where $\tilde{y} = mb - y$. Note that $\tilde{p}(r) := \mathbb{P}(\tilde{S}_1 = r) = p(b - r) = 0$ if $r < 0$. Since $\underline{y} \in \mathcal{E}_0$ we have $\mathbb{P}(\tilde{S}_m = \tilde{y}) > 0$, and since $\tilde{y}/m \rightarrow 0$ it follows that, when m is large enough,

each of $\mathbb{P}(\tilde{S}_{m-j} = \tilde{y})$, $j = 1, 2, \dots, l$, is also positive. (Just remove j zero steps of \tilde{S} , i.e., j steps of size b of S , from a path with $\mathbb{P}(\tilde{S}_m = \tilde{y}) > 0$.) So it suffices to prove (4.16) with $l = 1$. Let π denote a generic partition of \tilde{y} as a sum of *positive* integers, $\alpha(s)$ the multiplicity of s in π and $\beta(\pi) = \sum_{s \geq 1} \alpha(s)$ the ‘length’ of π . Observe that $\beta(\pi) \leq \tilde{y}$ for all π . Then

$$\begin{aligned} \mathbb{P}(\tilde{S}_m = \tilde{y}) &= \sum_{\pi} \tilde{p}(0)^{m-\beta(\pi)} \frac{m! \prod_{s \geq 1} \tilde{p}(s)^{\alpha(s)}}{(m - \beta(\pi))! \prod_{s \geq 1} \alpha(s)!} \\ &= \tilde{p}(0) \sum_{\pi} \frac{m}{m - \beta(\pi)} \tilde{p}(0)^{m-1-\beta(\pi)} \frac{(m - 1)! \prod_{s \geq 1} \tilde{p}(s)^{\alpha(s)}}{(m - 1 - \beta(\pi))! \prod_{s \geq 1} \alpha(s)!}. \end{aligned}$$

Comparing this with the corresponding expression for $\mathbb{P}(\tilde{S}_{m-1} = \tilde{y})$ we see that

$$\frac{1}{\tilde{p}(0)} \geq \frac{\mathbb{P}(\tilde{S}_{m-1} = \tilde{y})}{\mathbb{P}(\tilde{S}_m = \tilde{y})} \geq \frac{m - \tilde{y}}{m \tilde{p}(0)},$$

and hence (4.16) follows.

If $\underline{x} = (x, l) \in \mathcal{E}_0$ and $\underline{x} \notin l^+$ then $\mathbb{P}(S_l = x) > 0$ and $x < lb$. Then it is easy to see that $\exists \underline{x}^* = (x^*, l^*) \in \mathcal{E}_0$ such that $x - x^* = (l - l^*)b \geq 0$ and no partition of x^* of length l^* into integers with $p(\cdot) > 0$ contains any b 's. Then what we have already proved shows that

$$\frac{\mathbb{P}(S_{m-l^*} = y - x^*)}{\mathbb{P}(S_{m-l} = y - x)} \rightarrow p(b)^{l^*-l},$$

so it suffices to prove (4.15) in the case that $\underline{x} = \underline{x}^*$. Then it reduces to

$$\frac{\mathbb{P}(\tilde{S}_{m-l} = \tilde{y} - \tilde{x})}{\mathbb{P}(\tilde{S}_m = \tilde{y})} \rightarrow 0, \tag{4.17}$$

where $\tilde{x} = lb - x > 0$, $\mathbb{P}(\tilde{S}_l = \tilde{x}) > 0$, and $\mathbb{P}(\text{any of } \tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_l = 0 \mid \tilde{S}_l = \tilde{x}) = 0$. Thus there exists a partition $\tilde{\pi}$ of \tilde{x} into positive integers with mutiplicities $\tilde{\alpha}(s)$ with $\tilde{p}(s) > 0$ for each s such that $\tilde{\alpha}(s) > 0$, and with length $\beta(\tilde{\pi}) = l$. Then to each partition π of $\tilde{y} - \tilde{x}$ into positive integers there corresponds a partition π^* of \tilde{y} into positive integers which is formed by adjoining the members of $\tilde{\pi}$ and π . It follows that

$$\begin{aligned} \mathbb{P}(\tilde{S}_m = \tilde{y}) &\geq \sum_{\pi^*} \tilde{p}(0)^{m-l-\beta(\pi)} \frac{m! \prod_{s \geq 1} \tilde{p}(s)^{\alpha(s)+\tilde{\alpha}(s)}}{(m - l - \beta(\pi))! \prod_{s \geq 1} (\alpha(s) + \tilde{\alpha}(s))!} \\ &\geq \frac{(m - l)! \mathbb{P}(\tilde{S}_{m-l} = \tilde{y} - \tilde{x})}{\{\tilde{y} \tilde{p}(0)\}^l}, \end{aligned}$$

and (4.17) follows. \square

Remark 3. – In [7], it is shown that $\hat{h}_b(\cdot)$ is a minimal regular function for the unrestricted space time random walk \underline{S} , and Proposition 4.4 confirms that it is indeed the limit of relativised Green’s functions for \underline{S} .

5. Estimates for the Green’s functions

Turning now to the Green’s functions of \underline{S}^* and \underline{L}^+ , we show first that, for fixed \underline{x} , the asymptotic behaviour of $g^*(\underline{x}, \underline{y})$ when $\theta := y/m \in [\delta, b_0]$ with $\delta > 0$ can be determined by a variation of the argument used in [2]. We write τ and σ for T_1^- and T_1^+ respectively, and recall that

$$g^*(\underline{x}, \underline{y}) = \mathbb{P}_x(S_{m-l} = y, \tau > m - l). \tag{5.1}$$

PROPOSITION 5.1. – For fixed $x \geq 0$ and $\delta > 0$ we have, uniformly for $\theta \in [\delta, b_0]$,

$$\frac{\mathbb{P}_x^{(\theta)}(S_m = y, \tau > m)}{\mathbb{P}_x^{(\theta)}(S_m = y)} - \mathbb{P}_x^{(\theta)}(\tau = \infty) \rightarrow 0 \quad \text{as } m \rightarrow \infty. \tag{5.2}$$

Proof. – Recalling that $H^- = -S_\tau$, we write $\mathbb{P}_x^{(\theta)}(S_m = y, \tau \leq m)$ as $P_1 + P_2$, where $P_1 = \mathbb{P}_x^{(\theta)}(S_m = y, \tau \leq m^{1/3}, H^- \leq m^{1/3})$, and show that

$$\frac{(P_1 + P_2)}{\mathbb{P}_x^{(\theta)}(S_m = y)} - \mathbb{P}_x^{(\theta)}(\tau < \infty) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

We start by noting that Corollary 4.2 gives, uniformly for $\theta \in [\delta, b_0]$,

$$\begin{aligned} & \frac{P_1}{\mathbb{P}_x^{(\theta)}(S_m = y)} - \mathbb{P}_x^{(\theta)}(\tau \leq m^{1/3}, H^- \leq m^{1/3}) \\ &= \sum_{\substack{1 \leq z \leq m^{1/3} \\ 1 \leq i \leq m^{1/3}}} \mathbb{P}_x^{(\theta)}(\tau = i, S_i = -z) \left\{ \frac{\mathbb{P}_x^{(\theta)}(S_{m-i} = y + z)}{\mathbb{P}_x^{(\theta)}(S_m = y)} - 1 \right\} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Using Lemma 4.1 and noting that

$$P_2 \leq \mathbb{P}_x^{(\theta)} \{ (m^{1/3} < \tau < \infty) \cup (m^{1/3} < H^- < \infty) \},$$

we see that it suffices to show that, uniformly for $\theta \in [\delta, b_0]$,

$$\sqrt{m} \mathbb{P}_x^{(\theta)} \{ (m^{1/3} < \tau < \infty) \cup (m^{1/3} < H^- < \infty) \} \rightarrow 0. \tag{5.3}$$

However, since $s(\cdot)$ is bounded on $[\delta, b_0]$ and

$$M^{(\theta)}(w) := \mathbb{E}^{(\theta)}(w^{S_1}) = \frac{M(ws(\theta))}{M(s(\theta))},$$

it is clear that we can choose $w \in (0, 1)$ such that $M^{(\theta)}(w) \leq a < 1$ for all $\theta \in [\delta, b_0]$. But if $\tau(-r) = \inf\{i: S_i < -r\}$ we have, for $r \geq 0$,

$$\begin{aligned} w^{-r} \mathbb{P}_r^{(\theta)}(\tau < \infty) &= w^{-r} \mathbb{P}^{(\theta)}(\tau(-r) < \infty) = w^{-r} \sum_{i=1}^{\infty} \mathbb{P}^{(\theta)}(\tau(-r) = i) \\ &\leq \sum_{i=1}^{\infty} \mathbb{E}^{(\theta)}(w^{S_i}; \tau(-r) = i) \leq \sum_{i=1}^{\infty} a^i \leq (1 - a)^{-1}. \end{aligned}$$

So if we put $[m^{1/3}] = j$, we have

$$\begin{aligned} \mathbb{P}_x^{(\theta)}(m^{1/3} < \tau < \infty) &= \sum_{r=0}^{\infty} \mathbb{P}_x^{(\theta)}(S_j = r, \tau > j) \mathbb{P}_r^{(\theta)}(\tau < \infty) \\ &\leq (1 - a)^{-1} \sum_{r=0}^{\infty} w^r \mathbb{P}_x^{(\theta)}(S_j = r) \leq (1 - a)^{-1} w^x a^j. \end{aligned}$$

It is also straightforward to deduce, from the Wiener–Hopf factorisation (3.2) with $t = 1$, that $\mathbb{E}^{(\theta)}(w^{-H_1}) \leq 1$ for all $\theta \in [\delta, b_0]$. This easily leads to a uniform exponential bound for $\mathbb{P}_x^{(\theta)}(m^{1/3} < H^- < \infty)$. Then (5.3) follows and this finishes the proof. \square

The same technique works for Case B when $y/m \geq \mu_+$.

PROPOSITION 5.2. – *In Case B we have, for fixed x and uniformly for $y \in \mathcal{E}_0$ such that $(y - x, m) \in \mathcal{E}_0$ and $y/m \geq \mu_+$,*

$$\frac{\mathbb{P}_x^+(S_m = y, \tau > m)}{\mathbb{P}_x^+(S_m = y)} \rightarrow \mathbb{P}_x^+(\tau = \infty) \quad \text{as } m \rightarrow \infty. \tag{5.4}$$

Proof. – This follows the same lines as the previous proof, except that we work under the fixed measure $\mathbb{P}^+ = \mathbb{P}^{(\mu_+)}$, and we rely on the estimate (4.14) rather than (4.9). The details are omitted. \square

In Case B it is also necessary to deal with the situation that m is fixed, and $y \rightarrow \infty$, when of course $\theta \rightarrow \infty$.

LEMMA 5.3. – *In Case B when $m \geq 1$ and $\underline{x} = (x, l)$ are fixed with $0 \leq l \leq m - 1$,*

$$\mathbb{P}_x^+(S_{m-l} = y, \tau > m - l) \sim p^+(y) \sum_{j=0}^{m-l-1} \mathbb{P}_x^+(\tau > j). \tag{5.5}$$

Proof. – Recall from (4.4) that $p^+(y)$ is regularly varying of index $-\kappa$ at infinity, with $\kappa > 3$, and note that since $\mathbb{P}^+(X_1 = -y)$ is exponentially small, $\mathbb{P}^+(|X_1| > y)$ is regularly varying of index $1 - \kappa$ at infinity. It follows that we can choose $\varepsilon \in (0, 1)$ so that if $K = K(y) = [y^\varepsilon]$ then

$$\{\mathbb{P}^+(|X_1| > K)\}^2 = o(p^+(y)) \quad \text{as } y \rightarrow \infty. \tag{5.6}$$

A standard property of regularly varying functions gives

$$p^+(z) \sim p^+(y) \quad \text{as } y \rightarrow \infty, \text{ uniformly for } y - K \leq z \leq y + K. \tag{5.7}$$

Writing $m - l = k$, it follows from (5.6) that, as $y \rightarrow \infty$,

$$\mathbb{P}_x^+(S_k = y, \tau > k) = \mathbb{P}_x^+\{(S_k = y, \tau > k) \cap A\} + o\{p^+(y)\}$$

where

$$A = \{\text{exactly one of } |X_i|, 1 \leq i \leq k, \text{ exceeds } K\}.$$

Now write

$$A \cap (S_k = y) = \bigcup_1^k B^{(j)},$$

where

$$B^{(j)} = \{|X_i| \leq K, 1 \leq i \leq k, i \neq j, |S_j| > K, S_k = y\},$$

and note that $B^{(j)} \cap \{\tau > j - 1\} = B^{(j)} \cap \{\tau > k\}$, provided $y \geq Kk$. Thus, using (5.7),

$$\begin{aligned} & \mathbb{P}_x^+ \{(S_k = y, \tau > k) \cap A\} \\ &= \sum_{j=1}^k \sum_{|z_i| \leq K, i \neq j} \mathbb{P}_x^+ \{X_i = z_i, i \neq j, \tau > j - 1\} p^+ \left(y - \sum_{i \neq j} z_i \right) \\ &\sim p^+(y) \sum_{j=1}^k \mathbb{P}_x^+ \{|X_i| \leq K, i < j, \tau > j - 1\} \mathbb{P}^+ \{|X_i| \leq K, j < i \leq k\}, \end{aligned}$$

and since k is fixed, (5.5) follows. \square

Next we turn to the case that $\theta \rightarrow 0$, when the approach used above is not effective, since $\mathbb{P}_x(\tau = \infty) = 0$ for all x . In this situation it seems that the estimation of $g^*(\underline{x}, \underline{y})$ has to be approached indirectly, by first finding the asymptotic behaviour of $g^+(\underline{z}, \underline{y})$ and then using the relation (2.5). It also seems to be necessary to give different arguments, according as y/\sqrt{m} tends to zero or not. Recall that $g^+(\underline{z}, \underline{y}) = v(\underline{y} - \underline{z})$. Rather than using the obvious identity

$$v(\underline{y}) = \sum_{r=0}^{\infty} \mathbb{P}(H_r^+ = y, T_r^+ = m)$$

it turns out to be much more effective to use a new representation, taken from [1]. (Actually the result given in [1] is for the *strict* ladder process, but it is easy to adapt the arguments to the weak case.) The result is

$$mv(\underline{y}) = \mathbb{E}\{N_y; S_m = y\}, \quad m > 0, y \geq 0, \tag{5.8}$$

where

$$N_y = \sup\{k: H_k^+ \leq y\}.$$

The identity (5.8) can also be formulated as a renewal equation, and in [1] this was used to establish the following result. This improves an earlier result in [9], and gives us the required estimate in the first of these cases. (Note this result does *not* require the existence of $\mathbb{P}^{(\theta)}$ for $\theta > 0$.)

PROPOSITION 5.4. – *If S satisfies (4.1) then, uniformly in $y \geq 0$ such that $y/\sqrt{m} \rightarrow 0$,*

$$mv(\underline{y}) \sim V(y) \mathbb{P}(S_m = y) \sim \frac{V(y)}{\sigma_0 \sqrt{2\pi m}} \quad \text{as } m \rightarrow \infty, \tag{5.9}$$

where $V(y) = \mathbb{E}\{N_y\}$.

If we recall that $v(\underline{y}) = g^+(\underline{0}, \underline{y}) = g^*(\underline{0}, \underline{y})$ we can put $\underline{x} = \underline{0}$ in (5.2) and rewrite it as

$$mv^{(\theta)}(\underline{y}) \sim \frac{y}{\theta} \mathbb{P}^{(\theta)}(\tau = \infty) \mathbb{P}^{(\theta)}(S_m = y) \quad \text{as } m \rightarrow \infty. \tag{5.10}$$

However for $\theta > 0$ the Wiener–Hopf factorisation yields

$$\mathbb{P}^{(\theta)}(\tau = \infty) = \frac{1}{\mathbb{E}^{(\theta)}(\sigma)},$$

and Wald’s identity gives

$$\frac{\theta}{\mathbb{E}^{(\theta)}(\sigma)} = d(\theta) := \mathbb{E}^{(\theta)}(H_1^+).$$

Of course $V(y) \sim y/d(\theta)$ when $y \rightarrow \infty$, and d is continuous, so we see that, under our assumptions, (5.9) and (5.10) can be combined in the form

$$mv^{(\theta)}(\underline{y}) \sim \frac{y}{d(\theta)} \mathbb{P}^{(\theta)}(S_m = y) \sim \frac{1}{\sigma(\theta)d(\theta)\sqrt{2\pi m}} \quad \text{as } m \rightarrow \infty, y \rightarrow \infty \tag{5.11}$$

We will now show that (5.11) holds when $\theta \rightarrow 0$, $\sqrt{m}\theta = y/\sqrt{m} \rightarrow 0$.

PROPOSITION 5.5. – *For any $\eta > 0$, $\exists \delta \in (0, b_0)$ such that (5.11) holds uniformly for $y \in [\sqrt{m}\eta, m\delta]$.*

Proof. – Given $\varepsilon > 0$, we split the $\mathbb{P}^{(\theta)}$ version of (5.8) into three terms, by writing

$$y^{-1}mv^{(\theta)}(\underline{y}) = \sum_{k=1}^{\infty} \frac{k}{y} \mathbb{P}^{(\theta)}\{N_y = k, S_m = y\} = \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}, \tag{5.12}$$

where the summation in $\Sigma^{(i)}$ is over A_i , with

$$A_1 = \left\{ 1 \leq k < \frac{(1 - \varepsilon)y}{d(\theta)} \right\}, \quad A_2 = \left\{ \frac{(1 - \varepsilon)y}{d(\theta)} \leq k \leq \frac{(1 + \varepsilon)y}{d(\theta)} \right\},$$

and

$$A_3 = \left\{ k > \frac{(1 + \varepsilon)y}{d(\theta)} \right\}.$$

We also write

$$P_i = \mathbb{P}^{(\theta)}\{N_y \in A_i, S_m = y\}, \quad i = 1, 2, 3.$$

Clearly

$$1 - \varepsilon \leq d(\theta) \frac{\Sigma^{(2)}}{P_2} \leq 1 + \varepsilon,$$

so (5.11) will follow if we can show that both $\Sigma^{(1)} + \Sigma^{(3)}$ and $P_1 + P_3$ are $o\{\mathbb{P}^{(\theta)}(S_m = y)\}$, uniformly for $y \in [\sqrt{m}\eta, m\delta]$. Notice that $\Sigma^{(1)} \leq P_1$, and

$$\begin{aligned}
 P_1 &\leq \mathbb{P}^{(\theta)} \left\{ N_y < \frac{(1-\varepsilon)y}{d(\theta)} \right\} = \mathbb{P}^{(\theta)} (H_{k(\theta)}^+ \geq y) \\
 &= \mathbb{P}^{(\theta)} \{ H_{k(\theta)}^+ - k(\theta)d(\theta) \geq y - k(\theta)d(\theta) \},
 \end{aligned}$$

where $k(\theta) = \lfloor \frac{(1-\varepsilon)y}{d(\theta)} \rfloor$. Note that $y - k(\theta)d(\theta) \sim \varepsilon y$ as $m \rightarrow \infty$, and $\mathbb{E}^{(\theta)}(\lambda^{H_1^+}) \rightarrow \mathbb{E}(\lambda^{H_1^+}) < \infty$ for $1 \leq \lambda \leq s(b_0)$. In particular, $\exists \delta > 0, \lambda_0 > 1$ such that

$$\mathbb{E}^{(\theta)}(\lambda^{H_1^+}) \leq c \quad \text{for } \lambda \in [0, \lambda_0] \text{ and } \theta \in [0, \delta].$$

A standard exponential inequality then shows that P_1 is uniformly $O(e^{-cy})$ and hence $o\{\mathbb{P}^{(\theta)}(S_m = y)\}$ by Lemma 4.1.

A similar calculation shows that $\mathbb{P}^{(\theta)}(N_y > \frac{y}{d(\theta)} + z)$ is uniformly $O(e^{-cz})$ as $z \rightarrow \infty$, and this is enough to show that both $\Sigma^{(3)}$ and P_3 are $o\{\mathbb{P}^{(\theta)}(S_m = y)\}$; the result follows. \square

6. The full Martin boundaries

Our aim now is to find, in Case A and Case B, all possible limits of the relativised Green’s functions $k^+(\underline{x}, \underline{y})$ and $k^*(\underline{x}, \underline{y})$. We start with the easier case of M^+ , and we need to introduce $W^\#$, the renewal function in the process of increasing ladder times $(T_k^+, k \geq 0)$ under the measure \mathbb{P}^+ ; thus

$$W^\#(n) = \sum_{r=0}^n \sum_{k=0}^{\infty} \mathbb{P}^+(T_k^+ = r), \quad n \geq 0.$$

THEOREM 6.1. – Assume (4.1) and either (4.2) or (4.3) and (4.4). Then the only possible limits of the relativised Green’s functions $k^+(\underline{x}, \underline{y}^{(r)})$ as $\underline{y}^{(r)} \rightarrow \infty$ through \mathcal{E}_0 are

- (i) the regular functions $h_\mu(\cdot)$ for $0 \leq \mu < \mu_+ = b$ in Case A, and $0 \leq \mu \leq \mu_+$ in Case B;
- (ii) the regular function $\hat{h}_b(\cdot)$ in Case A;
- (iii) the super-regular functions defined in Case B, for $y = 0, 1, 2, \dots$, by

$$\tilde{h}_y(\underline{x}) = \begin{cases} \frac{V(y-x)}{V(y)} & \text{for } 0 \leq x \leq y, \quad l = 0, 1, \dots, \\ 0 & \text{otherwise;} \end{cases}$$

- (iv) the super-regular functions defined, for $m = 1, 2, \dots$, by

$$h_m^\#(\underline{x}) = \begin{cases} \frac{W^\#(m-l-1)}{W^\#(m-1)} & \text{for } 0 \leq l < m, \quad x = 0, 1, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $y^{(r)}/m^{(r)} \rightarrow \mu \leq \infty$, then

$$k^+(\underline{x}, \underline{y}^{(r)}) \rightarrow \begin{cases} \tilde{h}_y(\underline{x}) & \text{if } \mu = 0 \text{ and } y^{(r)} \rightarrow y; \\ h_{\mu \wedge \mu_+}(\underline{x}) & \text{if } y^{(r)} \rightarrow \infty \text{ and } \mu < \mu_+ = b \text{ in Case A}; \\ \hat{h}_b(\underline{x}) & \text{if } \mu = b \text{ in Case A}; \\ h_m^\#(\underline{x}) & \text{if } \mu = \infty \text{ and } m^{(r)} \rightarrow m \text{ in case B.} \end{cases} \quad (6.1)$$

Proof. – Of course if $y^{(r)} \rightarrow y$, then $y^{(r)} = y$ for all sufficiently large r , and since $k^+(\underline{x}, \underline{y}^{(r)}) = v(\underline{y}^{(r)} - \underline{x})/v(\underline{y}^{(r)})$, the first statement in (6.1) follows immediately from Proposition 5.4. If $y^{(r)} \rightarrow \infty$ and $y^{(r)}/m^{(r)} \rightarrow \mu \leq b_0$ then Propositions 5.4, 5.2, and 5.1 show that (5.11) holds. Thus, putting $\tilde{\theta} = (y - x)/(m - l)$, so that $|\theta - \tilde{\theta}| \leq c/m$ for large m , it follows from (5.11) and an argument similar to that in the proof of Corollary 4.2 that

$$\begin{aligned} \frac{v(\underline{y}^{(r)} - \underline{x})}{v(\underline{y}^{(r)})} &= s(\theta)^x t(\theta)^l \left\{ \frac{s(\theta)}{s(\tilde{\theta})} \right\}^{y-x} \left\{ \frac{t(\theta)}{t(\tilde{\theta})} \right\}^{m-l} \frac{v^{(\tilde{\theta})}(\underline{y}^{(r)} - \underline{x})}{v^{(\tilde{\theta})}(\underline{y}^{(r)})} \\ &\rightarrow s(\theta)^x t(\theta)^l = h_\mu(\underline{x}). \end{aligned}$$

If in Case B $y^{(r)}/m^{(r)} \rightarrow \mu > \mu_+$ (or $y^{(r)}/m^{(r)} \rightarrow \mu_+$ and $y^{(r)}/m^{(r)} \geq \mu_+$) we write

$$\begin{aligned} \frac{v(\underline{y}^{(r)} - \underline{x})}{v(\underline{y}^{(r)})} &= s(\mu_+)^x t(\mu_+)^l \frac{\mathbb{P}^+(S_{m-l} = y - x, \tau > m - l)}{\mathbb{P}^+(S_m = y, \tau > m)} \\ &\rightarrow s(\mu_+)^x t(\mu_+)^l = h_{\mu_+}(\underline{x}), \end{aligned}$$

where we have used Proposition 5.1 (with $x = 0$) and Proposition 5.2. In Case A when $y/m \rightarrow b$ we will use again the fact that $v(y, m) = \mathbb{P}(S_m = y, \tau > m)$. If $\underline{x} = (lb, l)$ we put $\tilde{y} = mb - y$, $\tilde{m} = [\tilde{y}/b] + l + 1$, and note that $S_k < 0$ for any $\tilde{m} \leq k \leq m$ is incompatible with $S_m = y$, because it entails

$$S_m < 0 + (m - \tilde{m})b < mb - \tilde{y} = y.$$

Similarly $S_k < 0$ for any $\tilde{m} \leq k \leq m - l$ is incompatible with $S_{m-l} = y - x = y - lb$. So

$$\begin{aligned} v(y, m) &= \sum_{z=0}^{\tilde{m}b} \mathbb{P}(S_{\tilde{m}} = z, \tau > \tilde{m}) \mathbb{P}_z(S_{m-\tilde{m}} = y, \tau > m - \tilde{m}) \\ &= \sum_{z=0}^{\tilde{m}b} \mathbb{P}(S_{\tilde{m}} = z, \tau > \tilde{m}) \mathbb{P}(S_{m-\tilde{m}} = y - z), \end{aligned} \quad (6.2)$$

and in the same way

$$v(y - x, m - l) = \sum_{z=0}^{\tilde{m}b} \mathbb{P}(S_{\tilde{m}} = z, \tau > \tilde{m}) \mathbb{P}(S_{m-l-\tilde{m}} = y - lb - z).$$

Since for $0 \leq z \leq \tilde{m}b$ we have $0 \leq \frac{y-z}{m-\tilde{m}} \rightarrow 0$, we can use Proposition 4.4 to show that, given arbitrary $\varepsilon > 0$,

$$(1 - \varepsilon)p(b)^l \leq \frac{\mathbb{P}(S_{m-\tilde{m}} = y - z)}{\mathbb{P}(S_{m-l-\tilde{m}} = y - lb - z)} \leq (1 + \varepsilon)p(b)^l$$

for $0 \leq z \leq \tilde{m}b$ and all sufficiently large y . Thus $k^+(\underline{x}, \underline{y}) \rightarrow \hat{h}_b(\underline{x})$. As for the case $\underline{x} \notin l^+$, since $\underline{x} \in \mathcal{E}_0$ we have $\mathbb{P}(S_l = x) > 0$. Thus

$$\begin{aligned} v(y, m) &= \mathbb{P}(S_m = y, m \text{ is a ladder epoch}) \\ &\geq \mathbb{P}(S_l = lb)v(y - lb, m - l) + \mathbb{P}(S_l = x)v(y - x, m - l), \end{aligned}$$

and it follows that

$$k^+(\underline{x}, \underline{y})\mathbb{P}(S_l = x) \leq 1 - p(b)^l k^+(lb, l, \underline{y}) \rightarrow 0.$$

Finally we consider the case that m is fixed and $y \rightarrow \infty$. (The case that $m \rightarrow \infty, y \rightarrow \infty$, and $y/m \rightarrow \infty$ is also dealt with by using Proposition 5.2.) First we recall that, by duality,

$$g^{(\mu_+)}(y, m) = \mathbb{P}^+(S_1 \geq 0, S_2 \geq 0, \dots, S_{m-1} \geq 0, S_m = y) = \mathbb{P}^+(\tau > m, S_m = y),$$

so that Lemma 5.3 applies and gives, using duality again,

$$\begin{aligned} g^{(\mu_+)}(y, m) &\sim p^+(y) \sum_{r=1}^m \mathbb{P}^+(\tau > r - 1) \\ &= p^+(y) \sum_{r=1}^m \mathbb{P}^+(r - 1 \text{ is a weak increasing ladder time}) \\ &= p^+(y)W^\#(m - 1). \end{aligned}$$

Since, for $l < m$,

$$k^+(\underline{x}, \underline{y}) = s(\mu_+)^x t(\mu_+)^l \frac{g^{(\mu_+)}(y - x, m - l)}{g^{(\mu_+)}(y, m)}$$

and $p^+(y - x)/p^+(y) \rightarrow 1$, the result follows. \square

The corresponding result for M^* is

THEOREM 6.2. – Assume (4.1) and either (4.2) or (4.3) and (4.4). Then the only possible limits of the relativised Green’s functions $k^*(\underline{x}, \underline{y}^{(r)})$ as $\underline{y}^{(r)} \rightarrow \infty$ through \mathcal{E}_0 are:

- (i) the regular functions $h_\mu^*(\cdot)$ for $0 \leq \mu < \mu_+ = b$ in Case A, and $0 \leq \mu \leq \mu_+$ in Case B;
- (ii) the regular function $\hat{h}_b(\cdot)$ in Case A;

(iii) the super-regular functions defined, in Case B, for $m = 1, 2, \dots$, by

$$h_m^S(\underline{x}) = \begin{cases} \frac{\sum_{j=0}^{m-l-1} \mathbb{P}_x^+(\tau > j)}{\sum_{j=0}^{m-l-1} \mathbb{P}^+(\tau > j)} & \text{for } 0 \leq l < m, x = 0, 1, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $y^{(r)}/m^{(r)} \rightarrow \mu \leq \infty$, then

$$k^*(\underline{x}, \underline{y}^{(r)}) \rightarrow \begin{cases} h_{\mu \wedge \mu_+}^*(\underline{x}) & \text{if } m^{(r)} \rightarrow \infty \text{ and } \mu < \mu_+ = b \text{ in Case A;} \\ \hat{h}_b(\underline{x}) & \text{if } \mu = b \text{ in Case A;} \\ h_m^S(\underline{x}) & \text{if } m^{(r)} \rightarrow m \text{ and } y^{(r)} \rightarrow \infty \text{ in Case B.} \end{cases} \tag{6.3}$$

Proof. – We show first that if y is fixed and $m \rightarrow \infty$ then (6.3) holds with $\mu = 0$. (This argument does not require the existence of a non-trivial exponential family.) In these circumstances we can apply Proposition 5.4 to S and Theorem 7 of [1] to $-S$ to see that, for fixed x, y , and z

$$\sigma \sqrt{2\pi} k^{3/2} v(y - z, k) \rightarrow V(y - z) \quad \text{as } k \rightarrow \infty, \tag{6.4}$$

and

$$\sigma \sqrt{2\pi} k^{3/2} u(y - z, k) \rightarrow U(x - z - 1) \quad \text{as } k \rightarrow \infty. \tag{6.5}$$

Now we note that, in the identity

$$g^*(\underline{x}, \underline{y}) = \sum_{z=0}^{x \wedge y} \sum_{k=0}^{m-l} u(x - z, k) v(y - z, m - l - k), \tag{6.6}$$

which is (2.5) in a different notation, the inner sum is the convolution of $u(x - z, \cdot)$ with $v(y - z, \cdot)$ evaluated at $m - l$. A standard result (e.g., Lemma 1 in [3]) shows that (6.4) and (6.5) together imply that

$$\sigma \sqrt{2\pi} m^{3/2} g^*(\underline{x}, \underline{y}) \rightarrow \sum_{z=0}^{x \wedge y} \{U(x - z - 1)v(y - z) + u(x - z)V(y - z)\}, \tag{6.7}$$

and a few lines of calculations show that the right hand side of (6.7) can also be written as $U(x)V(y)$. Since $g^*(\underline{0}, \underline{y}) = v(y)$ we deduce, using (6.4) again, that $k^*(\underline{x}, \underline{y}) \rightarrow U(x)$, and this is (6.3) with $\mu = 0$.

Next, suppose that $y \rightarrow \infty, m \rightarrow \infty$, and $y/m \rightarrow 0$. Then $y \geq x$ eventually, so we see from (6.6) that it suffices to show that, for fixed x, z , and l

$$\frac{\sum_{k=0}^{m-l} u(x - z, k) v(y - z, m - l - k)}{v(y, m)} \rightarrow \sum_{k=0}^{\infty} u(x - z, k) = u(x - z) \quad \text{as } m \rightarrow \infty. \tag{6.8}$$

Since $x - z$ is fixed, (6.5) shows that $k^{3/2} u(x - z, k) \leq c$ for all $k \geq 1$. Given any $0 < K < \infty$ we write $m_0 = m_0(y, m) = \lceil (K/y)^{2/3} m \rceil$, so that, with $\theta = y/m$,

$$\begin{aligned}
 & \sum_{k=m_0}^{m-l} u(x-z, k)v(y-z, m-l-k) \\
 & \leq c(m_0)^{-3/2} \sum_{k=1}^{m-l} v(y-z, m-l-k) \\
 & = c(m_0)^{-3/2} \sum_{k=1}^{m-l} M(s(\theta))^{m-l-k} v^{(\theta)}(y-z, m-l-k) s(\theta)^{z-y} \\
 & \leq c(m_0)^{-3/2} \frac{M(s(\theta))^m}{s(\theta)^y} \cdot \frac{s(\theta)^z}{M(s(\theta))^l} \mathbb{P}^{(\theta)}(y-z \text{ is a ladder height}) \\
 & \leq c(m_0)^{-3/2} \frac{s(\theta)^z}{M(s(\theta))^l} e^{-mH(\theta)} \leq c \frac{y}{Km^{3/2}} e^{-mH(\theta)} \\
 & \sim c \frac{y}{Km} \mathbb{P}(S_m = y) \sim \frac{c}{K} v(y, m). \tag{6.9}
 \end{aligned}$$

Here, of course, we have used the fact that $s(\theta) \downarrow 1$, Lemma 4.1, and (5.11). To deal with the remaining terms, put $j = k + l$ and $\tilde{\theta} = \frac{y-z}{m-j} = \theta + \gamma$, so that $\gamma = \frac{jy-mz}{m(m-j)}$. Next, we use (5.11) twice to see that, uniformly for $k < m_0$,

$$\begin{aligned}
 \frac{v(y-z, m-j)}{v(y, m)} & \sim \frac{\mathbb{P}(S_{m-j} = y-z)}{\mathbb{P}(S_m = y)} \\
 & = \frac{\exp\{-(m-j)H(\tilde{\theta})\}}{\exp\{-mH(\theta)\}} \cdot \frac{\mathbb{P}(S_{m-j} = y-z)}{\mathbb{P}(S_m = y)} \\
 & \sim \exp\{mH(\theta) - (m-j)H(\tilde{\theta})\} \quad \text{as } m \rightarrow \infty.
 \end{aligned}$$

Recall from (4.5) and (4.6) that H' is increasing and

$$\begin{aligned}
 mH(\theta) - (m-j)H(\tilde{\theta}) & = (m-j) \left\{ H(\theta) - H(\tilde{\theta}) + \frac{j}{m-j} H(\theta) \right\} \\
 & \leq (m-j) \left\{ H(\theta) - H(\tilde{\theta}) + \frac{j\theta}{m-j} H'(\theta) \right\} \\
 & = (m-j) \{ H(\theta) - H(\tilde{\theta}) + \gamma H'(\theta) \} + zH'(\theta). \tag{6.10}
 \end{aligned}$$

Now z is fixed and $H'(\mu) \rightarrow 0$ as $\mu \rightarrow 0$, and from the mean value theorem we see that

$$|H(\theta) - H(\tilde{\theta})| \leq |\gamma| H'(\theta),$$

so recalling the definition of γ we see that the right hand side of (6.10) is bounded by a constant for all $k < m_0$ and sufficiently large m , and tends to zero as $m \rightarrow \infty$ for each fixed k . Hence $v(y-z, m-l-k)/v(y, m)$ is bounded above by a constant for all $k < m_0$ and sufficiently large m , and tends to 1 as $m \rightarrow \infty$ for each fixed k . We can therefore apply dominated convergence to conclude that

$$\frac{\sum_{k=0}^{m_0} u(x-z, k)v(y-z, m-l-k)}{v(y, m)} \rightarrow \sum_{k=0}^{\infty} u(x-z, k) = u(x-z) \quad \text{as } m \rightarrow \infty,$$

and since we can choose K arbitrarily large this, together with (6.9), establishes (6.8).

The case that $\theta = y/m \rightarrow \mu > 0$, (where $\mu < b$ in Case A) is easily dealt with, in virtue of Propositions 5.1 and 5.2. Indeed, it is immediate from (5.2) and (5.4) that, with \mathbb{P}^+ and μ_+ replacing $\mathbb{P}^{(\theta)}$ and θ when $\theta \geq \mu_+$,

$$\begin{aligned} \frac{g^*(\underline{x}, \underline{y})}{g^*(\underline{0}, \underline{y})} &= \frac{\mathbb{P}_x(S_{m-l} = y, \tau > m-l)}{\mathbb{P}(S_m = y, \tau > m)} = s(\theta)^x t(\theta)^l \frac{\mathbb{P}_x^{(\theta)}(S_{m-l} = y, \tau > m-l)}{\mathbb{P}^{(\theta)}(S_m = y, \tau > m)} \\ &\sim s(\mu)^x t(\mu)^l \frac{\mathbb{P}_x^{(\theta)}(\tau = \infty) \mathbb{P}^{(\theta)}(S_{m-l} = y-x)}{\mathbb{P}^{(\theta)}(\tau = \infty) \mathbb{P}^{(\theta)}(S_m = y)} \\ &\rightarrow s(\mu)^x t(\mu)^l \frac{\mathbb{P}_x^{(\mu)}(\tau = \infty)}{\mathbb{P}^{(\mu)}(\tau = \infty)} \\ &= s(\mu)^x t(\mu)^l U^{(\mu)}(x) = h_\mu(x). \end{aligned}$$

Note here that the identity $U^{(\mu)}(x) = \mathbb{P}_x^{(\mu)}(\tau = \infty) / \mathbb{P}^{(\mu)}(\tau = \infty)$ is valid because S drifts to $+\infty$ under $\mathbb{P}^{(\mu)}$ when $\mu > 0$.

Next, we deal with Case A when $y/m \rightarrow b$. Retaining the notation of the previous proof, recall that Eq. (6.2) states that if $\underline{x} = (lb, l)$, then

$$g^*(\underline{0}, \underline{y}) = v(y, m) = \sum_{z=0}^{\tilde{m}b} \mathbb{P}(S_{\tilde{m}} = z, \tau > \tilde{m}) \mathbb{P}(S_{m-\tilde{m}} = y-z).$$

But note now that $S_r < x$ for any $\tilde{m} \leq r \leq m-l$ is incompatible with $S_{m-l} = y-x$, so that

$$\begin{aligned} g^*(\underline{x}, \underline{y}) &= \mathbb{P}_x(S_{m-l} = y, \tau > m-l) \\ &= \sum_{z=0}^{\tilde{m}b} \mathbb{P}_x(S_{\tilde{m}} = x+z, \tau > \tilde{m}) \mathbb{P}_{x+z}(S_{m-l-\tilde{m}} = y, \tau > m-l-\tilde{m}) \\ &= \sum_{z=0}^{\tilde{m}b} \mathbb{P}_x(S_{\tilde{m}} = x+z, \tau > \tilde{m}) \mathbb{P}(S_{m-l-\tilde{m}} = y-x-z) \\ &\geq \sum_{z=0}^{\tilde{m}b} \mathbb{P}(S_{\tilde{m}} = z, \tau > \tilde{m}) \mathbb{P}(S_{m-l-\tilde{m}} = y-x-z). \end{aligned}$$

We can then use Proposition 4.4 to deduce that, for any $\varepsilon > 0$,

$$\liminf k^*(\underline{x}, \underline{y}) \geq \hat{h}_b(\underline{x}) - \varepsilon.$$

But plainly

$$g^*(\underline{0}, \underline{y}) \geq \mathbb{P}(S_l = lb) \mathbb{P}_{lb}(S_{m-l} = y-lb, \tau > m-l) = p(b)^l g^*(\underline{x}, \underline{y}),$$

so that $k^*(\underline{x}, \underline{y}) \leq \hat{h}_b(\underline{x})$.

As for the case $\underline{x} \notin l^+$, note that $\underline{x} \in \mathcal{E}_0$ implies $\mathbb{P}(S_l = x, \tau > l) > 0$. So

$$g^*(\underline{0}, \underline{y}) \geq p(b)^l g^*((lb, l), \underline{y}) + \mathbb{P}(S_l = x, \tau > l)^l g^*(\underline{x}, \underline{y}),$$

and hence

$$k^*(\underline{x}, \underline{y}) \leq \frac{1 - p(b)^l k^*((lb, l), \underline{y})}{\mathbb{P}(S_l = x, \tau > l)} \rightarrow 0.$$

Finally, the result in the case $m(r) \rightarrow m$, $y^{(r)} \rightarrow \infty$ follows immediately from Lemma 5.3. \square

Remark 4. – The situation that $y^{(r)} \rightarrow y$ and $m \rightarrow \infty$ gives the promised examples with

$$\lim_{r \rightarrow \infty} k^*(\cdot, \underline{y}^{(r)}) \neq \Phi(\lim_{r \rightarrow \infty} k^+(\cdot, \underline{y}^{(r)})).$$

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