REMARQUES. Même si nous avions utilisé un mouvement brownien hyperbolique avec dérive (défini dans l'article suivant) ou une généralisation d-dimensionnelle des coordonnées équidistantes, où le rayon r reste défini par

$$\sinh(r) = \frac{x_1}{x_d},$$

nous aurions été incapable de faire apparaître un terme en $\nu/\cosh(r)$ dans la diffusion R. Par contre, nous pourrions atteindre d'autres valeurs de μ demi-entières.

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An explanation of a generalized Bougerol's identity in terms of hyperbolic Brownian motion

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Abstract. In this paper, we state some new results about exponential functionals of Brownian motion, and give an explanation relying on the three dimensional hyperbolic Brownian motion.

0. Introduction.

The aim of this paper is to investigate some relationships between hyperbolic geometry and exponential functionals of Brownian motion, *i.e.* the random variables

$$A_t^{(\nu)} = \int_0^t ds \exp 2(B_s + \nu s), \tag{0.1}$$

where B_t is a linear Brownian motion started at 0; in the case $\nu = 0$, this functional is simply denoted A_t. More precisely, we generalize the following identity obtained by Bougerol [B] and valid for each fixed time $t \geq 0$

$$\sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t}, \tag{0.2}$$

where $(\gamma_t,\, t \geq 0)$ is an independent linear Brownian motion started at 0. It may be worth recalling that Ph. Bougerol was concerned with convolution powers of probabilities on some solvable groups and that his identity is a property of Brownian motion on the Lie group $SO_0(2,1)$, whose solvable part NA is isomorphic to the hyperbolic plane.

In the third paragraph, we state a generalized Bougerol's identity for Brownian motion with drift; hence we don't restrict ourselves to the functional A_t. Contrary to Yor's paper [Y2], we don't use the complicated expression of the conditional density

$$a_t(x,y)\, dy \overset{\text{(def)}}{=} P(A_t \in dy \mid B_t = x)\,, \tag{0.3}$$

since we work directly with Laplace transforms. Hence we begin the paper with a new proof of the known double Laplace transform

$$\int_0^\infty dt \exp\Big(-\frac{\theta^2}{2}t\Big)\, \mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big)\,,$$

see for example Yor [Y2].

Next, we examine some consequences of the Kantorovich-Lebedev inversion formula. The use of Kantorovich-Lebedev transform in probability begun in McKean's paper [McK]. McKean studied the law of $(\tau, |B(\tau)|)$ where τ is the first hitting time of a level a by $\int_0^t B_s\, ds$. As a matter of fact, $(B_t, \int_0^t B_s\, ds)$ has something to do with (B_t, A_t) since the derivative at 0 of $\int_0^t \exp(2\varepsilon B_s)\, ds$ with respect to ε is $2\int_0^t B_s\, ds$.

In the second part, we interpret the generalized Bougerol's identity with the help of hyperbolic geometry in dimension 3. We also give a geometric proof of Dufresne's result on the law of $A_\infty^{(\nu)} = \lim_{t\to+\infty} A_t^{(\nu)}$ when $\nu = (1-n)/2$. In fact, a better way of recovering this result for every negative ν is to deduce it from the semigroup of a planar hyperbolic Brownian motion with drift. This can be done via the spectral analysis of the operator

$$y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + \Big(\frac{1}{2}+\nu\Big)\, y\,\frac{\partial}{\partial x}\,. \tag{0.4}$$

This yields

$$\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big) = \sum_{0<n<-\nu/2} \exp\big(t\,n(\nu+n)\big)\frac{2(\nu+2n)}{n!\,(n+\nu-1)}$$

$$\cdot \Big(\frac{u}{2}\Big)^{-\nu} K_{-\nu-2n}(u)$$
$$+\frac{1}{2\pi^2}\int_0^\infty e^{-(\nu^2+s^2)t/2}\, s\sinh(\pi s)\Big|\Gamma\Big(\frac{\nu+is}{2}\Big)\Big|^2 \tag{0.5}$$
$$\cdot\Big(\frac{u}{2}\Big)^{-\nu} K_{is}(u)\, ds\,.$$

For instance, we refer to C. Monthus [M]. M. Yor pointed out to us that similar computations already appeared in [W].

I. The generalized Bougerol's Identity.

I.1. Determination of the law of $A_t^{(\nu)}$.

We first recall some basic results about the laws of exponential Brownian functionals found in De Schepper and al. [SGD] and M. Yor [Y2]. For (θ,ν) reals, we put $\mu = \sqrt{\theta^2+\nu^2}$ and we define the function $G_\mu : \mathbb{R}^2 \to \mathbb{R}^+$ by

$$G_\mu(u,v) = \begin{cases} 2\, I_\mu(u) K_\mu(v)\,, & \text{if } u \leq v\,,\\ 2\, I_\mu(v) K_\mu(u)\,, & \text{if } v \leq u\,. \end{cases} \tag{1.0}$$

Theorem 1. *The following formula holds*

$$\int_0^\infty dt \exp\Big(-\frac{\theta^2}{2}t\Big)\, \mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big) = \int_{-\infty}^{+\infty} dy\, e^{\nu y} G_\mu(u, ue^y)\,. \tag{1.1}$$

To be quickly convinced of the validity of this result, we give below a new proof relying on the skew product representation of planar Brownian motion.

In fact, this proof which relies on the computation of a Green function (see [HW, Section 5]) avoids new calculations because the work on Green functions and winding numbers was already done in [IMcK].

Let $(Z_t,\, t \geq 0)$ be a complex Brownian motion starting from $Z_0 = a \neq 0$; we then have

$$Z_t = R_t \exp(i\,\theta_t) \equiv R_t \exp(i\,\beta_{H_t}) \qquad \text{with} \qquad H_t = \int_0^t \frac{ds}{R_s^2}\,, \tag{1.2}$$

where $R_t = |Z_t|$, $(\theta_t,\, t \geq 0)$ is the winding number of Z around zero, and $(\beta_u,\, u \geq 0)$ is a standard Brownian motion independent of R. Here

R is obviously a two dimensional Bessel process starting from $|a|$. We recall that the law of θ_t conditionally on R_t (see Revuz and Yor [RY]; also Yor [Y2]) is characterized by

$$\mathbb{E}_a\left(\exp\left(i\,\alpha(\theta_t-\theta_0)\right)\mid R_t=\rho\right)=\frac{I_{|\alpha|}}{I_0}\left(\frac{|a|\rho}{t}\right). \tag{1.3}$$

We also recall that (H_u) is the inverse process of (A_t). In order to simplify the notation, we call $F(\theta,u)$ the left side of the relation (1.1). By virtue of the Cameron-Martin relation, we can write, if $\mu=\sqrt{\theta^2+\nu^2}$

$$\begin{aligned}F(\theta,u)&=\int_0^\infty dt\,e^{-\theta^2t/2}\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t+\nu B_t-\frac{\nu^2}{2}t\Big)\Big)\\&=\int_0^\infty dt\,e^{-\mu^2t/2}\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t+\nu B_t\Big)\Big)\\&=\int_0^\infty dt\,\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t+\nu B_t+i\,\mu\beta_t\Big)\Big),\end{aligned}$$

where $(\beta_t,\,t\geq 0)$ is another Brownian motion independent of $(B_t,\,t\geq 0)$. The change of variables $A_t=v$ gives

$$\begin{aligned}F(\theta,u)&=\int_0^\infty dv\,e^{-u^2v/2}\mathbb{E}\big(e^{i\mu\theta_v}(R_v)^{(\nu-2)}\big)\\&=\int_0^\infty dz\,z^{\nu-1}\int_0^\infty\frac{dv}{v}\,e^{-u^2v/2}\exp\Big(-\frac{1+z^2}{2v}\Big)I_\mu\Big(\frac{z}{v}\Big),\end{aligned}$$

where we have conditioned on R_v and used relations (1.2) and (1.3). The integral in dv is precisely equal to $G_\mu(u,uz)$ and we only need to make the further change of variable $z=\exp(y)$ to find the result (1.1).

I.2. Some consequences of formula (1.1).

I.2.1. Bougerol's identity plays an important role in several domains. We recall, for example, the work of C. Monthus [M], where different applications to physics are developed. As another application, we show that the density of A_t may be deduced from (0.2). Taking Fourier transforms, we obtain

$$\mathbb{E}\Big(\exp\Big(-\frac{\xi^2}{2}A_t\Big)\Big)=\sqrt{\frac{2}{\pi t}}\int_0^\infty dx\exp\Big(-\frac{x^2}{2t}\Big)\cos(\xi\sinh(x)). \tag{1.4}$$

With the help of the Fourier-Plancherel formula, we may write

$$\mathbb{E}\Big(\exp\Big(-\frac{\xi^2}{2}A_t\Big)\Big)=\frac{2}{\pi}\int_0^{+\infty}dy\exp\Big(-\frac{y^2}{2}t\Big)\cosh\Big(\frac{\pi}{2}y\Big)K_{iy}(\xi).$$

Thus the preceding Laplace transform is also a Kantorovich-Lebedev transform; this fact will be generalized later to the drifted case in paragraph 4. Now, using again Fourier-Plancherel formula (but differently), we find that

$$\begin{aligned}&\mathbb{P}_0(A_t\in dx)\\&\quad=\frac{dx}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}dz\,\frac{\cosh(z)}{\sqrt{2\pi x^3}}\,e^{-\cosh^2(z)/2x}\,e^{-(z+i\pi/2)^2/2t}.\end{aligned} \tag{1.5}$$

Corollary. (*A local limit result*).

$$\lim_{t\to+\infty}\sqrt{2\pi t}\,\mathbb{P}_0(A_t\in dx)=\frac{1}{x}\exp\Big(-\frac{1}{2x}\Big)\,dx.$$

PROOF. Perform the change of variable $y=\sinh(z)$ in (1.5).

An open question about the law of the exponential functional A_t of a Brownian motion, is whether it is determined by its integral moments. In order to study this problem, we have considered a sequence of measures $\mu_n(dx)$ defined by

$$\begin{aligned}&\mu_n(dx)\\&\quad=\frac{dx}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}dz\,\frac{\cosh(z)}{\sqrt{2\pi x^3}}\,e^{-\cosh^2(z)/2x}\,e^{-(z+i(4n+1)\pi/2)^2/2t},\end{aligned} \tag{1.6}$$

for any integer n. It is easy to show that for every pair (m,n) of integers, we have

$$\mathbb{E}\big((A_t)^m\big)=\int_0^\infty x^m\mu_n(dx).$$

We now wish to show, from the injectivity of the Laplace transform, that the law of the functional A_t is different from $\mu_n(dx)$ if $n\neq 0$. Indeed, the Laplace transform

$$z\mapsto\int_0^{+\infty}e^{-zx}\mu_n(dx)=\int_{-\infty}^{+\infty}e^{-\sqrt{2z}\cosh(a)}e^{-(a+i(4n+1)\pi/2)^2/2t}\,\frac{da}{\sqrt{2\pi t}}$$

defines an analytic function on the vertical strip $\operatorname{Re} z \geq 1$, but this quantity can be expressed as

$$\int_1^{+\infty} e^{-(\cosh^{-1}(y)+i(4n+1)\pi/2)^2/2t} e^{-\sqrt{2z}y} \frac{dy}{\sqrt{2\pi t(y^2-1)}}.$$

The last step is the study of the positivity of $\mu_n(dx)$ for $n > 1$; unfortunately, numerical simulations show that the sign of the density is not constant.

REMARKS. a) The residue formula applied to the function

$$f(z) = \cosh(z) e^{-\cosh^2(z)/2x} e^{-(z+i(4n+1)\pi/2)^2/2t}$$

on a rectangular loop with edges $\operatorname{Re} z = 0$ or 2π, $\operatorname{Im} z = \pm R$ doesn't lead to the absurdity $\mu_n = \mu_{n+1}$ since the contributions of the vertical segments do not vanish when R increases to infinity.

b) Denote by f the density function of A_1. We obtain easily from (1.5)

$$-\int_0^{+\infty} \frac{\log(f(x^2))}{1+x^2}\, dx = +\infty.$$

Unfortunately G. D. Lin's [Li] new criteria about the moment determinacy (Theorem 4) is of no avail here since $-xf'(x)/f(x)$ has a finite limit when x increases to infinity.

I.2.2. As an application of formula (1.1) we compare with Monthus computations [M]. In the paper [OMM], the authors have shown the validity of formula (1.5), in the case $\nu = 0$, with the help of the Fokker-Planck equation. In order to generalize formula (1.5), C. Monthus [M] has obtained the following

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big) &= \sum_{0<n<-\nu/2} e^{tn(\nu+n)} \frac{2(\nu+2n)}{n!\,(n+\nu-1)} \Big(\frac{u}{2}\Big)^{-\nu} K_{-\nu-2n}(u) \\ &+ \frac{1}{2\pi^2}\int_0^\infty e^{-(\nu^2+s^2)t/2} \Big|\Gamma\Big(\frac{\nu+is}{2}\Big)\Big|^2 \Big(\frac{u}{2}\Big)^{-\nu} K_{is}(u)\, s\sinh(\pi s)\, ds.\end{aligned} \tag{0.5}$$

Now, we show how to obtain the preceding formula with the help of (1.5). We mention that this approach is carried out only in the case of a positive drift in which case the discrete sum over n such that $0 < n < -\nu/2$ disappears

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big] &= \frac{1}{2\pi^2}\int_0^\infty e^{-(\nu^2+s^2)t/2}\Big|\Gamma\Big(\frac{\nu+is}{2}\Big)\Big|^2\Big(\frac{u}{2}\Big)^{-\nu} \\ &\quad\cdot K_{is}(u)\, s\sinh(\pi s)\, ds.\end{aligned} \tag{1.7}$$

We begin with the fact that the function G_μ defined by (1.0) enjoys the following integral representation, see [DP, formula 11.279]

$$\frac{\pi^2}{4} G_\mu(u, ue^y) = \int_0^\infty \frac{d\rho\, \rho}{\rho^2+\mu^2} \sinh(\pi\rho) K_{i\rho}(u) K_{i\rho}(ue^y). \tag{1.8}$$

Hence, it is easy to obtain

$$\begin{aligned}\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big) &= \frac{2}{\pi^2}\int_0^\infty d\rho\, e^{-(\rho^2+\nu^2)t/2}\sinh(\pi\rho) K_{i\rho}(u) \\ &\quad\cdot\int_{-\infty}^{+\infty} dy\, e^{\nu y} K_{i\rho}(ue^y).\end{aligned} \tag{1.9}$$

From the well-known result

$$\int_0^\infty dx\, x^{\nu-1} K_{i\rho}(x) = 2^{\nu-2}\Big|\Gamma\Big(\frac{\nu+i\rho}{2}\Big)\Big|^2, \qquad \nu \geq 0, \tag{1.10}$$

we find the sought formula (1.7). It is convenient to remark the importance of the sign of the drift ν. As a matter of fact, the passage from (1.8) to (1.9) requires the use of Fubini's Theorem, easily justified. In the case $\nu < 0$, one cannot use this argument since the integral with respect to dy in (1.9) diverges.

I.3. The generalized Bougerol's Identity.

We give here a new representation of the conditional law of A_t given B_t. The following elementary proposition will help us to obtain a new identity in law.

Lemma 2.

i) *The law of the functional* $A_t^{(\nu)}$ *is characterized by: for all* $u \geq 0$,

$$\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(\nu)}\Big)\Big)$$

$$(1.11)\qquad = e^{-\nu^2 t/2}\int_{\mathbb{R}} dx\, e^{\nu x}\int_{|x|}^{+\infty} dz\,\frac{z}{\sqrt{2\pi t^3}}\,e^{-z^2 t/2}J_0(u\phi(x,z))\,,$$

where we have denoted

$$(*)\qquad \phi(x,z) = \sqrt{2\exp(x)\cosh(z) - \exp(2x) - 1}\,.$$

for $z \geq |x|$.

ii) *In particular, for* $u \geq 0$ *and* $x \in \mathbb{R}$, *we have*

$$(1.12)\qquad \begin{aligned}&\exp\Big(-\frac{x^2}{2t}\Big)\mathbb{E}\Big(\exp-\Big(\frac{u^2}{2}A_t\Big)/B_t = x\Big)\\ &\quad = \int_{|x|}^{+\infty} dz\,\frac{z}{t}\,e^{-z^2 t/2}J_0(u\phi(x,z))\,,\end{aligned}$$

PROOF. To prove the assertion i), we need to invert the double Laplace transform in (1.1). We may do this directly by using the following integral representation (see Lebedev [L, problem 8, p. 140])

$$(1.13)\qquad \begin{aligned}&I_\mu(x)K_\mu(y)\\ &\quad = \frac{1}{2}\int_{\log(y/x)}^{\infty} dr\, e^{-\mu r}J_0\big(\sqrt{2\cosh(r)\,xy - x^2 - y^2}\,\big)\,,\end{aligned}$$

with $y \geq x$. We get the second assertion with the help of the Cameron-Martin relation.

In order to give an interpretation to the preceding lemma, we introduce the following classical representation of the Bessel function of the first kind

$$(1.14)\qquad J_0(z) = \frac{1}{\pi}\int_{-1}^{+1}\frac{dr}{\sqrt{1-r^2}}\cos(zr)\,.$$

Let Z be an arcsine distributed variable. Then we have, for all real ξ

$$(1.15)\qquad J_0(\xi) = \mathbb{E}\big(\exp(i\xi(2Z-1))\big)\,,$$

In the following proposition, we present a generalization of Bougerol's identity which, in fact is an interpretation of Lemma 2.

Proposition 3. *We denote by* $(B_t^\nu, t \geq 0)$ *a Brownian motion with drift* ν. *Let* $(\gamma_t, t \geq 0)$ *be a standard Brownian motion independent of* B^ν, $(R_t, t \geq 0)$ *a two-dimensional Bessel process started at zero and* Z *an arcsine variable such that* B^ν, R *and* Z *are independent. Let* ϕ *be the function defined by*

$$(*)\qquad \phi(x,z) = \sqrt{2\,e^x\cosh(z) - e^{2x} - 1}\,,$$

for $z \geq |x|$. *Then, we have for any fixed time* t

$$(1.16)\qquad \gamma_{A_t^\nu} \overset{\text{(law)}}{=} (2Z-1)\,\phi\big(B_t^\nu, \sqrt{R_t^2 + (B_t^\nu)^2}\,\big)\,,$$

In particular if we put $\nu = 0$, *we recover Bougerol's identity in the form*

$$(1.17)\qquad \gamma_{A_t} \overset{\text{(law)}}{=} (2Z-1)\,\phi\big(B_t, \sqrt{R_t^2 + B_t^2}\big) \overset{\text{(law)}}{=} \sinh(B_t)\,.$$

I.4. The formula (1.1) as a Kantorovich-Lebedev transform.

In this paragraph, we write (1.1) as a Kantorovich-Lebedev transform; hence the inverse formula (†) below enables us to extend formula (1.4) to the drifted case. Finally we give another extension of Bougerol's identity, which is more developed in [ADY].

I.4.1. Although the formula (1.12) about the conditional density of A_t given B_t has nice consequences, it does not enable us to verify the simple identity in law

$$(1.18)\qquad \exp(-2B_t)\,A_t \overset{\text{(law)}}{=} A_t\,,$$

due to the independence of increments of B. It may be helpful to consider this fact as a consequence of the Kantorovich-Lebedev inversion formula. Below, we recall the basic results about this transform found in Lebedev [L] or [DP]. Given a regular function $f : \mathbb{R}^+ \to \mathbb{R}$ (see [L] for the hypotheses on f), its Kantorovich-Lebedev transform is defined on $\mathbb{R}^+$ by

$$F(\tau) = \int_0^{+\infty} f(x)\, K_{i\tau}(x)\, dx\,.$$

Its inverse may be defined by the integral formula

$$(\dagger) \qquad f(x) = \frac{2}{x\pi^2} \int_0^{+\infty} \tau \sinh(\pi\tau)\, K_{i\tau}(x)\, F(\tau)\, d\tau\,,$$

which is valid at every continuity point in $\mathbb{R} - \{0\}$ of the function f. Hence,

$$F(\tau) = \frac{2}{\pi^2} \int_0^{+\infty} \frac{dx}{x} K_{i\tau}(x) \int_0^{+\infty} d\rho\, F(\rho)\, \rho \sinh(\pi\rho)\, K_{i\rho}(x)\,.$$

I.4.2. Let us combine both (1.1) and (1.8). The left hand side of (1.1) is

$$\int_0^{\infty} dt \exp\Big(-\frac{(\theta^2+\nu^2)\,t}{2}\Big)\mathbb{E}\Big(\exp\Big(\nu B_t + \frac{u^2}{2} A_t\Big)\Big)\,,$$

whereas the right side is, by (1.8), equal to

$$\frac{4}{\pi^2}\int_{-\infty}^{+\infty} dy \exp(\nu y) \int_0^{+\infty} \frac{\rho\, d\rho}{\rho^2+\theta^2+\nu^2} \sinh(\pi\rho)\, K_{i\rho}(u)\, K_{i\rho}(u \exp(y))\,.$$

Hence, inverting the Laplace transform in t, we obtain that for every real c,

$$\mathbb{E}\Big[\exp\Big(-\frac{u^2}{2} A_t\Big)\Big| B_t = c\Big] p_t(c)$$

$$(1.19) \qquad = \frac{2}{\pi^2}\int_0^{\infty} \rho\, e^{-\rho^2 t/2} \sinh(\pi\rho)\, K_{i\rho}(u)\, K_{i\rho}(u\exp(c))\, d\rho\,,$$

where we have denoted by p_t the probability density of B_t. Fix now $z > 0$ and consider f such that

$$F(\rho) = \exp\Big(-\frac{\rho^2 t}{2}\Big) K_{i\rho}(z)\,.$$

Now, according to the Kantorovich-Lebedev inversion formula ($\dagger$), we get

$$(1.20) \qquad \mathbb{E}\Big(\exp\Big(-\frac{z^2}{2} e^{-2B_t} A_t\Big) K_{i\rho}(z e^{-B_t})\Big) = e^{-\rho^2 t/2} K_{i\rho}(z)\,.$$

Using time reversal, one can rewrite (1.20) in the form

$$(1.21) \qquad \mathbb{E}\Big(\exp\Big(-\frac{z^2}{2} A_t\Big) K_{i\rho}(z e^{B_t})\Big) = e^{-\rho^2 t/2} K_{i\rho}(z)\,,$$

which is readily confirmed by observing that the process $(M_t, t \geq 0)$ defined by

$$M_t = \frac{K_{i\rho}(z e^{B_t})}{K_{i\rho}(z)} \exp\Big(-\frac{z^2}{2} A_t + \frac{\rho^2}{2} t\Big)\,,$$

is a true martingale.

I.4.3. REMARK. As a matter of fact, (1.18) may be viewed as a direct consequence of the symmetry in z and u of the formula

$$p_t\Big(\log\Big(\frac{z}{u}\Big)\Big)\mathbb{E}\Big[e^{-u^2 A_t/2}\Big| B_t = \log\Big(\frac{z}{u}\Big)\Big]$$

$$= \frac{2}{\pi^2}\int_0^{\infty} \rho\, e^{-\rho^2 t/2} \sinh(\pi\rho)\, K_{i\rho}(u)\, K_{i\rho}(z)\, d\rho\,.$$

As a consequence of the preceding computations, we state the following

Proposition 4. *For every real a, the following identity in law holds*

$$(1.22) \qquad \gamma_{A_t} + \sinh(a)\, e^{B_t} \overset{(\text{law})}{=} \sinh(B_t + a)\,,$$

where $(\gamma_u, u \geq 0)$ is an auxiliary Brownian motion independent of B.

PROOF. With the help of Fubini's Theorem and the classical formula

$$(1.23) \qquad \int_0^{\infty} dx\, K_{ix}(y) \cos(ax) \cosh\Big(\frac{\pi}{2} x\Big) = \frac{\pi}{2} \cos(y \sinh(a))\,,$$

we obtain

$$\mathbb{E}\Big(\cos\Big(-\frac{z^2}{2} A_t\Big) \cos(z \sinh(a) e^{B_t})\Big)$$

$$(1.24) \qquad = \frac{2}{\pi}\int_0^{\infty} d\rho\, e^{-\rho^2 t/2} K_{i\rho}(z) \cosh\Big(\frac{\pi}{2}\rho\Big) \cos(a\rho)\,.$$

The last integral may be evaluated by the use of Fourier-Plancherel formula; we find

$$\mathbb{E}\Big(\cos\Big(-\frac{k^2}{2}A_t\Big)\cos\big(k\sinh(a)e^{B_t}\big)\Big) = E\big(\cos\big(k\sinh(B_t+a)\big)\big),$$

we conclude by the injectivity of the Fourier transform.

II. Links with hyperbolic Geometry.

II.0. Background on hyperbolic Brownian motions.

The n-dimensional half-space $\mathbb{H}_n = \mathbb{R}^{n-1}\times\mathbb{R}^+$ is equipped with its hyperbolic metric

$$ds^2 = x_n^{-2}\sum_{i=1}^{n} dx_i^2 \tag{2.1}$$

and its Laplace-Beltrami $\Delta_{\mathbb{H}_n}$ operator.

The hyperbolic Brownian motion infinitesimal generator is

$$\frac{1}{2}\Delta_{\mathbb{H}_n} = \frac{1}{2}x_n^2\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \Big(1-\frac{n}{2}\Big)x_n\frac{\partial}{\partial x_n}. \tag{2.2}$$

More precisely, if $\mathbf{B}$ denotes a n-dimensional Euclidean Brownian motion, we call $\mathbf{B}$-hyperbolic Brownian motion started at $p = (x_1,\dots,x_n)$ in $\mathbb{H}_n$ the unique strong solution of the following system

$$\begin{cases} X_t^{(i)} = x_i + \displaystyle\int_0^t X_s^{(n)} dB_s^{(i)}, & 1\le i\le n-1,\\ X_t^{(n)} = x_n + \displaystyle\int_0^t X_s^{(n)} dB_s^{(n)} + \Big(1-\frac{n}{2}\Big)\int_0^t ds\, X_s^{(n)}. \end{cases} \tag{2.3}$$

We find that $X_t^{(n)} = x_n \exp\big(B_t^{(n)} + \nu t\big)$ if $\nu = (1-n)/2$ and moreover there exists a $(n-1)$-dimensional Brownian motion $(\gamma_t^{(1)},\gamma_t^{(2)},\dots,\gamma_t^{(n-1)})$ independent from $B^{(n)}$ such that

$$\begin{aligned} X_t = \big(&x_1+\gamma^{(1)}(A_t^{(\nu)}), x_2+\gamma^{(2)}(A_t^{(\nu)}),\dots,\\ &x_{n-1}+\gamma^{(n-1)}(A_t^{(\nu)}), x_n\exp\big(B_t^{(n)}+\nu t\big)\big). \end{aligned} \tag{2.4}$$

More generally, if we replace $1-n/2$ by $\alpha+1-n/2$ the process $(X_t^{(\alpha)}, t\ge 0)$ is called a n-dimensional hyperbolic Brownian motion with drift α. Thus, for any real number ν, $\gamma^{(1)}(A_t^{(\nu)})$ is the horizontal component of a planar hyperbolic Brownian motion with drift $\alpha = 1/2+\nu$.

Note that Fourier inversion on $\mathbb{H}_n$ in rectangular coordinates can be deduced from the Kantorovich-Lebedev inversion formula (if $n=2$ see [T, p. 138]).

II.1. An explanation of Proposition 3.

Proposition 5. *Let $(X_t^{(\alpha)} = (x_t, y_t, z_t))$ be a 3-dimensional hyperbolic Brownian motion with drift $\alpha = 1-\mu$ started at $(0,0,1)$. Suppose that the third component satisfies the equation*

$$dz_t = z_t\, dB_t + \Big(\frac{1}{2}-\mu\Big) z_t\, dt.$$

Let D_t be the hyperbolic distance between $X_t^{(\alpha)}$ and $X_0^{(\alpha)}$, i.e.

$$D_t = \cosh^{-1}\Big(\frac{1}{2}\Big(\frac{x_t^2+y_t^2}{z_t} + z_t + \frac{1}{z_t}\Big)\Big). \tag{2.5}$$

Then, the pair (B_t, D_t) has the same law as $\big(B_t, \sqrt{R_t^2 + (B_t-\mu t)^2}\big)$ for every fixed time $t\ge 0$, where $(R_t, t\ge 0)$ is an independent Bessel process of dimension two started at 0.

REMARKS.

a) The function ϕ introduced in Lemma 2 appears again here since

$$\sqrt{x_t^2+y_t^2} = \phi(\log(z_t), D_t).$$

b) This result, like Bougerol's identity, is only valid for fixed times. For instance, if $\mu = 0$, $(D_t, t\ge 0)$ is not at all a 3-dimensional Bessel process.

c) Hence, we partially recover and only at fixed times that the process $(D_t, t\ge 0)$ has the same law as a Bessel process with drift as defined by Pitman and Rogers [PR].

d) In the sequel, we interpret the identity (1.12) as a Hankel transform of index 0 or equivalently as a radial Fourier transform in dimension two. Thus, our geometric interpretation critically relies on the rôle of the three dimensional hyperbolic space. We are unable to give a significant generalization to other dimensions.

Let us begin by explaining how Proposition 3 is a consequence of Proposition 5.

First step. If Θ denotes a uniform random variable on the circle S^1, then V and the projection of Θ on an axis have the same distribution. Hence the result contained in Proposition 2 is in fact a two-dimensional one: it should be read as

$$\hat{R}(A_t^{(\mu)}) \overset{(\text{law})}{=} \phi\Big(B_t - \mu t, \sqrt{R_t^2 + (B_t - \mu t)^2}\Big), \tag{2.6}$$

where $(\hat{R}_u,\, u \geq 0)$ is a two dimensional Bessel process independent from B.

Second step. Proposition 5 *implies* (2.6). Since the Euclidean distance between X_t and the (Oz) axis can be realized as $\hat{R}(A_t^{(\mu)})$ where $\hat{R}$ is a two dimensional Bessel process independent from B, we have

$$2\cosh(D_t) = \frac{1}{z_t}\,\hat{R}^2(A_t^{(\mu)}) + z_t + \frac{1}{z_t}\,.$$

Moreover $z_t = \exp(B_t - \mu t)$, thus $\hat{R}(A_t^{(\mu)}) = \phi(B_t - \mu t, D_t)$. By applying our Proposition 4, the result follows.

PROOF OF PROPOSITION 5. By Girsanov theorem, we only have to establish the result for a single μ. From now on, we choose $\mu = 0$, hence $(z_t,\, t \geq 0)$ is a geometric Brownian motion, $z_t = \exp(B_t)$.

On one hand, the conditional density $P(\sqrt{R_t^2 + x^2} \in dy / B_t = x)$ is

$$\mathbf{1}_{[|x|,+\infty[}(y)\exp\Big(\frac{x^2 - y^2}{2t}\Big)\frac{y}{t}\,dy$$

i.e. a two dimensional Bessel law at time t conditioned to be greater than $|x|$. On the other hand, we show that the conditional density $k_t(x,y) \overset{(\text{def})}{=} P(S \in dy / B_t = x)$ is equal to the former if we set $S \overset{(\text{def})}{=} \cosh^{-1}(\exp(-x) R^2(A_t^{(0)})/2 + \cosh(x))$.

We first deduce $k_t(x,y)$ from formula (1.12)

$$\mathbb{E}\Big(\exp\Big(-\frac{u^2}{2}A_t^{(0)}\Big) \Big/ B_t = x\Big) = \int_{|x|}^{\infty} dz\,\frac{z}{t}\exp\Big(\frac{x^2 - z^2}{2t}\Big) J_0(u\phi(x,z))\,.$$

By an obvious change of variable, we rewrite the right hand side as a Hankel transform

$$\int_0^{\infty} J_0(ua) f(a)\,da$$

with

$$f(a) = \frac{\exp(-x)}{t}\,\frac{z(a)}{\sinh z(a)}\exp\Big(\frac{-z^2(a)}{2t}\Big)\exp\Big(\frac{x^2}{2t}\Big)$$

and

$$z(a) = \cosh^{-1}\Big(\frac{1}{2}\,a^2\exp(-x) + \cosh(x)\Big)\,.$$

Since $\int_0^{+\infty}\sqrt{a}\,|f(a)|\,da$ is finite, the Hankel inversion formula with $\nu = 0$ (see [L, identity (5.14.11)]) gives

$$f(r) = \int_0^{+\infty} y\,J_0(yr)\,dy \int_0^{+\infty} J_0(ya) f(a)\,da\,.$$

Hence, $k_t(x,y)$ is identified with $f(y)$ and we obtain $P(S \in dy / B_t = x)$.

II.2. A geometric explanation of Dufresne's result.

We aim at giving a geometric explanation, at least in some special cases, of the following result

Theorem (Dufresne, [D]). *Let* (β_s) *a linear Brownian motion started at* 0. *If* ν *is a negative real number*

$$A_\infty^{(\nu)} = \int_0^{+\infty} \exp(2\beta_s + 2\nu s)\,ds$$

has the same law as $1/2Z_{-\nu}$, *where* $Z_{-\nu}$ *denotes a gamma variable with parameter* $-\nu$, *i.e.*

$$\mathbb{P}(Z_{-\nu} \in ds) = \frac{s^{-\nu-1}}{\Gamma(-\nu)}\exp(-s)\,\mathbf{1}_{\mathbb{R}^+}(s)\,ds\,.$$

Dufresne's complicated proof based on discrete approximations was simplified by M. Yor [Y1] who showed that this theorem is another formulation of the distribution of last exit times for transient Bessel processes.

We give here a new proof relying on the exit distribution of the hyperbolic Brownian motion. In fact, we are only able to prove for some values of ν the following equivalent form

Proposition 6. *If $2\nu = 1-n$, with n an integer greater than or equal to 2 then, for every positive real λ,*

$$\mathbb{E}_0\Big(\exp\Big(-\frac{\lambda^2}{2}A_\infty^{(\nu)}\Big)\Big) = \frac{2}{\Gamma(-\nu)}\Big(\frac{\lambda}{2}\Big)^{-\nu} K_\nu(\lambda) \tag{2.7}$$

REMARK. For instance, $A_\infty^{(-1/2)}$ follows the stable law of index 1/2, since

$$K_{-1/2}(\lambda) = K_{1/2}(\lambda) = e^{-\lambda}\sqrt{\frac{\pi}{2\lambda}}\,.$$

PROOF. Let $(X_t,\, t \geq 0)$ be a hyperbolic Brownian motion in $\mathbb{H}_n$ started at $(0,\dots,0,1)$. From the representation (2.4), when t increases to infinity,

$$X_t \xrightarrow{\text{a.s.}} X_\infty = (\gamma^{(1)}(A_\infty^{(\nu)}),\dots,\gamma^{(n-1)}(A_\infty^{(\nu)}),0)\,,$$

where $(\gamma^{(1)},\dots,\gamma^{(n-1)},\beta)$ is a Brownian motion in $\mathbb{R}^n$. Thus the projection X'_∞ of X_∞ on the hyperplane $\{x_n = 0\}$ is isotropic and is characterized by its Fourier transform

$$F(\|\mathbf{u}\|) = \mathbb{E}\Big(\exp\Big(-\frac{\|\mathbf{u}\|^2}{2}A_\infty^{(\nu)}\Big)\Big)$$

where $\mathbf{u}$ is an arbitrary $n-1$ dimensional vector. On the other hand, the law of X'_∞ is easily found if we consider the spherical model of the hyperbolic space, that is to say the open ball D of radius 1 equipped with the metric

$$ds^2 = \frac{4}{(1-\rho^2)^2}\sum_{i=1}^n dx_i^2$$

where $\rho = \|x\|$. Let J be the inversion with southern pole defined on

$\mathbb{R}^n - \{(0,\dots,0,-1)\}$ by

$$\begin{aligned} J(y_1,\dots,y_n) &= (x_1,\dots,x_n)\\ &= (0,\dots,0,-1)\\ &\quad + 2\Big(\sum_{k=1}^{n-1} y_k^2 + (y_n+1)^2\Big)^{-1}(y_1,\dots,y_{n-1},y_n+1)\,.\end{aligned}$$

The J-image of the spherical model is exactly the half space one, thus the hyperbolic Brownian motion X is the J-transform of the D-hyperbolic motion. See also in [H, p. 226] the equivalence of three models of $\mathbb{H}_n$. Now, by rotational invariance of the sphere, the law of X'_∞ is the image of the uniform measure on ∂B, a radial distribution on the hyperplane $\{x_n = 0\}$.

A straightforward computation shows that

$$r^2 = \sum_{k=1}^{n-1} x_k^2 = \operatorname{cotg}^2\Big(\frac{\theta}{2}\Big)$$

if $y_n = \cos(\theta)$. Since the density of the colatitude θ in $]0,\pi[$ is proportional to $\sin^{n-2}(\theta)$, the Euclidean norm of $J(X_\infty)$ has the density

$$f_n(r) = c_n\Big(\frac{1}{1+r^2}\Big)^{n-1} r^{n-2} \qquad \text{with} \qquad c_n = \frac{2^{n-1}\,\Gamma\big(\frac{n}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{n-1}{2}\big)}\,.$$

Recall that the characteristic function of an isotropic d-dimensional density k with respect to $d\rho$ is radial and given by the formula

$$F(\mathbf{v}) = \Gamma\Big(\frac{d}{2}\Big)\int_0^{+\infty}\Big(\frac{\rho\,\|\mathbf{v}\|}{2}\Big)^{1-d/2} J_{d/2-1}(\rho\,\|\mathbf{v}\|)\,k(\rho)\,d\rho\,, \tag{2.8}$$

where $J_{d/2-1}$ is the Bessel function of index $d/2-1$.

Then (2.7) is obtained with the help of (2.8) where $k(\rho) = f_n(\rho)$ and $d = n-1$. To finish the proof, we use the identity ([L, Example 3, p. 133]) with $a = n/2 - 3/2$

$$\int_0^{+\infty}\frac{\rho^{a+1}J_a(\rho u)}{(\rho^2+1)^{2a+2}}\,d\rho = \frac{u^{2a+1}}{2^{2a+1}\Gamma(2a+2)}K_{a+1}(u)$$

and the duplication formula for the gamma function.

REMARKS. a) Recall that the characteristic function of the Cauchy isotropic law in $\mathbb{R}^d$ is

$$F(\mathbf{u}) = \exp\left(-\|\mathbf{u}\|\right).$$

Hence, the density of the norm is proportional to

$$\left(\frac{1}{1+r^2}\right)^{(d+1)/2} r^{d-1}$$

which differs from our f_{d+1} unless $d = 1$. In fact, f_n is the radial density of a conformal random variable (invariant by symmetries and inversions) on $\mathbb{R}^{n-1}$. Moreover, an isotropic Cauchy random vector does not admit a first moment, whereas X'_∞ is integrable unless $n = 2$, precisely when it is Cauchy too.

b) Note also that the density of the random variable $X_\infty^{(1)} = \gamma^{(1)}(A_\infty^{(\nu)})$ is

$$\frac{\Gamma(1/2-\nu)}{\sqrt{\pi}\,\Gamma(-\nu)}\left(\frac{1}{1+x^2}\right)^{1/2-\nu}.$$

Beware of mistaking it for the radial $(n-1)$-dimensional Cauchy density, indeed the exponent of $1/(1+r^2)$ is the same.

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