

Further results on some singular linear stochastic differential equations

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Abstract

A class of Volterra transforms, preserving the Wiener measure, with kernels of Goursat type is considered. Such kernels satisfy a self-reproduction property. We provide some results on the inverses of the associated Gramian matrices which lead to a new self-reproduction property. A connection to the classical reproduction property is given. Results are then applied to the study of a class of singular linear stochastic differential equations together with the corresponding decompositions of filtrations. The studied equations are viewed as non-canonical decompositions of some generalized bridges.

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1. Introduction and preliminaries

Gaussian enlargement of filtrations has been extensively studied between the late 70's and the early 90's; see [10,20–22] and the references therein. Results stemming from the Gaussian nature of the underlying generalized Gaussian bridges are of interest not only in probability, also in financial mathematics, since they have appeared in an insider trading model developed in [6,25].

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Transforms of Volterra type allow one to construct interesting families of Gaussian processes. Volterra transforms are classified, both from the theory and applications points of view, according to whether their kernels are square-integrable or not. Those with square-integrable kernels play a crucial role in the study of equivalent Gaussian measures, stochastic linear differential equations and the linear Kalman–Bucy filter; see [18,24]. To our knowledge, less attention was given to Volterra transforms with non-square-integrable kernels. Such transforms naturally appear, for instance, in non-canonical representations of some Gaussian processes; see [23]. They also appear if one forces such transforms to preserve the Wiener measure. Most known examples have corresponding kernels of Goursat type. A few nontrivial ones originate from P. Lévy, see [26,27], and serve as a standard reference for showing the importance of the canonical decomposition of semi-martingales. Such constructions have been enriched by people from the Japanese school; see [13–15,17,19].

Let us now fix the mathematical setting and summarize results of this paper. We take $B := (B_t, t \geq 0)$ to be a standard Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$. Denote by $\{\mathcal{F}_t^B, t \geq 0\}$ the filtration it generates. Let $f = (f_1, \dots, f_n)^* \in L^2_{\text{loc}}(\mathbb{R}_+) = \{h; \int_0^t h^2(s)ds < \infty, \text{ for all } t \in [0, \infty)\}$, where $*$ stands for the transpose operator and n is a natural number. Although some of our results extend readily to the cases when $n = \infty$, to simplify the study, we only consider the cases where n is finite. We assume that, for any fixed $t > 0$, the covariance matrix m_t , of the Gaussian random variable $\int_0^t f^*(s)dB_s$, is invertible, i.e., the Gramian matrix $m_t = \int_0^t f(s) \cdot f^*(s)ds$ has an inverse α_t . We emphasize that, under the aforementioned condition, it is not difficult to see that $\alpha_t \rightarrow \alpha_\infty$, as $t \rightarrow \infty$, where α_∞ is a finite matrix. Furthermore, for any i , $(\alpha_\infty)_{ij} = 0$ for all j if and only if $\|f_i\| := (\int_0^\infty f_i^2(s)ds)^{1/2} = \infty$. With $\phi(t) = \alpha_t \cdot f(t)$ for $t > 0$, we shall establish in Theorem 2.2 that $(\alpha_t, t > 0)$ is given in terms of ϕ by $\alpha_t = \int_t^\infty \phi(u) \cdot \phi^*(u)du + \alpha_\infty$, for any $t > 0$. This relation has importance in its own right in this work and may have interesting applications to other fields where Gramian matrices together with their inverses are of prime importance; see for instance [8] and the references therein. In particular, we also refer the reader to [3] for applications to the theory of special functions and to [4,5] for applications to reproducing kernel Hilbert spaces and spline functions.

We define the Volterra transform Σ , associated with a Volterra kernel k , on the set of continuous semi-martingales X such that

$$\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^t \int_0^v k(u, v)dX_u dv < \infty, \quad 0 < t < \infty \text{ a.s.}, \tag{1}$$

by

$$\Sigma(X)_t = X_t - \int_0^t \int_0^u k(u, v)dX_v du, \quad 0 < t < \infty. \tag{2}$$

Following [15], the kernel $k(t, s) = \phi^*(t) \cdot f(s)$, for $0 < s \leq t < +\infty$, is a self-reproducing Volterra kernel. That is equivalent to saying that Σ , when applied to the Brownian motion B , satisfies the following two conditions:

- (i) $\Sigma(B)$ is a standard Brownian motion;
- (ii) for any fixed $t \geq 0$, $\mathcal{F}_t^{\Sigma(B)}$ is independent of $\int_0^t f(u)dB_u$.

Existence of $\Sigma(B)$ may be established by using a generalized Hardy inequality discovered in [15]; see Remark 2.3 given below. We call k and Σ , respectively, a *Goursat–Volterra kernel*

and transform, with reproducing basis f . The dimension of $\text{Span}\{f\}$ is called the order of the Goursat–Volterra kernel k . This terminology is formally fixed in Definition 2.1.

Next, we bring our focus onto conditions (i) and (ii) and think of them in terms of enlargement of filtrations and stochastic differential equations. Condition (ii) says that the orthogonal decomposition

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma \left(\int_0^t f(u)dB_u \right) \tag{3}$$

holds true, for any $t \geq 0$. Here, by $\mathcal{F} \otimes \mathcal{G}$ we mean $\mathcal{F} \vee \mathcal{G}$ with independence between \mathcal{F} and \mathcal{G} . We shall show that, for Goursat–Volterra transforms, Eq. (3) can in fact be rewritten as

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma \left(Y - \int_t^\infty \phi(u)d\Sigma(B)_u \right) \tag{4}$$

valid for any $t \geq 0$, where $Y = (Y_1, \dots, Y_N)^*$ is a Gaussian random vector which is independent of $\mathcal{F}_\infty^{\Sigma(B)}$ with covariance matrix $E[Y \cdot Y^*] = \alpha_\infty = \lim_{t \rightarrow \infty} \alpha_t$ in the case $\alpha_\infty \neq 0$, and $Y \equiv 0$ otherwise. We allow here Y to have some null or constant components. Going back to condition (i), we observe that the determination of all continuous semi-martingales which satisfy it amounts to solving the equation

$$X_t = W_t + \int_0^t \int_0^s \phi^*(s) \cdot f(u)dX_u ds, \quad X_0 = 0, \quad t > 0, \tag{5}$$

considered on a possibly enlarged probability space, where W is a standard Brownian motion. Note that we only assume

$$\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^t \int_0^v \phi^*(v) \cdot f(u)dX_u dv < \infty, \quad 0 < t < \infty \text{ a.s.}, \tag{6}$$

and the latter is not absolutely convergent. Because of the singularity at time 0, we call (5) a singular linear stochastic differential equation. If we take $W = \Sigma(B)$ then, by construction, the original Brownian motion B is one solution. A second one coincides with the associated f -generalized bridge on the interval of its finite lifetime, introduced in [1]. It follows that the Goursat–Volterra transform Σ , when defined as above, is not invertible in the sense that (5) has many solutions. This is not a surprising fact. Indeed, k being a self-reproducing kernel implies that it is not square-integrable, as seen in [11]. Next, Theorem 3.1 deals with the investigation of all continuous semi-martingale solutions to (5). In particular, we show that a necessary and sufficient condition for the existence of a strong solution that is Brownian and \mathcal{F}_∞^B -measurable is $\alpha_\infty \equiv 0$. In that case $\mathcal{F}_\infty^{\Sigma(B)} = \mathcal{F}_\infty^B$. When $\alpha_\infty \neq 0$, Theorem 3.1 leads to the conclusion that there exists still a strong solution which is a Brownian motion, in an enlarged space, that involves an independent centered Gaussian vector Y with covariance matrix α_∞ . Another natural aim is a characterization of all continuous semi-martingales that satisfy both conditions (i) and (ii). This is partially obtained in Theorem 4.1 for the case $\alpha_\infty \equiv 0$ and the analysis exhibits some connections to certain space–time harmonic functions. The latter are functions $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ such that $h(\cdot, \int_0^\cdot f^*(s)dB_s)$ is a continuous $(\mathbb{P}_0, \mathcal{F})$ -martingale with expectation 1, where \mathbb{P}_0 stands for the Wiener measure.

The main results of this paper extend a part of the first chapter of [30] and some results found in [22]. Our work offers explicit examples of conditionings and conditioned stochastic differential equations introduced and studied in [7]. Furthermore, singular equations of type (5)

and the progressive enlargement of a filtration given in Corollary 3.2 can easily be applied to insider trading models elaborated in [6,7,25].

2. Goursat–Volterra kernels and transforms

With a Brownian motion B we associate the centered Gaussian process $\Sigma(B)$ defined by (2), which we assume is well defined, where k is a continuous Volterra kernel. That is to say, $k : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ satisfies

$$k(u, v) = 0, \quad 0 < u \leq v < \infty,$$

and is continuous on $\{(u, v) \in (0, +\infty) \times (0, +\infty) : u > v\}$. We know from [11] that Σ preserves the Wiener measure, or $\Sigma(B)$ is a Brownian motion, if and only if k satisfies the self-reproducing property

$$k(t, s) = \int_0^s k(t, u)k(s, u)du, \quad 0 < s \leq t < \infty. \tag{7}$$

For a connection with reproducing kernels, in the usual sense, we refer the reader to the end of this section. Observe that (2), when applied to B , can be viewed as the semi-martingale decomposition of $\Sigma(B)$ with respect to the filtration $(\mathcal{F}_t^B, t \geq 0)$. Now, as a consequence of the Doob–Meyer decomposition of $\Sigma(B)$ in its own filtration, we must have the strict inclusion

$$\mathcal{F}_t^{\Sigma(B)} \subsetneq \mathcal{F}_t^B, \quad 0 < t < \infty.$$

It is shown in [22] that the missing information, called the reproducing Gaussian space, is given in the orthogonal decomposition

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma(\Gamma_t^{(k)}),$$

where

$$\Gamma_t^{(k)} = \left\{ \int_0^t f(u)dB_u; f \in L^2((0, t]), f(s) = \int_0^s k(s, u)f(u)du \text{ a.e.} \right\}$$

for any $t > 0$. Given a kernel k , it is not an easy task to determine a basis of $\Gamma_t^{(k)}$ for each fixed $t > 0$, because this amounts to solving explicitly the integral equation

$$f(t) = \int_0^s k(t, u)f(u)du, \quad 0 < t < \infty.$$

It is easier to fix the family of spaces $(\Gamma_t^{(k)}, t > 0)$ and work out the corresponding Volterra kernel. This procedure, in fact, corresponds to decomposing the Wiener measure over the interval $[0, t]$, for any fixed $t > 0$, along $\Gamma_t^{(k)}$. Recall that a Goursat kernel is a kernel of the form

$$k(t, s) = \phi^*(t) \cdot f(s), \quad 0 < s \leq t < \infty,$$

where $\phi = (\phi_1, \dots, \phi_n)^*$ and $f = (f_1, \dots, f_n)^*$ are two vectors of functions defined on $(0, \infty)$ and $n \in \mathbb{N}$. For such kernels it is natural to introduce the following definition.

Definition 2.1. A Goursat–Volterra transform Σ of order $(n_t, t > 0)$ is a Volterra transform preserving the Wiener measure such that, for any Brownian motion B and $t > 0$, $\mathcal{F}_t^{\Sigma(B)}$ is

independent of $\int_0^t f(u)dB_u$ for some vector $f \equiv (f_1, \dots, f_{n_t})^*$ of n_t linearly independent $L^2_{loc}(\mathbb{R}_+)$ functions. The associated kernel is called a Goursat–Volterra kernel. The objects f , $\text{Span}\{f\}$ and $\text{Span}\{\int_0^\cdot f(s)dB_s\}$ are called reproducing basis, space and Gaussian space, respectively.

Because for each fixed $t > 0$, m_t is positive definite, it can be seen that $t \rightarrow n_t$ is nondecreasing. However, in our setting, we always take the order to be constant and finite. The simplest known example of a Goursat–Volterra kernel is $k_1(t, s) = t^{-1}$ and this gives

$$\Sigma(B)_\cdot = B_\cdot - \int_0^\cdot \frac{B_u}{u} du.$$

That corresponds to setting $n = 1$ and taking $f_1 \equiv 1$. It is observed in [22] that Σ when iterated takes a remarkably simple form. That is with $\Sigma^{(0)} = Id$, $\Sigma^{(1)} = \Sigma$ and $\Sigma^{(m)} = \Sigma^{(m-1)} \circ \Sigma$, for $m \geq 2$, where \circ stands for the composition operation, we have

$$\Sigma^{(n)}(B)_\cdot = \int_0^\cdot L_n \left(\log \frac{\cdot}{s} \right) dB_s,$$

where $(L_n, n \in \mathbb{N})$ is the sequence of Laguerre polynomials. As a generalization of the above kernel, we quote the following result from [15].

Theorem 2.1 (Hibino–Hitsuda–Muraoka [15]). *Let f be a vector of n functions of $L^2_{loc}(\mathbb{R}_+)$ such that for any $t > 0$ the Gramian matrix $m_t = \int_0^t f(s) \cdot f^*(s)ds$ has an inverse denoted by α_t . Then, with $\phi(\cdot) = \alpha \cdot f(\cdot)$, the kernel k , defined by $k(t, s) = 0$ if $s > t$ and $k(t, s) = \phi^*(t) \cdot f(s)$ otherwise, is a Goursat–Volterra kernel of order n .*

For a proof of this result, we refer the reader to [15]. Some arguments of the proof are sketched in Remark 2.3. In the remainder of this paper, unless otherwise specified, we work under the setting of Theorem 2.1. The objective of the next result is to obtain an expression for α in terms of $\phi(\cdot)$. As a straightforward application, we shall show that it allows us to obtain a new self-reproducing property satisfied by the kernel k . To our knowledge the following result is not known.

Theorem 2.2. α_t converges to a finite matrix α_∞ as $t \rightarrow \infty$. Moreover, we have

$$\alpha_t = \int_t^\infty \phi(u) \cdot \phi^*(u)du + \alpha_\infty, \quad 0 < t < \infty. \tag{8}$$

Consequently, the self-reproduction property

$$k(t, s) = \int_t^\infty k(u, t)k(u, s)du + f^*(t) \cdot \alpha_\infty \cdot f(s), \quad 0 < s \leq t < \infty, \tag{9}$$

holds true.

Proof. Fix $t > 0$. Observe that the matrices α_t and m_t are symmetric positive definite with absolutely continuous entries. Next, the identity $\alpha_t \cdot m_t = Id_n = m_t \cdot \alpha_t$, when differentiated, yields $\alpha'_t \cdot m_t = -\alpha_t \cdot m'_t$. It follows that

$$\phi(t) \cdot f^*(t) = \alpha_t \cdot f(t) \cdot f^*(t) = \alpha_t \cdot m'_t = -\alpha'_t \cdot m_t.$$

Consequently, we have $\alpha'_t = -\phi(t) \cdot f^*(t) \cdot \alpha_t = -\phi(t) \cdot \phi^*(t)$. For any $1 \leq j \leq n$, $(\alpha'_t)_{j,j} = -\phi_j^2(t)$ is negative. Hence, $(\alpha_t)_{j,j}$ is decreasing. Because $(\alpha_t)_{j,j} > 0$ we get that

$\int_r^\infty \phi_j^2(s)ds < \infty, r > 0$. Since, for $t \geq r$, we can write $\alpha_t = \alpha_r - \int_r^t \phi(s) \cdot \phi^*(s)ds$; by letting $t \rightarrow +\infty$, we find $\lim_{t \rightarrow \infty} \alpha_t = \alpha_r - \int_r^\infty \phi(s) \cdot \phi^*(s)ds = \alpha_\infty$. Thus, α_∞ is a matrix with finite entries. The last statement follows from $k(t, s) = f^*(t) \cdot \alpha_t \cdot f(s)$ where we use the expression for α_t given in (8). \square

Self-reproducing kernels, in particular Goursat–Volterra kernels, are different from but related to kernel systems and reproducing kernel Hilbert spaces. Our next objective is to outline this connection. For, let us start by fixing a time interval $[0, t]$, for some $t > 0$. Let the vector $q_t(u) := (q_{m,t}(u), 0 < u \leq t; 1 \leq m \leq n)$ be formed by the orthonormal sequence associated with f_1, f_2, \dots, f_n over the interval $[0, t]$. This system is uniquely characterized by

$$\int_0^t q_{m,t}(r)q_{k,t}(r)dr = \delta_{m,k}, \quad 1 \leq m, k \leq n,$$

with the requirement that for each integer $1 \leq m \leq n, q_{m,t}$ is a linear combination of f_1, \dots, f_m with a positive leading coefficient associated with f_m . We refer the reader to Lemma 6.3.1, p. 294, in [3] for an expression for the latter in terms of a determinant. The classical kernel system is then given by the symmetric kernel

$$\kappa_t(u, v) = q_t(u) \cdot q_t^*(v), \quad 0 < u, v \leq t.$$

This is a reproducing kernel in the sense that

$$\kappa_t(u, v) = \int_0^t \kappa_t(u, r)\kappa_t(v, r)dr, \quad 0 < u, v \leq t.$$

For $1 \leq i, j \leq n, (\alpha_t)_{i,j}$ is seen to be the coefficient of $f_i(u)f_j(v)$ in the expansion of κ_t . To be more precise, $(\alpha_t)_{i,j} = (b_t \cdot b_t^*)_{i,j}$ where b is an upper diagonal matrix whose entry $(b_t)_{i,k}$ is the coefficient of $f_i(u)$ in $q_{k,t}(u)$ for $i \leq k$. We clearly have $\phi_i^2(t) = -2(b_t^* \cdot b_t)_{i,i}$ for all i and it would be interesting to express the matrix b_t in terms of $\phi(t)$. Now, we are ready to state the following result.

Proposition 2.1. *For each fixed $t > 0$, the kernel system associated with f , over the time interval $[0, t]$, is given by $\kappa_t(u, v) = \int_r^\infty k(r, u)k(r, v)dr + f^*(u) \cdot \alpha_\infty \cdot f(v)$ for $0 < u, v \leq t$. In particular, we have $k(t, s) = \kappa_t(t, s)$ for all $0 < s \leq t < \infty$.*

Proof. As in the proof of Theorem 2.2, the first part of the result follows from the well-known relationship $\kappa_t(u, v) = f^*(u) \cdot \alpha_t \cdot f(v)$ for any $0 < u, v \leq t$. The second part follows by taking the limit and using continuity. \square

Remark 2.1. To see an example where $\alpha_\infty \neq 0$, let us discuss the case $n = 2$. Assume that f_1 and f_2 are two functions in $L^2_{loc}(\mathbb{R}_+)$. We distinguish four cases and three different forms for α_∞ . The first corresponds to $\alpha_\infty \equiv 0$ when $\|f_1\| = \|f_2\| = +\infty$. The second corresponds to the case when $\|f_1\|$ and $\|f_2\|$ are finite which implies that α_∞ is positive definite. Observe that the off-diagonal entries are zero only when $\int_0^\infty f_1(s)f_2(s)ds = 0$. The latter integral is zero if, for instance, we take $f_1 = \varphi - \psi$ and $f_2 = \varphi + \psi$, where $\|\varphi\| = \|\psi\| < \infty$. In the third case, all the entries of α_∞ are zero but $(\alpha_\infty)_{1,1} = 1/\|f_1\|^2$ if $\|f_1\| < +\infty$ and $\|f_2\| = +\infty$. The remaining case is similar by symmetry.

Remark 2.2. We shall now discuss examples of kernels of order $n, n \in \mathbb{N}$, whose reproducing spaces are Müntz spaces; see [9] for Müntz polynomials and spaces. We refer the reader to [2]

for proofs of results given below. Take $f_i(s) = s^{\lambda_i}$, $i = 1, 2, \dots$, where $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ is a sequence of reals such that $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i > -1/2$. For a fixed $n < \infty$, the kernel k_n defined by $k_n(t, s) = 0$ if $s > t$ and

$$k_n(t, s) = t^{-1} \sum_{j=1}^n a_{j,n}(s/t)^{\lambda_j}, \quad a_{j,n} = \frac{\prod_{i=1}^n (\lambda_i + \lambda_j + 1)}{\prod_{i=1, i \neq j}^n (\lambda_i - \lambda_j)}, \quad j = 1, \dots, n, \tag{10}$$

if $0 < s \leq t$ is a Goursat–Volterra kernel of order n . Its reproducing Gaussian space, at time $t > 0$, is $\text{Span}\{\int_0^t s^i dB_s; i = 1, 2, \dots, n\}$. Going back to the Gramian matrix $(m_t, t \geq 0)$, observe that it has the entries

$$(m_t)_{i,j} = (\lambda_i + \lambda_j + 1)^{-1} t^{\lambda_i + \lambda_j + 1}, \quad i, j = 1, \dots, n.$$

Thus if $t = 1$ then m_1 is a Cauchy matrix. When $\lambda_i = ci$, for some constant $c \neq 0$, and $n = \infty$, m_1 is the well-known Hilbert matrix. Note that because $\|f_i\| = +\infty$, $i = 1, \dots, n$, we have $\alpha_\infty \equiv 0$. So we have $\phi_i(t) = a_{i,n} t^{-\lambda_i - 1}$, $i = 1, 2, \dots, n$. Furthermore, the entries of α_t are given by

$$(\alpha_t)_{i,j} = a_{i,n} a_{j,n} (\lambda_i + \lambda_j + 1)^{-1} t^{-\lambda_i - \lambda_j - 1}, \quad i, j = 1, \dots, n,$$

which follows from the expression for the kernels when compared with [Theorem 2.2](#). Note that α_t , for $t \neq 1$, can easily be constructed from α_1 which is known and can be found in [\[29\]](#). Finally, we mention that some results have been obtained concerning infinite order kernels in the Müntz case; see [\[2,16\]](#).

Remark 2.3. Observe that we can write

$$\Sigma(B)_t = \int_0^\infty (I - K_f^*) 1_{[0,t]}(u) dB_u, \quad 0 < t < \infty$$

where K_f^* is the adjoint of the bounded integral operator K_f defined on $L^2_{\text{loc}}(\mathbb{R}_+)$ by

$$K_f \alpha(t) = \int_0^t k(t, r) \alpha(r) dr, \quad \alpha \in L^2_{\text{loc}}(\mathbb{R}_+).$$

That $I - K_f$ is a partial isometry, with initial subspace $L^2_{\text{loc}}(\mathbb{R}_+) \ominus \text{Span}\{f\}$ and final subspace $L^2_{\text{loc}}(\mathbb{R}_+)$, follows from the generalized Hardy inequality

$$\|K_g \alpha\| \leq 2\|\alpha\|, \quad \alpha \in L^2_{\text{loc}}(\mathbb{R}_+).$$

Consequently, the operator $I - K_f^*$, when defined on $L^2_{\text{loc}}(\mathbb{R}_+)$, is isometric which implies the statement of [Theorem 2.1](#). For the above results, we refer the reader to [\[15\]](#). We also refer the reader to the comments of [Section 3](#) therein because here we are working with $L^2_{\text{loc}}(\mathbb{R}_+)$ instead of $L^2_{\text{loc}}([0, 1])$.

Remark 2.4. Many authors work under the condition

$$\int_0^t \left(\int_0^u k^2(u, v) dv \right)^{1/2} du < \infty \tag{11}$$

for all $t > 0$, which is sufficient for $\Sigma(B)$, where B is a standard Brownian motion, to be well defined; see for instance [11]. However, condition (11) is too strong for $\Sigma(B)$ to be well defined. To see that, let us fix $b \in L^2_{\text{loc}}(\mathbb{R}_+)$. The associated Goursat–Volterra kernel of order 1 is then found to be

$$k(t, v) = b(t)b(v) \Big/ \int_0^t b^2(r)dr.$$

This satisfies (11) if and only if $\int_0^t |b(s)|/(\int_0^s b^2(r)dr)^{1/2}ds < \infty$ for all $t < \infty$. For example, the kernel associated with $b(t) = t^{-1}e^{-1/t}$ fails to satisfy (11).

3. On some singular linear stochastic differential equations

Consider the singular linear stochastic equation (5). Our interest lies in the set of all its continuous semi-martingale solutions which may be defined on a possibly enlarged space. For a particular solution X , we recall that (5) is well defined in the sense that (6) holds. If we set $W = \Sigma(B)$, where B is a Brownian motion, then the set includes at least two solutions which we will now briefly describe. First, B is a solution. Second, there is a solution which is defined on \mathbb{R}_+ and coincides with the f -generalized bridge over its lifetime. The latter process, denoted by $(B^y_u, u \leq t_1)$, for some $t_1 > 0$ and a column vector of reals y , is defined by

$$B^y_u = B_u - \psi^*(u) \cdot \int_0^{t_1} f(s)dB_s + \psi^*(u) \cdot y, \quad 0 < u < t_1,$$

where ψ is the unique solution to the linear system

$$\int_0^u f(s)ds = \psi(u) \cdot \int_0^{t_1} f(s) \cdot f^*(s)ds = \psi(u) \cdot m_{t_1}, \quad 0 < u < t_1.$$

Thus $\psi(u) = \alpha_{t_1} \cdot \int_0^u f(s)ds$ which implies that $\int_0^{t_1} f(s)dB_s^y = y$, since α_{t_1} is the inverse of m_{t_1} . This is why the above process is called an f -generalized bridge over $[0, t_1]$ with endpoint y . Now, we have $\Sigma(B^y) = \Sigma(B)$ which is true because Σ is linear and $\Sigma(\int_0^t f(r)dr) \equiv 0$ since $f(t) = \int_0^t k(t, v)f(v)dv$ for all $0 < t < \infty$. This shows that B^y is also a solution to (5) which, in fact, is a non-canonical decomposition. For further results on these processes, such as their canonical decomposition in their own filtrations, we refer the reader to [1]. Now, we consider Eq. (5) where the driving Brownian motion W is taken to be arbitrary.

Theorem 3.1. (1) X solves Eq. (5) if and only if there exists a random vector $Y = (Y_1, \dots, Y_n)^*$ such that

$$X = X^0 + \int_0^\cdot f^*(u)du \cdot Y \tag{12}$$

where

$$X^0 = W - \int_0^\cdot \int_u^\infty \phi^*(v) \cdot f(u)dW_v du.$$

In terms of X, Y it is given by $Y = \lim_{t \rightarrow \infty} \alpha_t \cdot \int_0^t f(u)dX_u$.

(2) X^0 is a Brownian motion if and only if $\alpha_\infty \equiv 0$. In the case $\alpha_\infty \neq 0$, a process X solving Eq. (5) is a Brownian motion if and only if Y is centered Gaussian with covariance matrix α_∞ and is independent of $\mathcal{F}_\infty^{X^0}$.

Proof. (1) We proceed by checking first that X_t^0 is a particular solution to (5). Using the stochastic Fubini theorem, found for instance in [28], we perform the decompositions

$$\begin{aligned} X_t^0 - \int_0^t \int_0^u k(u, v) dX_v^0 du &= W_t - \int_0^t \int_u^\infty k(v, u) dW_v du \\ &\quad - \int_0^t \int_0^u k(u, v) \left(dW_v - \int_v^\infty k(\rho, v) dW_\rho dv \right) du \\ &= W_t - \int_0^t \int_u^\infty k(v, u) dW_v du - \int_0^t \int_0^u k(u, v) dW_v du \\ &\quad + \int_0^t \int_0^u \int_0^\rho k(u, v) k(\rho, v) dv dW_\rho du + \int_0^t \int_u^\infty \int_0^u k(u, v) k(\rho, v) dv dW_\rho du. \end{aligned}$$

Since k is self-reproducing, the last four terms in the last equation cancel showing that X_t^0 solves (5). Next, if X is a solution then by setting $X = X^0 + Z$ we see that Z has to satisfy

$$dZ_r = \int_0^r k(r, v) dZ_v dr, \quad 0 < r < \infty.$$

Multiplying both sides by $f(r)$ and integrating with respect to r , along $[0, t]$, yields

$$\begin{aligned} \int_0^t f(v) dZ_v &= \int_0^t f(v) \phi^*(v) \cdot \int_0^v f(r) dZ_r dv \\ &= \int_0^t m_v \cdot \phi(v) \phi^*(v) \cdot \int_0^v f(r) dZ_r dv \\ &= - \int_0^t m_v \cdot \frac{d}{dv} \alpha_v \cdot \int_0^v f(r) dZ_r dv \end{aligned}$$

where we used the expression for α' given in the proof of Theorem 2.2 to obtain the last equality. Because α is the inverse of m , the latter relation can be written as $\frac{d}{dt} \alpha_t \cdot \int_0^t f(s) dZ_s = 0$. This, when integrated, yields $\alpha_t \cdot \int_0^t f(s) dZ_s = Y$ for some random vector Y . Hence $\int_0^t f(r) dZ_r = m_t \cdot Y$ which implies that $Z_t = Y^* \cdot \int_0^t f(s) ds$. This completes the proof of the first part of the first assertion. For the second part, by using Theorem 2.2 we obtain

$$\begin{aligned} \phi(t) dW_t &= \phi(t) dX_t - \phi(t) \phi^*(t) \cdot \int_0^t f(u) dX_u dt \\ &= \alpha_t \cdot d \left(\int_0^t f(u) dX_u \right) - \phi(t) \phi^*(t) \cdot \int_0^t f(u) dX_u dt \\ &= d \left(\alpha_t \cdot \int_0^t f(u) dX_u \right). \end{aligned}$$

Integrating on both sides over $[s, t]$ we obtain

$$\int_s^t \phi(u) dW_u = \alpha_t \cdot \int_0^t f(u) dX_u - \alpha_s \cdot \int_0^s f(u) dX_u.$$

Next, observe that as $t \rightarrow \infty$ the left hand side converges almost surely. So the right hand side converges as well to some limit which we denote by \tilde{Y} . To be more precise, setting

$$\tilde{Y} = \lim_{t \rightarrow \infty} \alpha_t \cdot \int_0^t f(u) dX_u,$$

we have shown that

$$\int_t^\infty \phi(u) dW_u = \tilde{Y} - \alpha_t \cdot \int_0^t f(u) dX_u, \quad 0 < t \leq \infty. \tag{13}$$

Consequently, we have

$$\begin{aligned} \int_0^t \int_u^\infty f^*(u) \cdot \phi(v) dW_v du - \tilde{Y}^* \cdot \int_0^t f(u) du &= \int_0^t f^*(u) \cdot \alpha(u) \cdot \int_0^u f(v) dX_v du \\ &= \int_0^t \int_0^u \phi^*(u) \cdot f(v) dX_v du. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_0^t \int_u^\infty k(v, u) dW_v du - \tilde{Y}^* \cdot \int_0^t f(u) du &= - \int_0^t \int_0^u k(u, v) dX_v du \\ &= W_t - X_t. \end{aligned}$$

Comparing with previous calculations yields $Y = \tilde{Y}$, \mathbb{P}_0 -almost surely.

(2) [Theorem 2.2](#) implies that

$$E[X_s^0 X_t^0] = s \wedge t - \int_0^{s \wedge t} \int_0^t f^*(r) \cdot \alpha_\infty \cdot f(v) dv dr. \tag{14}$$

This clearly shows that X^0 is a Brownian motion if and only if $\alpha_\infty \equiv 0$. Next, if X is as prescribed then by virtue of (14), and the fact that α_∞ is the covariance matrix of Y , we have

$$\begin{aligned} E[X_s X_t] &= s \wedge t - \int_0^s \int_0^t f^*(u) \cdot \alpha_\infty \cdot f(v) dv du \\ &\quad + \int_0^s \int_0^t E[(Y^* \cdot f(u))(Y^* \cdot f(v))] dv du \\ &= s \wedge t. \end{aligned}$$

Because X is a continuous Gaussian process we conclude that it is a Brownian motion. Conversely, if X is a Brownian solution to (5) then it has to be of the form (12). By virtue of the orthogonal properties of the Goursat–Volterra transform, we see that Y is independent of $\mathcal{F}_t^{\Sigma(X)} = \mathcal{F}_t^W$ for any fixed $t > 0$. Next, by letting t go to ∞ , we get that Y is independent of $\mathcal{F}_\infty^{X^0} \subseteq \mathcal{F}_\infty^W$. Thus, Y is Gaussian vector, with covariance matrix α_∞ , which is independent of $\mathcal{F}_\infty^{X^0}$ as required. \square

Thanks to the importance of the symmetric matrix α_∞ , for instance in [Theorem 2.2](#), it is natural to look for a description of its structure. The following result, which is hidden in the proof of [Theorem 3.1](#), gives a necessary and sufficient condition for a column or a row to be zero.

Corollary 3.1. *For $1 \leq i \leq n$, $(\alpha_\infty)_{i,j} = (\alpha_\infty)_{j,i} = 0$ for all j , if and only if $\|f_i\| = \infty$.*

Proof. For a fixed $t > 0$, α_t is the covariance matrix of $\alpha_t \cdot \int_0^t f(s) dB_s$. Furthermore, due to [Theorem 3.1](#), we conclude that $\alpha_t \cdot \int_0^t f(s) dB_s$ converges to a Gaussian vector Y , possibly with some null components, such that $E(Y \cdot Y^*) = \alpha_\infty$. Thus, $Y_i \equiv 0$ for some i if and only if $(\alpha_\infty)_{i,i} = 0$ and if and only if $\|f_i\| = \infty$. Now, $(\alpha_\infty)_{i,i} = 0$ if and only if $(\alpha_\infty)_{i,j} = 0$ for all j . In order to see that, we let $t \rightarrow \infty$ and use continuity in the well-known inequality $|(\alpha_t)_{i,j}|^2 \leq (\alpha_t)_{i,i}(\alpha_t)_{j,j}$ valid for symmetric positive definite matrices. \square

Now, we take a look at the orthogonal decompositions of filtrations which arise from Goursat–Volterra transforms and provide their interpretation.

Corollary 3.2. *The orthogonal decomposition given by (4) holds true. Furthermore, the progressive decomposition*

$$\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma \left(Y - \int_t^\infty \phi(u) d\Sigma(B)_u \right), \quad 0 < t < \infty$$

holds true, where $Y \equiv 0$ if $\alpha_\infty \equiv 0$ and Y is a Gaussian vector independent of $\mathcal{F}_\infty^{\Sigma(B)}$ with covariance matrix α_∞ otherwise. Thus, we have $\mathcal{F}_\infty^B = \mathcal{F}_\infty^{\Sigma(B)}$ in the case $\alpha_\infty \equiv 0$ and $\mathcal{F}_\infty^B = \mathcal{F}_\infty^{\Sigma(B)} \vee \sigma\{Y\}$ otherwise.

Proof. For a fixed $t > 0$, Theorem 3.1 implies that

$$B_t = \Sigma(B)_t - \int_0^t \int_u^\infty k(v, u) d\Sigma(B)_v du + Y^* \cdot \int_0^t f(u) du$$

where Y is a Gaussian vector with covariance α_∞ which is independent of $\mathcal{F}_\infty^{\Sigma(B)}$. Hence, we have

$$\int_0^t f(u) dB_u = m_t \cdot \left(Y - \int_t^\infty \phi(u) d\Sigma(B)_u \right)$$

which gives

$$\sigma \left\{ \int_0^t f(u) dB_u \right\} = \sigma \left\{ Y - \int_t^\infty \phi(u) d\Sigma(B)_u \right\}.$$

This implies the first assertion while the last one follows on letting t tend to $+\infty$. □

Remark 3.1. Recall that \mathcal{F}_0^B and $\mathcal{F}_0^{\Sigma(B)}$ are trivial. So by letting t converge to 0, in Corollary 3.2, we see that $\phi^* \in L^2([\varepsilon, \infty)^n)$ for all $\varepsilon > 0$ but $\phi_i \notin L^2((0, +\infty))$, for $i = 1, \dots, n$. This fact can also be shown by a combination of Theorem 2.2 and the inequality

$$(\alpha_t)_{i,i} \geq 1/(m_t)_{i,i} = 1/\|f_i\|^2$$

which follows from the orthogonal diagonalization of m_t and may be found in Exercise 8, p. 274, in [12].

Remark 3.2. It is clear that if the choice of the vector f allows the use of integration by parts for the integrand in the right hand side of (5) then we obtain a stochastic differential equation which does not involve a stochastic integral. For instance, that is the case for the examples given by P. Lévy, found in [26,27]. These go back to around the middle of the last century when stochastic integration was not yet developed worldwide.

4. Connections to some positive martingales

Let $(k(t, s), t \geq s > 0)$ be a Goursat–Volterra kernel of order n , where n is a natural number. Assume that f is a reproducing basis for k , or for the associated Volterra transform Σ , and let us keep the notation used in the Introduction. Consider the singular stochastic differential equation (5) associated with k and driven by a given standard Brownian motion W . Our aim here is to describe the set

$$\mathcal{Y}^{(k)} = \left\{ \mathbb{P} \text{ is the probability law of a continuous semi-martingale } X \right. \\ \left. \text{on } (\mathcal{C}([0, \infty), \mathbb{R}), \mathcal{F}_\infty^*) \text{ solving (5) s.t. } \Sigma(X) \text{ is a Brownian motion} \right. \\ \left. \text{and } \mathcal{F}_t^{\Sigma(X)} \text{ is independent of } \int_0^t f(s) dX_s, \text{ for all } 0 < t < \infty \right\}.$$

We read from Corollary 3.1 that $\alpha_\infty \equiv 0$ if and only $\|f_i\| = \infty$ for all i . Now, we are ready to state the following unified characterization of the set $\mathcal{Y}^{(k)}$.

Theorem 4.1. *If $\alpha_\infty \equiv 0$ then the following assertions are equivalent:*

- (1) $\mathbb{P} \in \mathcal{Y}^{(k)}$.
- (2) \mathbb{P} is the law of $B + Y^* \cdot \int_0^\cdot f(s) ds$, where B is a standard Brownian motion and Y is a vector of random variables which is independent of \mathcal{F}_∞^B .
- (3) There exists a positive function $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ such that $h(\cdot, \int_0^\cdot f^*(s) dB_s)$ is a continuous $(\mathbb{P}_0, \mathcal{F})$ -martingale with expectation 1, and $\mathbb{P} = \mathbb{P}_0^h$ with

$$\mathbb{P}_0^h|_{\mathcal{F}_t} = h \left(t, \int_0^t f^*(s) dB_s \right) \cdot \mathbb{P}_0 \Big|_{\mathcal{F}_t}, \quad 0 < t < \infty,$$

where \mathbb{P}_0 stands for the Wiener measure,

Proof. We split the proof into several steps where we show that (1) \iff (2) and (2) \iff (3). Let us show that (1) \implies (2). Let $\mathbb{P} \in \mathcal{Y}^{(k)}$. Theorem 3.1 implies that there exists a vector Y such that \mathbb{P} is the law of $X_t^0 + Y^* \cdot \int_0^t f(u) du$. That combined with the assumption $\alpha_\infty \equiv 0$ leads to the fact that X^0 is a Brownian motion. Hence, it suffices to show that Y is independent of X^0 . From (13) we see that $Y = \int_t^\infty \phi(u) dB_u + \alpha_t \cdot \int_0^t f(u) dX_u$; the vector $\int_0^t f(u) dX_u$ is independent of B and, consequently, it is also independent of X^0 . Thus, whenever $Z \in L^2(\mathcal{F}_\infty^{X^0})$, for any fixed $t \geq 0$, we have

$$E \left[E \left[Z | \mathcal{F}_t^{X^0} \right] \phi \left(Y - \int_t^\infty \phi(u) dB_u \right) \right] = E [Z] E \left[\phi \left(Y - \int_t^\infty \phi(u) dB_u \right) \right]$$

for any bounded function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$. By letting $t \rightarrow \infty$ we conclude that $E [Z \cdot \phi(Y)] = E [Z] E [\phi(Y)]$ which implies the required independence. We shall now show that (2) \implies (1). To this end, let k be a Goursat–Volterra kernel. Denote by f a reproducing basis associated with k and put $X_t = B_t + Y^* \cdot \int_0^t f(s) ds$ for $t > 0$. For a fixed $t > 0$, because $\int_0^t f(u) dB_u \in \Gamma_t^{(k)}$, we can write

$$X_t - \int_0^t \int_0^u k(u, v) dX_v du = B_t - \int_0^t \int_0^u k(u, v) dB_v du = \Sigma(B)_t$$

which is, of course, a Brownian motion. Furthermore, using once more the above argument we can easily see that $\int_0^t f(u) dX_u$ is independent of $\mathcal{F}_t^{\Sigma(B)}$. Next, we deal with (2) \implies (3). Denote by $\nu(dy)$ the distribution of Y . For any measurable functional ϕ , we then have that

$$E \left[\phi \left(B_s + Y^* \cdot \int_0^s f(u) du : s \leq t \right) \right] \\ = \int_{\mathbb{R}^n} E \left[\phi \left(B_s + y^* \cdot \int_0^s f(u) du : s \leq t \right) \right] \nu(dy) \\ = \int_{\mathbb{R}^n} E \left[\exp \left(\int_0^t y^* \cdot f(u) dB_u - \frac{1}{2} \int_0^t (y^* \cdot f(u))^2 du \right) \phi(B_s : s \leq t) \right] \nu(dy)$$

where the last equality is obtained by the Girsanov theorem. The required space–time harmonic function is thus given on $\mathbb{R}^+ \times \mathbb{R}^n$ by

$$h(t, x) = \int_{\mathbb{R}^n} \exp\left(y^* \cdot x - \frac{1}{2} \int_0^t (y^* \cdot f(s))^2 ds\right) \nu(dy).$$

It remains to show that (3) \implies (2). For fixed $0 < u \leq t < +\infty$, set $\psi(u, t) = \alpha_t \cdot \int_0^u f(s) ds$. Let us write the obvious decomposition

$$B_u = \left(B_u - \psi^*(u, t) \cdot \int_0^t f(s) dB_s \right) + \psi^*(u, t) \cdot \int_0^t f(s) dB_s$$

and denote by H_u^t the first term of its right hand side. We observe that the process $(H_u^t, u < t)$ has then the same law under \mathbb{P}_0 as under \mathbb{P}_0^h . Next, to simplify notation, write

$$\hat{H}_u^t = \psi^*(u, t) \cdot \int_0^t f(s) dB_s = Y_t^* \cdot \int_0^u f(s) ds,$$

where we set $Y_t^* = \alpha_t \cdot \int_0^t f(r) dB_r$. For any $0 \leq s \leq u \leq t$, we have $E[H_s^t H_u^t] = s - \psi^*(s, t) \cdot \int_0^u f(v) dv$ and $\psi^*(s, t) \cdot \int_0^u f(v) dv = \int_0^u f^*(v) dv \cdot \alpha_t \cdot \int_0^s f(r) dr \rightarrow 0$ as $t \rightarrow \infty$ because $\alpha_\infty \equiv 0$. We conclude that the convergence in distribution $H_u^t \rightarrow B^{(h)}$ holds, where $B^{(h)}$ is a \mathbb{P}_0^h -Brownian motion. That implies the convergence of \hat{H}_u^t as well to a finite limit. But that can happen if and only if Y_t^* converges to a finite limit which we denote by Y^* . Finally, from the above arguments we see that Y^* is independent of $\mathcal{F}_\infty^{B^{(h)}}$ which ends the proof. \square

Remark 4.1. Unfortunately, for the case $\alpha_\infty \neq 0$, the second statement in the above theorem is too strong. For example, X^0 satisfies the assertion (1) but it is easily seen that it does not satisfy (2). The implications (2) \implies (1) and (2) \implies (3) still work in this case. We also can replace B by X^0 in statement (2) and prove that (1) \iff (2) still holds true. However, (2) \implies (3) fails.

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