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1. MOTIVATION

Rough path theory, broadly speaking, aims to extend Stieltjes-type calculus from paths with finite variation to paths with finite p-variation. In the literature, while there have been plenty of bounds on p-variation, there have been very few examples where the exact values of p-variation are calculated. One situation where it would be helpful to have such exact value is the study of isoperimetric-type inequality

$$\left(\int\int_{\mathbb{R}^2} |\eta(\gamma,(x,y))|^q \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\zeta\left(\frac{2}{pq}\right)-1\right)\left(\|\gamma\|_{p-var}\right)^{\frac{2}{q}},$$

where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous loop, $\eta(\gamma,(x,y))$ is its winding number around the point (x,y), $\|\gamma\|_{p-var}$ is the p-variation of γ and ζ is the classical zeta function. The constant $\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\zeta\left(\frac{2}{pq}\right)-1\right)$ is likely to not be optimal, so in order to find the optimal constant, it would be useful to have the exact value of $\|\gamma\|_{p-var}$ for a reasonably large class of γ. The goal of this two-day project is to try and calculate $\|\gamma\|_{p-var}$ when γ is the boundary of a regular polygon.

2. The essential problem

Let

$$\mathbf{X} = ((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n))$$

be a sequence of n points in the x-y plane. We define the norm of a vector (u,v) as

$$\|(u,v)\| = \sqrt{u^2+v^2}.$$

Here the square root is the positive square root.

For each ordered subset

$$I = ((x_{i_1},y_{i_1}),\cdots,(x_{i_k},y_{i_k})) \tag{2.1}$$

of $((x_1,y_1),\cdots,(x_n,y_n))$, with $i_1 = 1,\ i_k = n$ and $i_1 < i_2 < \cdots < i_k$, the p-variation with respect to I is defined as

$$\|\mathbf{X}|_I\|_{p-var} = \left(\sum_{j=1}^{k-1} \|\left(x_{i_{j+1}}-x_{i_j}, y_{i_{j+1}}-y_{i_j}\right)\|^p\right)^{\frac{1}{p}} \tag{2.2}$$

$$= \left(\sum_{j=1}^{k-1}\left(\left(x_{i_{j+1}}-x_{i_j}\right)^2+\left(y_{i_{j+1}}-y_{i_j}\right)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}. \tag{2.3}$$

The p-variation of $\mathbf{X}$, $\|\mathbf{X}\|_{p-var}$, is defined as the maximum of $\|\mathbf{X}|_I\|_{p-var}$ across all the different subsets I in (2.1) such that $i_1 = 1$ and $i_k = n$.[1]

Example 1. Let $\mathbf{X} = ((0,0), (1,2), (2,3))$. Since we only consider subsets of $\{(0,0), (1,2), (2,3)\}$ where $(0,0)$ and $(2,3)$ are in the subset, so there are only two possible subsets that we will consider

$$I_1 = \{(0,0), (2,3)\}$$

and

$$I_2 = \{(0,0), (1,2), (2,3)\}.$$

Note that

$$\begin{aligned}\|\mathbf{X}|_{I_1}\|_{2-var} &= \left(\| (2,3) - (0,0) \|^2\right)^{\frac{1}{2}} \\ &= \left(\left((2-0)^2 + (3-0)^2\right)\right)^{\frac{1}{2}} \\ &= \sqrt{13}\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{X}|_{I_2}\|_{2-var} &= \left(\| (2,3) - (1,2) \|^2 + \| (1,2) - (0,0) \|^2\right)^{\frac{1}{2}} \\ &= \left(\left((2-1)^2 + (3-2)^2\right) + \left((1-0)^2 + (2-0)^2\right)\right)^{\frac{1}{2}} \\ &= (2+5)^{\frac{1}{2}}.\end{aligned}$$

Since $\sqrt{13} > \sqrt{7}$, we have

$$\|\mathbf{X}\|_{2-var} = \sqrt{13}.$$

In this two-day workshop, we investigate the p-variation of a regular n-gon with side lengths 1. From now on, let $(x_1, y_1), \ldots, (x_n, y_n)$ be the coordinates of the vertices of a regular n-gon, arranged in the anti-clockwise direction. Let $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. With such a choice of $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$, the quantity (2.2) is called the p-variation of the regular n-gon.[2]

Research Question: Can we calculate the p-variation of a regular n-gon exactly for certain range of p?

Example 2. For regular 3-gon, or equivalently an equilateral triangle, one way to choose $\mathbf{X}$ is as

$$\mathbf{X} = \left((0,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), (1,0), (0,0)\right).$$

For later calculation purpose, we denote

$$\begin{aligned}(x_1, y_1) = (0,0), (x_2, y_2) &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ (x_3, y_3) = (1,0), (x_4, y_4) &= (0,0).\end{aligned}$$

[1] If we consider the piecewise linear path connecting the n-points in $\mathbf{X}$ in order, then $\|\mathbf{X}\|_{p-var}$ coincides with the p-variation of the piecewise linear path as defined in [2].

[2] Although there is more than one way to choose which vertex can be the starting vertex, the choice of starting vertex does not affect the calculation of p-variation as it is rotationally invariant. The $p-$variation is also translation invariant, which means that the location of the n-gon does not affect the p-variation.

Here there are four points so we will need to think about the possible subsets for $(1,2,3,4)$ that contains 1 and 4. They are

$$\begin{aligned}&I_1 = (1,4)\,, I_2 = (1,2,4)\,, I_3 = (1,3,4)\\&I_4 = (1,2,3,4)\,.\end{aligned}$$

We denote

$$\begin{aligned}\|\mathbf{X}|_{I_1}\|_{p-var} &= \left(\left((x_4 - x_1)^2 + (y_4 - y_1)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\&= \left(\left((0-0)^2 + (0-0)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\&= 0.\end{aligned}$$

$$\begin{aligned}&\|\mathbf{X}|_{I_2}\|_{p-var}\\=&\left(\left((x_4 - x_2)^2 + (y_4 - y_2)^2\right)^{\frac{p}{2}} + \left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\=&\left(\left(\left(0-\frac{1}{2}\right)^2 + \left(0-\frac{\sqrt{3}}{2}\right)^2\right)^{\frac{p}{2}} + \left(\left(\frac{1}{2}-0\right)^2 + \left(\frac{\sqrt{3}}{2}-0\right)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\=&2^{\frac{1}{p}}.\end{aligned}$$

Similarly,

$$\|\mathbf{X}|_{I_3}\|_{p-var} = 2^{\frac{1}{p}}$$

and

$$\|\mathbf{X}|_{I_4}\|_{p-var} = 3^{\frac{1}{p}}.$$

The p-variation $\|\mathbf{X}\|_{p-var}$ is the biggest of these numbers, so

$$\|\mathbf{X}\|_{p-var} = 3^{\frac{1}{p}}.$$

Example 3. For a regular 4-gon, or equivalently a unit square, one way to choose the vertices is as

$$\mathbf{X} = ((0,0)\,, (0,1)\,, (1,1)\,, (1,0)\,, (0,0))\,,$$

where $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (0,1), \ldots, (x_5, y_5) = (0,0)$.

Possible ordered subsets of $(1,2,3,4,5)$ that includes 1 and 5 are $I_1 = (1,5)$, $I_2 = (1,2,5)$, $I_3 = (1,3,5)$, $I_4 = (1,4,5)$, $I_5 = (1,2,3,5)$, $I_6 = (1,2,4,5)$, $I_7 = (1,3,4,5)$ and $I_8 = (1,2,3,4,5)$. To calculate the p-variation of $\mathbf{X}$ note in particular that

$$\begin{aligned}&\|\mathbf{X}|_{I_3}\|_{p-var}\\=&\left(\left((x_3 - x_1)^2 + (y_3 - y_1)^2\right)^{\frac{p}{2}} + \left((x_5 - x_3)^2 + (y_5 - y_3)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\=&\left(\left(1^2 + 1^2\right)^{\frac{p}{2}} + \left((0-1)^2 + (0-1)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\\=&2^{\frac{1}{p}}2^{\frac{1}{2}}.\end{aligned}$$

and

$$\|\mathbf{X}|_{I_5}\|_{p-var} = 2^{\frac{2}{p}}.$$

It can be checked that $\|\mathbf{X}|_{I_1}\|_{p-var}, \|\mathbf{X}|_{I_2}\|_{p-var}, \ldots$ will all be less than or equal to $2^{\frac{1}{p}}2^{\frac{1}{2}}$ or $2^{\frac{2}{p}}$. Therefore,

$$\|\mathbf{X}\|_{p-var}$$

would be equal to $2^{\frac{1}{p}}2^{\frac{1}{2}}$ or $2^{\frac{2}{p}}$, whichever one is bigger. This means in particular that

$$\|\mathbf{X}\|_{p-var} = \begin{cases} 2^{\frac{2}{p}}, & \text{if } p < 2 \\ 2^{\frac{1}{p}}2^{\frac{1}{2}}, & \text{if } p \geq 2. \end{cases}$$

2.1. **Regular Pentagon or Hexagon.** In this report, we will use log to denote the natural logarithm $\log_e$ to the base e. This is different from A-level textbooks where the notation was usually ln.

A direct calculation has concluded that, for regular pentagon, the p-variation is

$$\|\mathbf{X}\|_{p-var} = \begin{cases} 5^{\frac{1}{p}}, & \text{if } p \leq \frac{\log 2}{\log \varphi}; \\ (2\varphi^p + 1)^{\frac{1}{p}} & \text{if } p > \frac{\log 2}{\log \varphi}. \end{cases}$$

For regular hexagon, the p-variation is

$$\|\mathbf{X}\|_{p-var} = \begin{cases} 6^{\frac{1}{p}}, & \text{if } p \leq \frac{\log 4}{\log 3}; \\ 3^{\frac{1}{p}}3^{\frac{1}{2}} & \text{if } \frac{\log 4}{\log 3} < p \leq \frac{2\log\left(\frac{3}{2}\right)}{\log\left(\frac{4}{3}\right)}; \\ 2 \cdot 2^{\frac{1}{p}} & \text{if } \frac{2\log\left(\frac{3}{2}\right)}{\log\left(\frac{4}{3}\right)} < p. \end{cases}$$

2.2. **General values of n.** The study of regular pentagon and hexagon in the previous section suggest a pattern, that for a general regular n-gon and for p sufficiently close to 1, we should expect

$$\|\mathbf{X}\|_{p-var} = n^{\frac{1}{p}}.$$

2.2.1. *Distance between vertices of regular n-gons.* Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the vertices of a regular n-gon ordered in the anti-clockwise direction. Let $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Recall that if $I = (i_1, \cdots, i_k)$ where $i_1 = 1$ and $i_k = n+1$ we have

$$\|\mathbf{X}|_I\|_{p-var} = \left(\sum_{j=1}^{k-1} \| \left(x_{i_{j+1}} - x_{i_j}, y_{i_{j+1}} - y_{i_j} \right) \|^p \right)^{\frac{1}{p}}.$$

Therefore it would be helpful to compute

$$\| \left(x_{i_{j+1}} - x_{i_j}, y_{i_{j+1}} - y_{i_j} \right) \|.$$

Note that $(x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})$ all lies on the circle with radius

$$\frac{1}{2\sin\left(\frac{\pi}{n}\right)}$$

and a centre which we will call O. Note that $\| (x_m, y_m) - (x_l, y_l) \|$ is the length of the cord for the sector with vertices (x_m, y_m), O and (x_l, y_l). By applying trigonometry to the triangle with vertices (x_m, y_m), O and (x_l, y_l), we have that for $l > m$,

$$\| (x_m, y_m) - (x_l, y_l) \| = \frac{\sin\left(\frac{(l-m)\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}.$$

Therefore, given a subset $I = \{(x_{i_1}, y_{i_1}), \cdots, (x_{i_k}, y_{i_k})\}$ of $\{(x_1, y_1), \cdots, (x_{n+1}, y_{n+1})\}$ with $i_1 = 1,\ i_k = n+1$ and $i_1 < i_2 < \cdots < i_k$, we have

$$\begin{aligned}\|\mathbf{X}|_I\|_{p-var} &= \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\sin^p\left(\frac{\pi}{n}\right)}\right)^{\frac{1}{p}} \\ &= \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{k-1} \sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)\right)^{\frac{1}{p}}\end{aligned}$$

2.2.2. *Maximising over the different partitions.* In order to work out the subsets I that would maximise

$$\|\mathbf{X}|_I\|_{p-var}$$

we first consider

$$\max_{1\le k\le n} \frac{\sin^p\left(\frac{k\pi}{n}\right)}{k},$$

which is

$$\begin{aligned}&= \frac{\pi}{n} \max_{1\le k\le n} \frac{\sin^p\left(\frac{k\pi}{n}\right)}{k\frac{\pi}{n}} \\ &= \frac{\pi}{n} \max_{x\in\left\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{n\pi}{n}\right\}} \frac{\sin^p(x)}{x}.\end{aligned}$$

To find the maxmiser, consider the function

$$f(x) = \frac{\sin^p(x)}{x}.$$

Lemma 4. *Let $p \ge 1$. Let z_0 be the unique real number such that $0 < z_0 < \frac{\pi}{2}$ and*

$$pz_0 - \tan(z_0) = 0.$$

The function

$$f(x) = \frac{\sin^p(x)}{x}$$

is strictly increasing for $0 < x < z_0$ and strictly decreasing for $z_0 < x \le \pi$.

Proof. To maximise the function, we compute its derivative

$$\begin{aligned}f'(x) &= \frac{px\sin^{p-1}(x)\cos(x) - \sin^p(x)}{x^2} \\ &= \sin^{p-1}(x)\left[\frac{px\cos(x) - \sin(x)}{x^2}\right].\end{aligned}$$

Now if $\frac{\pi}{2} \le x \le \pi$, then $\cos(x) \le 0$ and $\sin(x) > 0$, so

$$px\cos(x) - \sin(x) < 0.$$

If $0 \le x_0 \le \pi$ and satisfies $f'(x_0) = 0$, then $0 \le x_0 \le \frac{\pi}{2}$. We now re-write $f'(x)$ as

$$f'(x) = \sin^{p-1}(x)\cos(x)\left[\frac{px - \tan(x)}{x^2}\right].$$

Let

$$g(x) = px - \tan(x).$$

Then

$$g'(x) = p - \sec^2(x).$$

Since $\cos(x)$ is strictly decreasing for $0 \le x \le \frac{\pi}{2}$, the function $x \to g'(x)$ is strictly decreasing. There is therefore only one y_0 such that $g'(y_0) = 0$ and

$$g'(x) \begin{cases} > 0 & \text{if } x < y_0; \\ < 0 & \text{if } x > y_0. \end{cases}$$

This means g is strictly increasing for $x < y_0$ and strictly decreasing for $x > y_0$.

Since $g(0) = 0$ and g is strictly increasing for $x < y_0$, we have $g(x) > 0$ for $0 < x \le y_0$. Since $g(y_0) > 0$ and $x \to g(x)$ is strictly decreasing when $y_0 < x < \frac{\pi}{2}$ and $g(x)$ approaches $-\infty$ as x approaches $\frac{\pi}{2}$, there must only be one z_0 that satisfies $y_0 < z_0 < \frac{\pi}{2}$ and $g(z_0) = 0$.

In summary, we have

$$g(x) \begin{cases} = 0 & x = 0 \\ > 0 & 0 < x < z_0 \\ = 0 & x = z_0 \\ < 0 & z_0 < x < \frac{\pi}{2}. \end{cases}$$

Since g has the same sign as f' for $0 < x < \frac{\pi}{2}$, we have

$$f'(x) \begin{cases} = 0 & x = 0 \\ > 0 & 0 < x < z_0 \\ = 0 & x = z_0 \\ < 0 & z_0 < x < \frac{\pi}{2}. \end{cases}$$

This means the function f has a maximum at the point z_0, where z_0 is the unique point where $0 < z_0 < \frac{\pi}{2}$ and

$$pz_0 - \tan(z_0) = 0.$$

□

2.2.3. *Location of z_0.*

Lemma 5. *If*

$$p \le \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}},$$

then the maximum

$$\max_{x \in \left\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{n\pi}{n}\right\}} \frac{\sin^p(x)}{x}$$

is attained at $x = \frac{\pi}{n}$ or $x = \frac{2\pi}{n}$.

Proof. In the previous section, we saw that the function

$$f(x) = \frac{\sin^p(x)}{x}$$

is maximised at a unique point $0 < z_0 < \frac{\pi}{2}$ satisfying

$$pz_0 - \tan(z_0) = 0.$$

Since for $p \ge 1$ the function $z \to pz - \tan(z)$ is non-negative for small and positive z, if x is a number such that $px - \tan(x) \le 0$, then $z_0 \le x$.

Note that

$$p\frac{2\pi}{n} - \tan\left(\frac{2\pi}{n}\right) \leq 0$$

if and only if

$$p \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}.$$

Therefore if $p \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$, then $z_0 \leq \frac{2\pi}{n}$.

Since $f(x)$ is increasing for $x \leq z_0$ and decreasing for $x \geq z_0$, the maximum

$$\max_{x \in \left\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{n\pi}{n}\right\}} \frac{\sin^p(x)}{x}$$

must be attained at $\frac{k\pi}{n}$ or $\frac{(k+1)\pi}{n}$, where k is the unique integer such that

$$\frac{k\pi}{n} \leq z_0 < \frac{(k+1)\pi}{n}.$$

If $p \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$, then $z_0 \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$. This means the maximum

$$\max_{x \in \left\{\frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{n\pi}{n}\right\}} \frac{\sin^p(x)}{x} \tag{2.4}$$

is attained at $\frac{\pi}{n}$ or $\frac{2\pi}{n}$. □

Lemma 6. *Let z_0 be the unique real number $0 < z_0 < \frac{\pi}{2}$ satisfying*

$$pz_0 - \tan(z_0) = 0.$$

Then z_0 increases with p.

Proof. Note that if

$$h(z) = \frac{\tan z}{z},$$

then for $0 < z < \frac{\pi}{2}$

$$\begin{aligned} h'(z) &= \frac{z\sec^2(z) - \tan(z)}{z^2} \\ &= \frac{z\sec(z) - \sin(z)}{z^2\cos(z)} > 0. \end{aligned}$$

We have that as p increases, z_0 increase. □

3. Conclusion

We assume throughout this section that $p \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$.

Case 1: If n is even.

Case 1a: $\frac{\log 2}{\log\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)} \geq p$.

In this case

$$\frac{\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \geq \frac{\sin^p\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$$

and so, by Lemma 5, the maximum (2.4) is attained at $x = \frac{\pi}{n}$.

Since

$$\begin{aligned}
\|\mathbf{X}|_I\|_{p-var} &= \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\sin^p\left(\frac{\pi}{n}\right)}\right)^{\frac{1}{p}} \\
&= \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\frac{(i_{j+1}-i_j)\pi}{n}} \frac{(i_{j+1}-i_j)\pi}{n}\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \frac{(i_{j+1}-i_j)\pi}{n}\right)^{\frac{1}{p}} \\
&= n^{\frac{1}{p}}.
\end{aligned}$$

The maximum is attained by choosing

$$i_j = \frac{j\pi}{n}$$

and so

$$\begin{aligned}
\|\mathbf{X}\|_{p-var} &= \max_I \|\mathbf{X}|_I\|_{p-var} \\
&= n^{\frac{1}{p}}.
\end{aligned}$$

Case 1b: $\frac{\log 2}{\log\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)} < p \leq \frac{\tan\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$.

In this case the maximum (2.4) is attained at $x = \frac{2\pi}{n}$, then a similar argument gives

$$\begin{aligned}
\|\mathbf{X}|_I\|_{p-var} &\leq \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \frac{(i_{j+1}-i_j)\pi}{n}\right)^{\frac{1}{p}} \\
&\leq \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{k-1} \frac{\sin^p\left(\frac{2\pi}{n}\right)}{2} (i_{j+1}-i_j)\right)^{\frac{1}{p}} \\
&\leq \frac{n^{\frac{1}{p}} \sin\left(\frac{2\pi}{n}\right)}{2^{\frac{1}{p}} \sin\left(\frac{\pi}{n}\right)}.
\end{aligned}$$

This maximum is attained by the partition

$$i_j = \frac{2j\pi}{n}$$

for $j = 1, 2, \ldots, \frac{n}{2}$. Therefore, in this case,

$$\|\mathbf{X}\|_{p-var} = \frac{n^{\frac{1}{p}} \sin\left(\frac{2\pi}{n}\right)}{2^{\frac{1}{p}} \sin\left(\frac{\pi}{n}\right)} = \frac{n^{\frac{1}{p}} \cos\left(\frac{\pi}{n}\right)}{2^{\frac{1}{p}}}$$

Case 2: If n is odd and $n \geq 3$.

Case 2a: $\frac{\log 2}{\log\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)} \geq p$.

This is exactly the same as Case 1a. The maximum (2.4) is attained at $x = \frac{\pi}{n}$ and leads to

$$\|\mathbf{X}\|_{p-var} = n^{\frac{1}{p}}.$$

Case 2b: $\frac{\log 2}{\log\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)} < p < \frac{\log 3}{\log\left(\frac{\sin\left(\frac{3\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)}$

The $\frac{\log 2}{\log\left(\frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)} < p$ condition ensures that

$$\frac{\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} < \frac{\sin^p\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$$

and the

$$p < \frac{\log 3}{\log\left(\frac{\sin\left(\frac{3\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)}$$

condition ensures that

$$\frac{\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} > \frac{\sin^p\left(\frac{3\pi}{n}\right)}{\frac{2\pi}{n}}.$$

In this case by Lemma 5, the maximum (2.4) is attained at $x = \frac{2\pi}{n}$. Since n is odd, there is at least one j such that $i_{j+1} - i_j$ is odd. Choose one such j. Then

$$\begin{aligned}
\sin^p\left(\frac{(i_{j+1} - i_j)\,\pi}{n}\right) &= \frac{\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\frac{(i_{j+1}-i_j)\pi}{n}} \frac{(i_{j+1} - i_j)\,\pi}{n} \\
&= \frac{2\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\frac{(i_{j+1}-i_j)\pi}{n}} \frac{(i_{j+1} - i_j - 1)}{2n}\pi \\
&+ \frac{2\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\frac{(i_{j+1}-i_j)\pi}{n}} \frac{1}{2n}\pi \\
&\leq \frac{2\sin^p\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \frac{(i_{j+1} - i_j - 1)}{2n}\pi \\
&+ \frac{2\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \frac{1}{2n}\pi.
\end{aligned}$$

In the final inequality, we used that because $i_{j+1} - i_j$ is odd, it cannot be 2 and so

$$\frac{\sin^p\left(\frac{(i_{j+1}-i_j)\pi}{n}\right)}{\frac{(i_{j+1}-i_j)\pi}{n}} \leq \frac{\sin^p\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}.$$

We therefore have

$$\|\mathbf{X}|_I\|_{p-var} \leq \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\left(\frac{n-1}{2}\right)\sin^p\left(\frac{2\pi}{n}\right) + \sin^p\left(\frac{\pi}{n}\right)\right)^{\frac{1}{p}}$$

and since this upper bound is attained with the partition $i_1 = 1, i_2 = 3, i_3 = 5, \ldots, i_{\frac{n+1}{2}} = n$, $i_{\frac{n+3}{2}} = n+1$.Therefore

$$\|\mathbf{X}\|_{p-var} = \frac{1}{\sin\left(\frac{\pi}{n}\right)} \left(\left(\frac{n-1}{2}\right)\sin^p\left(\frac{2\pi}{n}\right) + \sin^p\left(\frac{\pi}{n}\right)\right)^{\frac{1}{p}}.$$

References

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