Value of money

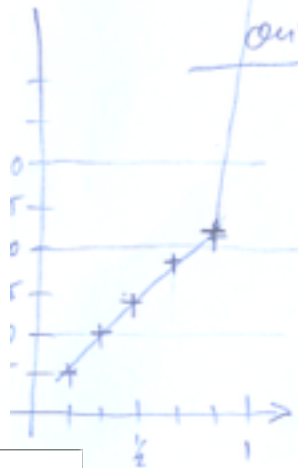
What do you prefer,

$b(\frac{1}{10{,}000}, 100{,}000{,}000)$ or $10{,}000$ for sure?

$E[b] = 10{,}000$, but EMV Strategy does not represent many people's preferences here. Use utility!

Ex: Die bets

How much would you pay $m(p)$ for a bet on outcomes of a die? Elicited from s.o. w/o training in decision theory:



outcome	CME $m(p)$	p
6	5	1/6
5,6	10	2/6
even	14	3/6
3,4,5,6	18	4/6
not 6	22	5/6
all	99	1

(Person asked: is this correct? There is no correct. It is a subjective value of the bet!)

This person's CME $m(p)$ much lower than $E[b(p)]$. It is linear for the most part and then explodes, as p equals 1 (certainty). This person seems to have a good intuition for probabilities, but does not want to gamble.

Review CME: (subjective)

$b(p,s,t)$: with probability p £t, with probability $1-p$ £s

$p \in [0,1]$ given

CME $m(p)$ is the maximal amount of money the person is prepared to forfeit to bet $b(p,s,t)$.

$m(p)$ monotone (modest assumption)

Assume strictly monotone to make things easy

Then inverse exists.

Utility is the inverse of m: $U(y) = m^{-1}(y)$

Often used in mathematical models:

$$m(p) = p^{\alpha}, \quad \alpha > 0, \text{ so } U(y) = y^{1/\alpha}$$

In previous example, approx., $m(p) = p^2 \cdot c$, where c is some constant

Interpretation of CME: Compare $m(p)$ with the expected value of the bet $E[b(p,s,t)] = pt + (1-p)s$

$m(p) < E[b(p,s,t)]$: prefers smaller but certain amount to expected but uncertain equivalent, i.e. willing to pay to remove risk — "risk averse"

$m(p) > E[b(p,s,t)]$: assigns amount higher than its expected value to the bet, i.e. pays to gamble — "risk seeking"

$m(p) = E[b(p,s,t)]$: "risk neutral"

The shape of utility

fix s, t

b bet: $\begin{array}{l} \nearrow^{p}\ \pounds t \\ \searrow_{1-p}\ \pounds s \end{array}$ OR $m(p)$

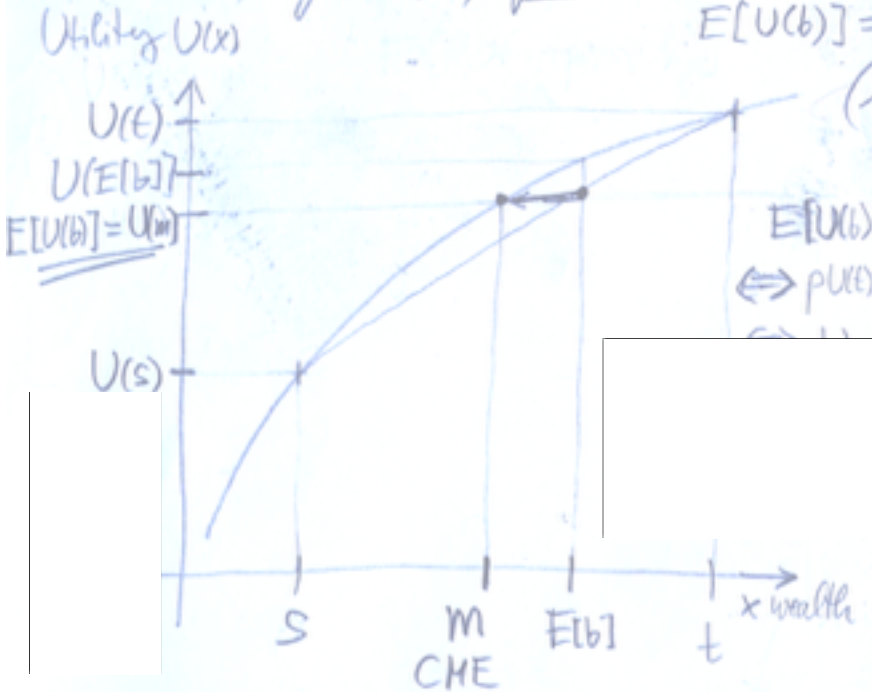
(Before, for $m(p)$ shape: $E[b(p)] = pt + (1-p)s$

Compare $E[b(p)]$ and $m(p)$)

Now, given U, find m with $U(m) = E[U(b)]$

$E[U(b)] = pU(t) + (1-p)U(s)$

(linear interpol. on y-axis!)



$E[U(b)] < U(E[b])$

$\Leftrightarrow pU(t) + (1-p)U(s) < U(pt + (1-p)s)$

...ncave

...onvex

...linear

Risk premium $= m - E[b]$

$E[b] = pt + (1-p)s$

U strictly monotone

$= m^{-1}$

$m < E[b]$	risk averse	U concave
$m > E[b]$	risk seeking	U convex
$m = E[b]$	risk neutral	U linear

Example : Fire insurance

value of house 100 (in £100 units)

value of house after fire 25

$q = 0.2$

$p = 1 - q$

(d_2) b: no insurance $\begin{array}{l} \nearrow^{p}\ 100 \\ \searrow_{1-p}\ 25 \end{array}$

$b(0.8, 25, 100)$

$E[b] = 0.2 \cdot 25 + 0.8 \cdot 100 = 85 = EMV(d_2)$

(d_1) Buy insurance

How much are you willing to pay for insurance?

(Owner's perspective) Assume $U(x) = \sqrt{x}$

Find m such that $U(m) = E[U(b)]$ risk averse (for owner, amount large)

$E[U(b)] = 0.2 \cdot U(25) + 0.8 \cdot U(100)$

$= 0.2\sqrt{25} + 0.8\sqrt{100} = 0.2 \cdot 5 + 0.8 \cdot 10 = 9$

$U(m) \stackrel{!}{=} 9 \Rightarrow m = 81$

Hence, owner is willing to pay up to

$(19 = 15 + 4)$ $100 - 81 = 19$ for insurance

risk premium $= m - E[b] = 81 - 85 = -4$

insurance premium $= -$ risk premium $= 4$

How much would insurer charge? $U(x) = x$ risk neutral (for insurer, amount small)

$E[\text{loss}] = 0.2 \cdot \text{damage} = 0.2 \cdot 75 = 15$

Wants at least 15.

$\Rightarrow$ interval for deal: $[15, 19]$

About interpretation / terminology:

Owner's perspective: Risk avoiding.
Owner's wealth is not really 100, he only owns that house including the risk for fire. So, the owner's wealth is just $E[b] = 85$. Hence, the owner should already be willing to pay 15.

In addition, for the sake of removing uncertainty, the owner is willing to pay even more than 15 for insurance, in fact, the insurance premium of (up to) 4.

Insurer's perspective: Risk neutral.
Needs to ask for at least 15 to cover potential costs. Due to owner's willingness to pay more, there is room for a deal. Any value between 15 and 19 should be an acceptable price for both of them.

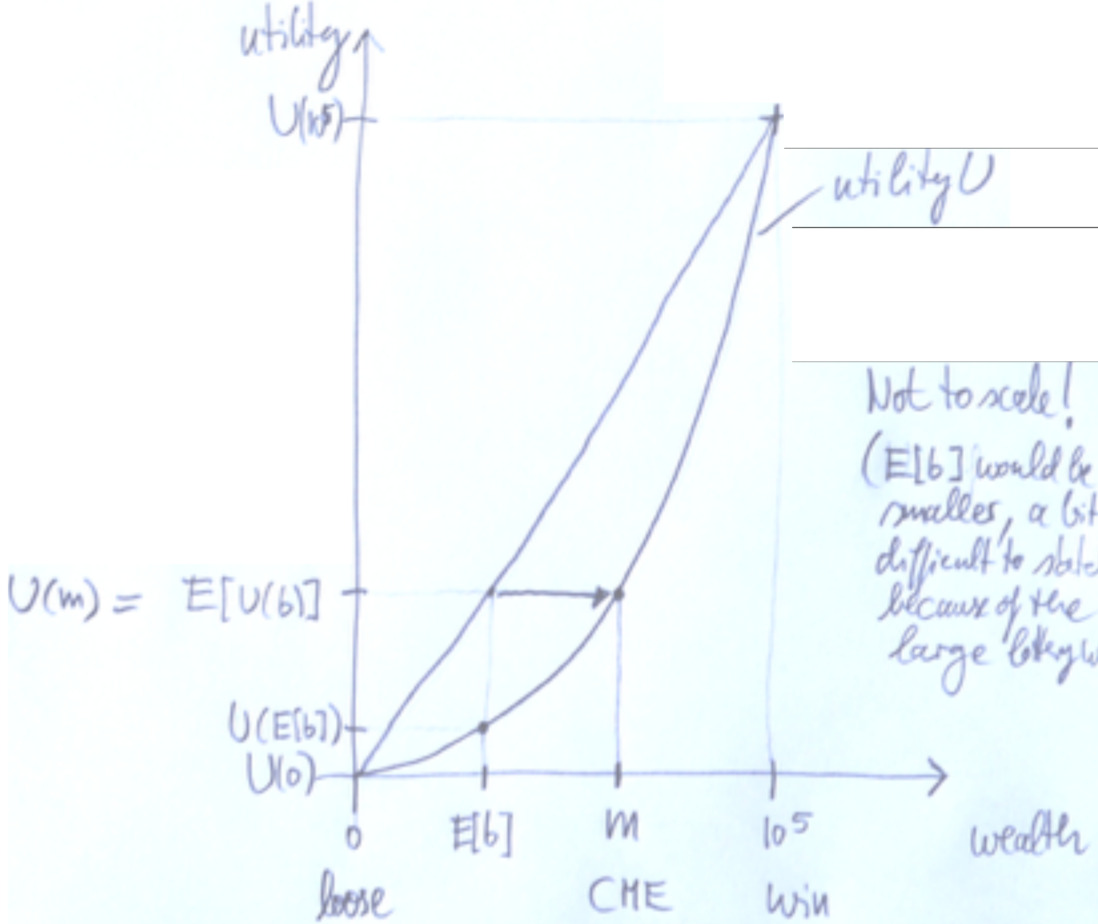
Example: Lottery

Very small proba. p to win very large amount £M (for a small ticket price). For example, $M = 10^5$, and $p = 10^{-8}$. Bet $b(10^{-8}, 10^5)$

$E[b] = p \cdot M = 10^{-8} \cdot 10^5 = 10^{-3}$

$E[U(b)] = p\,U(M) + (1-p) \cdot U(0)$

Assumption: $U(x) = x^2$ (risk seeking), initial wealth 0 (simplification)



Not to scale! ($E[b]$ would be smaller, a bit difficult to sketch because of the large lottery win)

$E[U(b)] = 10^{-8} \cdot (10^5)^2 + (1-10^{-8}) \cdot 0^2 = 10^2$

Find m such that $U(m) = E[U(b)]$

That means $m^2 = 10^2$, hence $m = 10$

A person with this utility (risk seeking) is willing to pay up to £10 to play this lottery.

$$
\begin{aligned}
\text{Risk premium} &= CME - EMV(b) \\
&= m - E[b] \\
&= 10 - 10^{-3} \\
&= 10 - 0.001 = 9.999
\end{aligned}
$$

This person's risk premium is £9.999.

In contrast, a person with risk neutral attitude $U(x) = x$ would only pay $E[b]$. Hence, his risk premium would be 0.