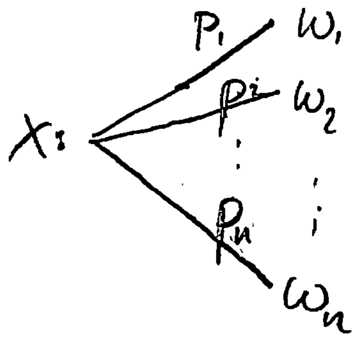
Review/extension of definitions:

Ω outcome space

$\mathcal{A}$ action space

For Ω finite: $x \in \mathcal{A}$ has form $b(\omega_1, p_1; \dots; \omega_n, p_n)$

$(p_i)_{i=1,\dots,n}$ PMF



For Ω countable: similar

$(p_i)_{i \in \mathbb{N}}$ PMF, $b(\omega_i, p_i; i \in \mathbb{N})$

Essentially, action x is a prob. measure on Ω, so $\mathcal{A}$ can be identified with a set of probability measures on Ω (with some algebra $\mathcal{F}$).

For Ω continuous: $(\Omega, \mathcal{F})$ measurable space,
i.e. Ω outcome space with σ-algebra $\mathcal{F}$

$\mathcal{A}$ is a set of probability measures on $(\Omega, \mathcal{F})$.

<u>Def</u>: A binary relation $\succ$ on an action space $\mathcal{A}$ is called preference relation if it has properties (C), (A) and (NT).

<u>Def</u>: A <u>numerical representation</u> of a preference relationship $\succ$ is a function

$u : \mathcal{A} \longrightarrow \mathbb{R}$ such that

$\forall x, y \in \mathcal{A} \quad x \succ y \iff u(x) > u(y)$

We showed last time

<u>Theorem</u>: Let $\mathcal{A}$ be a finite action space with preference relation $\succ$. Then there is a numerical representation of $\succ$.

<u>Note</u>: u is not unique.
For example, if u is a numerical presentation and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\tilde{u} = f \circ u$ is also a numerical presentation

What about infinite $\mathcal{A}$?
Countable? Think of proof, does it go through?
Continuous? Trace back to countable ...

<u>Def</u> $\mathcal{Z} \subset \mathcal{A}$ is called <u>order dense</u> if
$\forall x, y \in \mathcal{A}$ with $x \succ y$ $\exists$ $z \in \mathcal{Z}$ with $x \succeq z \succeq y$.

<u>Theorem</u>: Let $\mathcal{A}$ be an action space with preference relation $\succ$. If $\mathcal{A}$ has a countable order dense subset then there is a numerical representation of $\succ$.

<u>Cor</u>: If $\mathcal{A}$ is countable then $\succ$ has a numerical representation.

<u>Proof of theorem</u>:
(abstract version of the proof seen before)
Let $\mathcal{Z}$ be an order dense subset of $\mathcal{A}$.

$$\forall x \in \mathcal{A} \quad \overline{\mathcal{Z}}(x) = \{ z \in \mathcal{Z} \mid z \succ x \}$$

$$\underline{\mathcal{Z}}(x) = \{ z \in \mathcal{Z} \mid z \prec x \}$$

$$x \succeq y \Rightarrow \overline{\mathcal{Z}}(x) \subseteq \overline{\mathcal{Z}}(y) \;\wedge\; \underline{\mathcal{Z}}(x) \supseteq \underline{\mathcal{Z}}(y)$$

$x \succ y \Rightarrow$ at least one of the subset relationships is strict, because $(*)$

(using properties of $\succ$)

$$\exists z \in \mathcal{Z} : x > z \geq y \;\vee\; x \geq z > y$$

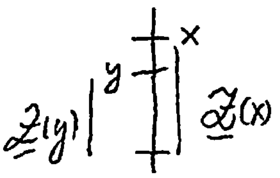
$$\Rightarrow \quad z \in \underline{\mathcal{Z}}(x) \setminus \underline{\mathcal{Z}}(y) \;\vee\; z \in \overline{\mathcal{Z}}(y) \setminus \overline{\mathcal{Z}}(x)$$

Take a $(p_z)_{z \in \mathcal{Z}}$ PMF with $p_z > 0 \;\forall z \in \mathcal{Z}$

Define
$$U(x) = \underbrace{\sum_{z \in \underline{\mathcal{Z}}(x)} p_z}_{\geq \sum_{z \in \underline{\mathcal{Z}}(y)} p_z} - \underbrace{\sum_{z \in \overline{\mathcal{Z}}(x)} p_z}_{\leq \sum_{z \in \overline{\mathcal{Z}}(y)} p_z} \geq U(y)$$

$x \succ y \Rightarrow$

because $\underline{\mathcal{Z}}(y) \subseteq \underline{\mathcal{Z}}(x)$ because $\overline{\mathcal{Z}}(y) \supseteq \overline{\mathcal{Z}}(x)$



And because of $(*)$ we actually have $> U(y)$

$\square$

<u>Note</u>: The theorem was about a sufficient condition for the existence of a numerical representation. It can also be shown that the order dense set is <u>neccessary</u>. The proof, however, is a rather tricky construction of such as set (not examinable).

Ex: Lexicographical order on $\mathbb{R}^2$ (ct.)

Recall for $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$\underbrace{x \succ y}_{\text{lexicographical order (binary relation on } \mathbb{R}^2)} \iff x_1 \underbrace{>}_{\text{regular "larger than" order on } \mathbb{R}} y_1 \vee (x_1 = y_1 \wedge y_1 > y_2)$$

<u>Proposition</u>: The lexicographical order on $\mathbb{R}^2$ does not have a numerical representation.

<u>Proof</u> (indirect): Assume u was a numerical repres.

Then, $\forall r \in \mathbb{R}, r \geq 0$: $(r,2) \succ (r,1) \Rightarrow u(r,2) > u(r,1)$

$\forall r \geq 0 \; \exists q_r \in \mathbb{Q}$: $\quad u(r,2) \overset{(1)}{>} q_r \overset{(2)}{>} u(r,1) \quad (*)$

because $\mathbb{Q}$ is dense in $\mathbb{R}$

$r \longmapsto q_r$ defines a map from $\mathbb{R}^+$ to $\mathbb{Q}$ $\quad (**)$

$\forall r > \tilde{r} \geq 0$: $\quad u(r,1) > u(\tilde{r},2)$

(1) in $(*) \Rightarrow \exists q_r \in \mathbb{Q}$: $q_r > u(r,1)$

(2) in $(*) \Rightarrow \exists q_{\tilde{r}} \in \mathbb{Q}$: $\quad u(\tilde{r},2) > q_{\tilde{r}}$

Hence, we have shown: $r > \tilde{r} \Rightarrow q_r > q_{\tilde{r}}$,

and thereby also that $q_r = q_{\tilde{r}}$ implies $r = \tilde{r}$ (use previous statement with $r, \tilde{r}$ swapped), so the map is injective from $\mathbb{R}^+$ to $\mathbb{Q}$, a contradiction to the countability of $\mathbb{Q}$. $\square$