Question (by ST222 student):

Is the map in (**) well defined?

Answer: Yes it is, but it is not given explicitly and it is not unique. Let φ be such a map. For each $r \in \mathbb{R}^+$ its value $\varphi(r)$ was chosen from the countable set $(u(r,1), u(r,2)) \cap \mathbb{Q}$. Different choices would have resulted in a different map $\tilde{\varphi}$. Both φ and $\tilde{\varphi}$ are well defined maps. However, the non-uniqueness of the construction may result in other object not being well defined, as they may depend on the choice of φ. (As an explicit example, imagine you define a function f as follows: $f(x) = \varphi(x) + \varphi(x/2)$, where φ is the map defined in (**). Then f is not well defined. The mistake was to refer to φ as "the" map, because φ as in (**) is not unique, so there is no "the" map φ.)

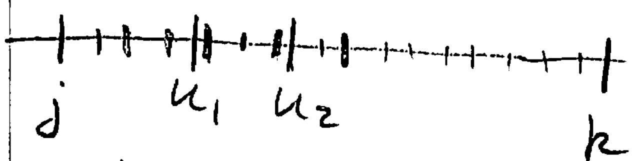
It is, however, possible to make φ unique by including a (constructive) algorithms to obtain q_r in (*). Can you think of any? Try, or read some suggestions on the following page.

<u>Task</u>: For $u_1 < u_2$ find algorithm to select
$$q \in (u_1, u_2) \cap \mathbb{Q}$$

In other words, describe a way to determine explicitly $q \in \mathbb{Q}$ such that $u_1 < q < u_2$.

(Note that $u_1, u_2 \in \mathbb{R}$ but it is not known whether or not $u_1 \in \mathbb{Q}$ or $u_2 \in \mathbb{Q}$, so the primitive choice $q = (u_1 + u_2)/2$ does not solve the problem.)

<u>Algorithm 1</u>:



Define $j := \max\{n \in \mathbb{N} \mid n \leq j\}$, $k := \min\{n \in \mathbb{N} \mid n \geq k\}$

$\forall n \in \mathbb{N}_0 \quad D_n := \{j + 2^{-n} \cdot m \cdot (k - j) \mid m = 0, \dots, 2^m\}$

$D_n \subset D_{n+1} \ \forall n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} D_n$ is dense in $[j, k]$.

Hence, there exists $n_0 \in \mathbb{N}_0$ with
$$(u_1, u_2) \cap D_{n_0} = \emptyset \quad \text{but} \quad (u_1, u_2) \cap D_{n_0+1} \neq \emptyset$$

Let q be the minimum of $(u_1, u_2) \cap D_{n_0+1}$.

(Actually, $(u_1, u_2) \cap D_{n_0+1}$ has only one element, which can be shown with a little more reasoning, but simply using the minimum also does the trick.)

$\square$

<u>Algorithm 2:</u>

The Archimedian axiom in $\mathbb{R}$ implies there is $n \in \mathbb{N}$ with $n(u_2 - u_1) > 1$. Let n_0 be the smallest n with this property. Since this means $n_0 u_2 > n_0 u_1 + 1$, there must be $m \in \mathbb{N}\setminus\{0\}$ with $n_0 u_2 > m > n_0 u_1$. Let m_0 be the smallest m with this property. Hence, $q := \frac{m_0}{n_0} \in \mathbb{Q}$ solves the task.

$\square$

<u>Algorithm 3:</u>

Let $\tilde{u}_1 = u_1 + \frac{u_2 - u_1}{4}$ and $\tilde{u}_2 = u_2 - \frac{u_2 - u_1}{4}$

Write $\tilde{u}_2$ in decimal representation:

$$\tilde{u}_2 = \sum_{n=0}^{\infty} d_2^{(n)} \cdot 10^{k-n}, \quad k \in \mathbb{N}_0, \; d_2^{(n)} \in \{0, 1, \ldots, 9\},$$

and represent $\tilde{u}_1$ accordingly using $d_1^{(n)} \in \{0, 1, \ldots, 9\}$.

Let n_0 be the smallest n such that $d_2^{(n)} > d_1^{(n)}$.

Then $q := \sum_{n=0}^{n_0} d_2 \quad 10^{k-n}$ fulfills $u_1 < \tilde{u}_1 \leq q \leq \tilde{u}_2 < u_2$

$\square$

<u>Algorithm n:</u>

A countably infinite number of algorithms can be constructed by replacing base 10 by other bases in algorithm 3.