More properties of preferences:

(Needed for v. Neumann–Morgenstern theory)

Archimedean property (Arch)

$\forall x, y, z \in \mathcal{A}$ with $x \succ y \succ z$ $\exists \alpha, \beta \in ($

such that $\alpha x + (1-\alpha) z \succ y \succ \beta x + (1-\beta) z$

Interpretation: $x \succ y$ is not so strong that mix it with z can't lead to reversal of preferences.

x can not be incommensurably better than y.

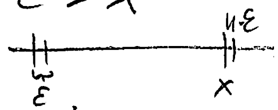
z can not be incommensurably worse than y

Homework: Lexicographical order is not (Arc

Compare with Archimedean on $\mathbb{R}$:

$\forall \varepsilon > 0 \ \forall x \in \mathbb{R} \ \exists n \in \mathbb{N}: \ n \cdot \varepsilon > x$

aka "continuity axiom"



- Can act as substitute for continuity.
- continuity of $\succ$ implies (Arch)

Example for incommensurability

$x = (£1000, 1)$ $\quad x \succ y \succ z$

$y = (£10, 1)$

$z = (\text{dead}, 1)$

$\alpha x + (1-\alpha) z \succ y$?

No, no matter how small α is.

Hence, **Archimedean** property is **not true**.

Interpretation money and life/death are **incommensurable**

However, people **do** take small risks for gaining or saving money,

e.g. walk/drive/cycle across town to save £20.

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Independence (Ind):

$\forall x, y, z \in \mathcal{A}$ and $\forall \alpha \in (0,1]$:

$$x \succ y \Rightarrow (1-\alpha) z + \alpha x \succ (1-\alpha) z + \alpha y$$

In plain English: Adding the same stuff to both sides does not change the preference.

Practical example for situation where independence is not true: You are in the desert and hungry. You have nothing and you are being offered a choice:

x: dry bread
y: can with soup

Your preference will be $x \succ y$

But if another choice is added:

z: can opener

Your preference will be $x + z \prec y + z$

von Neumann - Morgenstern theorem is similar to the previous theorem about preferences and utilities in that it connects them, but goes beyond this by providing an explicit representation.

We previously showed theorems about the representation of preference relations through utility function. Now, we will add some assumption that allow to specify the form of such a representation more detailled.

von Neumann – Morgenstern representation theorem:

Let $\mathcal{A}$ be an action space on a countable outcome space Ω and let $\succ$ be a preference relation on $\mathcal{A}$ that also satisfies (Ind) and (Arch).

Then there is a function $u: \Omega \longrightarrow \mathbb{R}$ such that for all $x, y \in \mathcal{A}$ we have

$$x \succ y \iff \sum_{\omega \in \Omega} p_x(\omega)\, u(\omega) > \sum_{\omega \in \Omega} p_y(\omega)\, u(\omega)$$

(using the representations of the elements in $\mathcal{A}$ through $x = b(\omega, p_x(\omega); \omega \in \Omega)$, $y = b(\omega, p_y(\omega); \omega \in \Omega)$ with PMFs $(p_x(\omega))_{\omega \in \Omega}$ and $(p_y(\omega))_{\omega \in \Omega}$)

Furthermore, the representation is unique up to positive linear transformations.

Note: The sums are actually expectations w.r.t. the PMFs that belong to x and y.

Note: Writing the (countable) Ω as $\{\omega_1, \omega_2, \omega_3, \ldots\}$ objects above can be written as $x = b(\omega_n, p_n; n \in \mathbb{N})$, $(p_n)_{n \in \mathbb{N}}$, $\sum_{n \in \mathbb{N}} p_n u(\omega_n)$ etc.

Expected utility principle:

Pick decision that maximises $E[u(X)]$

Allais (1953) considered the lotteries

$X: \xrightarrow{1.00} \$500{,}000$

$X': \xrightarrow{0.01} \$0, \quad \xrightarrow{0.89} \$500{,}000, \quad \xrightarrow{0.10} \$2{,}500{,}000$

$y: \xrightarrow{0.89} \$0, \quad \xrightarrow{0.11} \$500{,}000$

$y': \xrightarrow{0.90} \$0, \quad \xrightarrow{0.10} \$2{,}500{,}000$

People often have the following preferences:

$$x \succ x' \quad \wedge \quad y \prec y'$$

Units of $\$100{,}000$. Utility u (any)

(I) $x \succ x' \iff u(5) > 0.1 \cdot u(25) + 0.89 \cdot u(5) + 0.01 u(0)$

(II) $y' \succ y \iff 0.1\, u(25) + 0.9\, u(0) > 0.11\, u(5) + 0.89\, u(0)$

Assume $u(0) = 0$ (also works in general)

(I) $0.11\, u(5) > 0.1\, u(25)$

(II) $0.1\, u(25) > 0.11\, u(5)$

(*)

↯ Paradox!