Methodology: Normative theory versus descriptive theory

	Normative approaches	**Descriptive approaches**
Probabilistic judgement Uncertainty, risk	Subjective probability	Perceived probabilities and observed processing (axioms may not hold)
Decision theory Preferences, choices	Expected utility maximisation	Observed choice behaviour
Game theory Strategies, moves	Reward maximisation	Observed moves and motives

Methodology: Sequence of coin tosses

Sequence #1

T HHHHT T T T HHHHT HHHHHHHHT T THHT T HHHHHT T T T T HHT HHT HHH T
T T HT T HHHHT HT T T HT T T HHT T T T HHHHHHT T T HHT T HHHT HHHHHT T T T
T HT T T HHT T HT T HHT T T HHT T T HH THHT HHT T T T T HHT HHHHHHT HT HT T
HT HT T HHHT T HHT HT HHHHHHHHT THT T HHHT HHT T HT T T T T T HHHT HHH

Sequence #2

T HT HT T T HT T T T T HT HT T T HT T HHHT HHT HT HT HT T T T HHT T HHT T HHH T
HHHT T HHHT T T HHHT HHHHT T T HT HT HHHHT HT T T HHHT HHT HT T T HHT H
HHT HHHHT T HT HHT HHHT T T HT HHHT HHT T T HHHT T T T HHHT HT HHHHT H
T,T HHT T T T HT HT HT T HT HHT T HT T THT T T T HHHHT HT HHHT T HHHHHT HH

One of these sequences was generated by a sequence of coin tosses, the other one was generated by students being told to write down a sequence generated by coin tosses.

Which one is the sequence generated by coin tosses?

Methodology: Sequence of coin tosses

Sequence #1

T HHHHT T T T HHHHT HHHHHHHHHT T THHT T HHHHHT T T T T HHT HHT HHH T
T T HT T HHHHT HT T T HT T T HHT T T T HHHHHHT T T HHT T HHHT HHHHHT T T T
T HT T T HHT T HT T HHT T T HHT T T HH THHT HHT T T T T HHT HHHHHHT HT HT T
HT HT T HHHT T HHT HT HHHHHHHHHT THT T HHHT HHT T HT T T T T T HHHT HHH

Sequence #2

T HT HT T T HT T T T T HT HT T T HT T HHHT HHT HT HT HT T T T HHT T HHT T HHH T
HHHT T HHHT T T HHHT HHHHT T T HT HT HHHHT HT T T HHHT HHT HT T T HHT H
HHT HHHHT T HT HHT HHHT T T HT HHHT HHT T T HHHT T T T HHHT HT HHHHT H
T,T HHT T T T HT HT HT T HT HHT T HT T THT T T T HHHHT HT HHHT T HHHHHT HH

How can you tell? Which features can you look at?

- number of heads, number of tails
- number of alternations
- numbers of runs

Methodology: Sequence of coin tosses

Sequence #1

T H H H H T T T T H H H H T H H H H H H H H T T T H H T T H H H H H T T T T T T H H T H H T H H H T
T T H T T H H H H T H T T T H T T T H H T T T T H H H H H H T T T H H T T H H H T H H H H H H T T T T
T H T T T H H T T H T T H H T T T H H T T T H H T H H T H H T T T T T H H T H H H H H H H T H T H T T
H T H T T H H H T T H H T H T H H H H H H H H H T T H T T H H H T H H T T H T T T T T T H H H T H H H

Sequence #2

T H T H T T T H T T T T T H T H T T T H T T H H H T H H T H T H T H T T T T H H T T H H T T H H H T
H H H T T H H H T T T H H H T H H H H T T T H T H T H H H H T H T T T H H H T H H T H T T T H H T H
H H T H H H H T T H T H H T H H H T T T H T H H H T H H T T T H H H T T T T H H H T H T H H H H T H
T T H H T T T T H T H T H T T H T H H T T H T T T H T T T T H H H H T H T H H H T T H H H H H T H H

Runs of H of lengths r

H	3	4	5	6	7	8
#1	6	3	2	2	0	2
#2	11	5	1	0	0	0

Runs of T of lengths r

T	3	4	5	6	7	8
#1	6	1	2	2	0	0
#2	9	4	2	0	0	0

Methodology:
Expected number of runs in a sequence of coin tosses

$X_i (i = 1, \ldots, n)$ 0-1 sequence of length n,
independent and identically distributed with $P(X_i = 1) = 0.5$

$Z_r =$ number of runs of exactly length r in n tosses

$$
\begin{aligned}
E\big[Z_r\big] &= 2 \cdot E\left[\sum_{i=1}^{n-r+1} 1_{\{X_i = X_{i+1} = \ldots = X_{i+r-1} = 1, X_{i-1} = X_{i+r} = 0\}}\right] \\
&= 2 \cdot \sum_{i=1}^{n-r+1} P\Big(X_i = X_{i+1} = \ldots = X_{i+r-1} = 1, X_{i-1} = X_{i+r} = 0\Big) \\
&\approx 2 \cdot (n - r + 1) P\Big(X_2 = X_3 = \ldots = X_{r+1} = 1, X_1 = X_{r+2} = 0\Big) \\
&= 2 \cdot (n - r + 1) \cdot 2^{-(r+2)} \\
&= (n - r + 1) \cdot 2^{-r-1}
\end{aligned}
$$

```
R <- vector(length=10)
N=200
for (r in 1:10){R[r]=(N-r+1)*2^{-1-r}}
```

```
> round(R, digits=1)
 [1] 50.0 24.9 12.4  6.2  3.1  1.5  0.8
0.4  0.2  0.1
```

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R <- vector(length=10)
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```
> round(R, digits=1)
 [1] 50.0 24.9 12.4  6.2  3.1  1.5  0.8
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```

Compare with sum of the entries in the tables below

H	3	4	5	6	7	8
#1	6	3	2	2	0	2
#2	11	5	1	0	0	0

T	3	4	5	6	7	8
#1	6	1	2	2	0	0
#2	9	4	2	0	0	0

Methodology:
Distribution of the longest run

$X_i (i = 1, \ldots, n)$ 0-1 sequence of length n,
independent and identically distributed with $P(X_i = 1) = 0.5$

$R_n =$ length of the longest run of heads in n tosses

CDF $F_n(x) = P(R_n \leq x)$

$A_n =$ number of sequences of length n with longest run at most x

$F_n = 2^{-n} A_n$

Strategy:

- Partition the set of these sequences
- derive a recursive formula

Methodology:
Distribution of the longest run

Strategy:

- Partition the set of these sequences
- derive a recursive formula

Key idea:

To see how this works, consider the case in which the longest head run consists of three heads or fewer. If $n \leq 3$ then clearly $A_n(3) = 2^n$ since any outcome is a favorable one. For $n > 3$, each favorable sequence begins with either T, HT, HHT, or HHHT and is followed by a string having no more than three consecutive heads. Thus

$$A_n(3) = A_{n-1}(3) + A_{n-2}(3) + A_{n-3}(3) + A_{n-4}(3) \quad \text{for } n > 3.$$

Methodology: Distribution of the longest run

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$$A_n(3) = A_{n-1}(3) + A_{n-2}(3) + A_{n-3}(3) + A_{n-4}(3) \quad \text{for } n > 3.$$

Using the recursion, the values of $A_n(3)$ can easily be computed:

n	0	1	2	3	4	5	6	7	8	$\cdots$
$A_n(3)$	1	2	4	8	15	29	56	108	208	$\cdots$

Using the recursion, the values of $A_n(3)$ can easily be computed:

n	0	1	2	3	4	5	6	7	8	...
$A_n(3)$	1	2	4	8	15	29	56	108	208	...

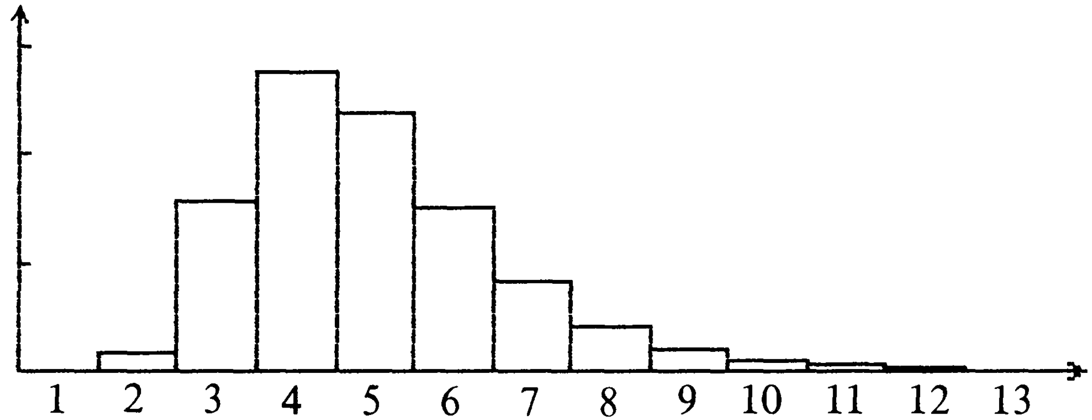
Thus for, say, $n = 8$ tosses of a fair coin, the probability is $208/2^8 = 0.8125$ that the longest head run has length no greater than 3. In the general case we obtain

$$A_n(x) = \begin{cases} \sum_{j=0}^{x} A_{n-1-j}(x) & \text{for } n > x; \\ 2^n & \text{for } n \le x. \end{cases} \tag{1}$$

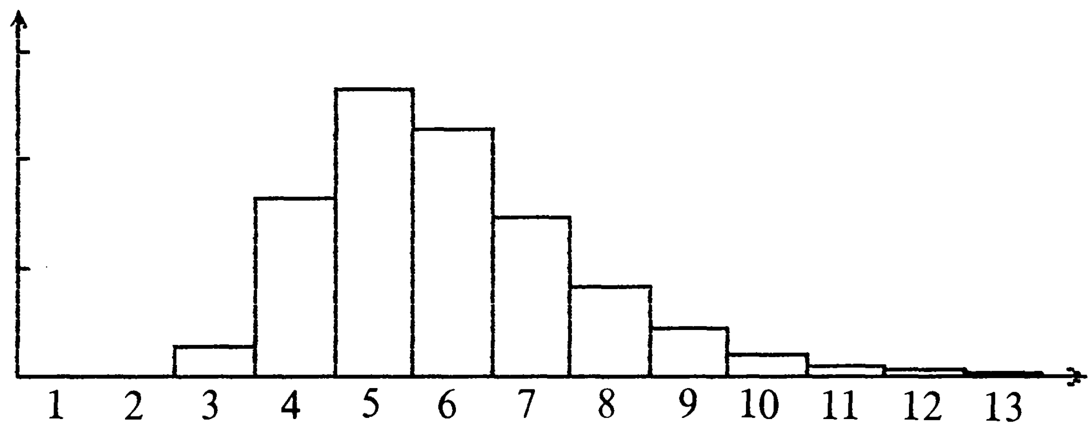
Note that for $n = 1, 2, 3, \ldots,$ the number $A_n(1)$ of sequences of length n that contain no two consecutive heads is the $(n+2)$nd Fibonacci number.

Distribution of longest run lengths

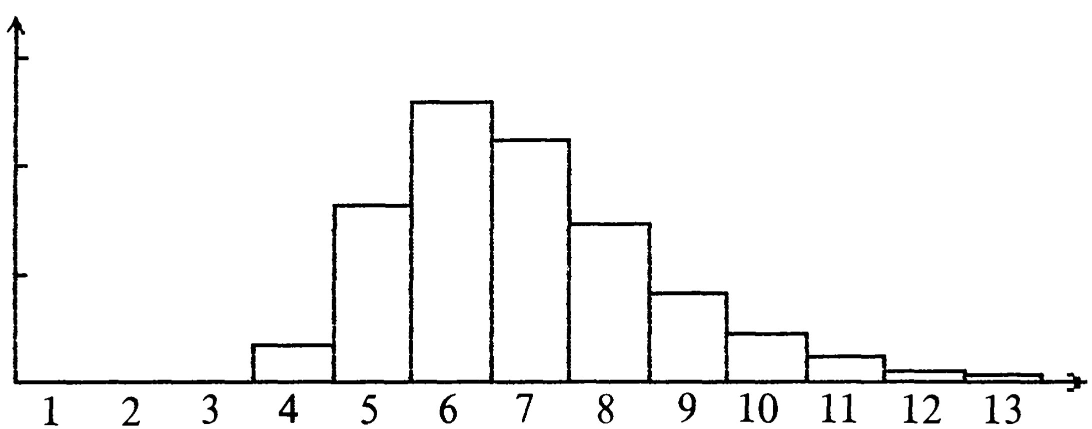
n=50



n=100



n=200



Human perception: Coin tossing

High density in heads or tails in repeated chain tossing. If a coin toss is repeated several times and the majority of the results consists of "heads", the assumption of local representativeness will cause the observer to believe the coin is biased toward "heads".

Try yourself. If you don't have any change to toss, use the online coin flip simulator at:

https://www.random.org/coins/