ST222 2017 GAMES, DECISIONS AND BEHAVIOUR
EXERCISE SHEET 3

1. Joe owes £100,000 to the Mafia. They want the money immediately. He has £60,000. He has the opportunity to place a bet in a casino in which with probability $\frac{1}{2}$ he will make a profit of £40,000 and with probability $\frac{1}{2}$ he will lose £60,000.
   (a) What is his EMV decision?
   (b) Does that coincide with common sense?
   (c) What utility function would you advise Joe to use in these circumstances?
   (d) What is the expected utility of the two possible decisions? And what is the expected utility decision?

2. The St. Petersburg Paradox shows that EMV decisions don't always lead to behaviour which we might consider sensible. It concerns the following game: You pay a fixed fee to enter, and then a fair coin will be tossed repeatedly until a tail first appears, ending the game. The pot starts at 1 dollar and is doubled every time a head appears. You win whatever is in the pot after the game ends.
   (a) Show that the expected monetary value of this game is infinite.
   (b) Show that, if your utility function U is bounded: $\forall x : U(x) \leq A$, then the expected utility of this game is also bounded.
   (c) If your utility function is the (unbounded, but in this case we can still obtain sensible results) risk-averse $U(x) = \log(x)$ then what is the expected utility of the game[1]?

3. An investor has \$1,000 to invest in speculative stocks. The investor is considering investing \$$a$ in stock A and \$$1,000 - a$ in stock B. An investment in stock A has a 0.6 chance of doubling in value, and a 0.4 chance of being lost. An investment in stock B has a 0.7 chance of doubling in value, and a 0.3 chance of being lost. Assume the stocks are independent of each other. The investor's utility function for a change in fortune, z, is $u(z) - \log(0.0007z + 1)$ for $-1,000 \leq z \leq 1,000$.
   (a) As a function of a, what are the monetary values of all four potential scenarios? What are their probabilities?
   (b) What is the optimal value of a in terms the investor's expected utility?

4. Suppose a decision maker has constant absolute risk aversion of the range −\$100 to \$1,000, that is, $u(x) = -ae^{-\lambda b} + b$, for all $x \in [-100, 1,000]$, for some constants $a, b \in \mathcal{R}$. We ask for her certainty equivalent for a gamble with prizes \$0 and \$1,000, each with probability 0.5, She says that her certainty equivalent for the gamble is 488. What, then, should she choose, if faced with the choice of:
   a gamble with prizes −\$100, \$300, and \$1,000, each with probability 1/3;
   a gamble with prizes \$530 with probability 3/4 and \$0 with probability 1/4;
   a gamble with are sure thing payment of \$385?

[1] *Hint: It may be useful to note that for* $p \in (0,1) : \sum_{n=1}^{\infty} n(1-p)p^n = \frac{p}{1-p}$.

5. You are being offered the choice between gamble A_1 and gamble A_2 and between gamble B_1 and gamble B_2 described below. Your preference is $A_1 \succ A_2$ and $B_2 \succ B_1$. Show that they are incompatible with the principle of maximising expected utility, no matter what your utility of money happens to be.
   A_1 : £50, £50, and £50, each with probability 1/3;
   A_2 : £100, £50, and £0, each with probability 1/3;
   B_1 : £50, £0, and £50, each with probability 1/3;
   B_2 : £100, £0, and £0, each with probability 1/3.

6. Let x be a bet that gives you £10,000,001 for sure, let y be a bet that gives you £10,000,000 for sure and let z be a bet that gives you 50 years in prison for sure. Your preferences are $x \succ y \succ z$.
   (a) State what the Archimedean axiom says for this situations.
   (b) What does is actually mean in terms of people's behaviour?
   (c) Discuss whether or not this is realistic. In particularly, consider that according to empirical evidence, people can not distinguish between very small probabilities.

7. The lexicographical order relation on $\mathcal{R}^2$ is defined as follows

$$(x_1, x_2) \succ (y_1, y_2) \quad \Longleftrightarrow \quad x_1 > y_1 \ \vee \ (x_1 = y_1 \ \wedge \ x_2 > y_2).$$

   (This is using the notation $x = (x_1, x_2)$ for $x \in \mathcal{R}^2$.)
   (a) Show that lexicographical order relation is complete and transitive.
   (b) Is it independent? Proof it or demonstrate that it is not true.
   (c) Does it have the Archimedean property? Proof it or demonstrate that it is not true.
   (d) (*Not examinable*) Show that it is not continuous using the following definition for continuuity:
   A preference relation on a topological space $\mathcal{A}$ is called *continuous* if for all $x \in \mathcal{A}$

$$\underline{\mathcal{B}} := \{y \in \mathcal{A} \,|\, x \succ y\} \qquad \text{and} \qquad \overline{\mathcal{B}} := \{y \in \mathcal{A} \,|\, y \succ x\}$$

   are open subsets in $\mathcal{A}$.