L3

Ex: Two-headed coin

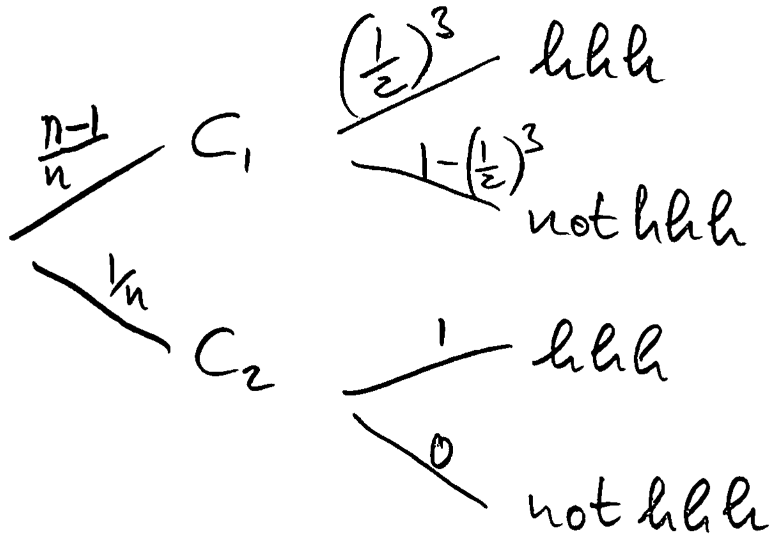
Bag with n coins, $n \geq 2$, one of which has a head on each side and the others are fair coins. Pick a coin at random and flip it three times. You observe 3 heads.
Dou you think this was the two-headed coin? How sure are you?

Model: $\Omega_1 = \{c_1, c_2\}$ c_1 fair coin, c_2 two-headed
$\Omega_2 = \{H, T\}^3$ H = heads, T = tails

$\mathcal{F}_1 = \mathcal{P}(\Omega_1) = \{\emptyset, \{c_1\}, \{c_2\}, \Omega_1\}$
(all subsets)

$\mathcal{F}_2 = \sigma(\{hhh\})$ smallest (σ-)algebra containing $\{hhh\}$
$= \{\emptyset, \{hhh\}, \{hhh\}^c, \Omega_2\}$
(we do not need all subsets, but just this case $\mathcal{F}_2$)

$\Omega = \Omega_1 \times \Omega_2$ product space
$\mathcal{F} = \sigma(\{C \times B \mid C \in \mathcal{F}_1, B \in \mathcal{F}_2\}) = \mathcal{F}_1 \otimes \mathcal{F}_2$
smallest (σ-)algebra containing all product sets of the specified form
(Ask yourself why $\sigma(\cdot)$ is needed, why not simply $\mathcal{F} = \{C \times B \mid C \in \mathcal{F}_1, B \in \mathcal{F}_2\}$?)



Compare proba is question through ratio:

$$R_n = \frac{P(C_2 \text{ and } hhh)}{P(C_1 \text{ and } hhh)} \overset{\text{not indep!}}{=} \frac{P(C_2)\cdot P(hhh \mid C_2)}{P(C_1)\, P(hhh \mid C_1)}$$

$$= \frac{1/n \cdot 1}{n-1/n \cdot (1/2)^3} = \frac{8}{n-1}$$

Concepts from mathem. statistics:
- inference
- likelyhood ratio

equally likely $\iff \frac{8}{n-1} = 1$

$\iff n = 9$

So, for $n \leq 8$ more likely coin was C_2

for $n = 9$ equally likely coin was C_1 as C_2

for $n \geq 10$ more likely coin was C_1

Ask: Can you think of ways to generalise this example?

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Ex: Class sizes from different perspectives

Simplified assumptions:

Each student takes exactly one module.

University offers 16 small modules with 10 students each.

2 large " " 100 " "

What is the average class size?

(i) X = size of a class sampled at random

$$E[X] = P(X=10)\cdot 10 + P(X=100)\cdot 100$$
$$= \frac{16}{18}\cdot 10 + \frac{2}{18}\cdot 100 = \underline{\underline{20}}$$

(ii) Y = size of the class of a randomly sampled student

$$E[Y] = P(Y=10)\cdot 10 + P(Y=100)\cdot 100$$
$$= \frac{160}{360}\cdot 10 + \frac{200}{360}\cdot 100 = \underline{\underline{60}}$$

Report by VC: "Average class size is 20."

Report by SU: "Average class size is 60."

Both are right, it depends on the perspective!

Related to the "Bus paradox".

Mathematical statistics phenomenon of size bias in sampling.

L3

<u>Ex:</u> Conny flips a coin repeatedly

p = proba for heads

<u>Model</u> (for unlimited number of trials)

heads = 1, tails = 0

$\Omega = \{0,1\}^{\mathbb{N}}$ all binary sequences

to build $\mathcal{F}$ an P first create versions for finite versions

RV $X_k : \Omega \longrightarrow \{0,1\}$ kth flip ($k \in \mathbb{N}$)

$\{X_k = 1\} = \{\omega \in \Omega \mid X_k(\omega) = 1\}$ kth flip is heads

$$\mathcal{F}_k = \sigma(X_k) = \{\emptyset, \{X_k=1\}, \{X_k=0\}, \Omega\}$$

$$\mathcal{F} = \bigotimes_{k=1}^{\infty} \mathcal{F}_k$$

P_k on $\mathcal{F}_k$: $P(X_k=1) = p$, for any finite $I \subset \mathbb{N}$ let

$$P = \bigotimes_{k=1}^{\infty} P_k : \quad P\Big(\bigcap_{k\in I} A_k\Big) = \prod_{k \in I} P(A_k) \quad A_k \in \mathcal{F}_k \ \forall k \in I$$

defines uniquely a proba measure on $(\Omega, \mathcal{F})$ (Kolmogorov-Daniell-Theorem)

$S_n = \sum_{k=1}^{n} X_k$ Distribution ?

$S_n \sim$ Binomial(n,p) (# success in n trials)

Waiting time until success = W (unbounded)

$W \sim$ Geometrical (p)

$$\begin{aligned} P(W=n) &= P(X_1 = X_2 = \dots = X_{n-1} = 0, X_n = 1) \\ &= P(X_1=0)^{n-1} P(X_n = 1) \\ &= (1-p)^{n-1} \cdot p \end{aligned}$$