W2 - L1

Ex Two people flip coins

Nick flips a nickel with proba for heads $p^{(1)}$: $X_k^{(1)}$ $(k \in \mathbb{N})$

Penny " " penny " " " " $p^{(2)}$: $X_k^{(2)}$ $(k \in \mathbb{N})$

Models as above: $(\Omega^{(1)}, \mathcal{F}^{(1)}, P^{(1)})$

$(\Omega^{(2)}, \mathcal{F}^{(2)}, P^{(2)})$

Joint model – so we can refer to events concerning both Penny and Nick

$\Omega = \Omega^{(1)} \times \Omega^{(2)}$, $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$

$= \sigma(\{A^{(1)} \times A^{(2)} \mid A^{(1)} \in \mathcal{F}^{(1)}, A^{(2)} \in \mathcal{F}^{(2)}\})$

For $A^{(1)} \in \mathcal{F}^{(1)}, A^{(2)} \in \mathcal{F}^{(2)}$

$P = P^{(1)} \otimes P^{(2)}$ is $P(A^{(1)} \times A^{(2)}) = P^{(1)}(A^{(1)}) \cdot P^{(2)}(A^{(2)})$

which models Nick and Penny as independent

(i) N = first time both are successful (get heads)

Distribution? $E[N] = ?$

$N = \min\{n \in \mathbb{N} \mid X_n^{(1)} = X_n^{(2)} = 1\}$

$A_n = \{X_n^{(1)} = X_n^{(2)} = 1\}$, N = Waiting time for A_n

$P(A_n) = P(\{X_n^{(1)} = 1\} \cap \{X_n^{(2)} = 1\})$

$= P(X_n^{(1)} = 1) P(X_n^{(2)} = 1) = p^{(1)} p^{(2)}$

$I_{A_n} \sim \text{Bernoulli}(p^{(1)} \cdot p^{(2)})$ $\qquad I_{A_n}(\omega) = \begin{cases} 1 & \omega \in A_n \\ 0 & \omega \notin A_n \end{cases}$

$N \sim \text{Geometric}(p^{(1)} \cdot p^{(2)})$

$E[N] = 1/p^{(1)} \cdot p^{(2)}$

Homework: $M = \min\{n \in \mathbb{N} \mid X_n^{(1)} = 1 \text{ \underline{or} } X_n^{(2)} = 1\}$

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(ii) M = time until at least one succeeds

$$= \min\{n \in \mathbb{N} \mid X^{(1)}_n = 1 \text{ or } X^{(2)}_n = 1\}$$

Distribution? Expectation?

Do this as homework.

Hints: Similarly to (i), define subsets in terms of $X^{(1)}, X^{(2)}$, so M can be expressed as waiting time for success for these. Calculate the probability using rules about set operations such as $(A \cap B)^C = A^C \cup B^C$ and the axioms of probability.

Solution below (upside down, try yourself first!):

$\forall n \in \mathbb{N}$ $\quad B_n = \{X_n = 1 \text{ or } Y_n = 1\}$

$$= \{X_n = 1\} \cup \{Y_n = 1\} = (\{X_n = 0\} \cap \{Y_n = 0\})^C$$

So, $P(B_n) = 1 - P(\{X_n = 0\} \cap \{Y_n = 0\})$

$= 1 - P(X_n = 0) \cdot P(Y_n = 0)$ using independence!

$= 1 - (1 - p^{(1)})(1 - p^{(2)})$ indep. of n!

M is the waiting time for B_n to occur.

$I_{B_n} \sim \text{Bernoulli}(1 - (1 - p^{(1)})(1 - p^{(2)}))$

So $M \sim \text{Geometric}(1 - (1 - p^{(1)})(1 - p^{(2)}))$

In particular, $E[M] = 1 / (1 - p^{(1)})(1 - p^{(2)})$

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Frequency interpretation of probability

X_n $(n \in \mathbb{N})$ outcomes of independently repeated random experiments

$(\Omega, \mathcal{F})$ outcome space with (σ-) algebra

$A \in \mathcal{F}$ an event (subset) of interest

What is $P(A)$?

How can we obtain this from observing $X_n (n \in \mathbb{N})$? (in stats: data)

Intuitively: $P(A) \approx \frac{1}{n} \sum_{k=1}^{n} 1_A(X_k)$ for n large

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Can we define probability like this as a limit? $P(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_A(X_k)$ $(*)$

No, because in order for this to be well defined we need to first prove that the limit exists, for which we already need to have P defined. So that is a logical circle.

However, $(*)$ can be obtained as a theorem following from the axiomatic definition and it can still serve as an interpretation of what probability actually means.

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Definition of subjective probability

People make statements like
"there's a 30% chance the UK will leave the EU with no deal"
Often, there is no way to calculate this.
So, this type of statement is a belief about the probability of an event and it is subjective.

We need a mathematical framework for this.
Crucial points:

- How can we quantify and elicit such beliefs?
- How can we link this to behaviour?
  (Attitudes and behaviour can differ. Think of the phrase: "Put your money where your mouth is!")
- Are the resulting systems of probabilities for various events actually consistent, i.e. do they define a probability measure obeying the axioms?

  E.g. Tom's beliefs did not
  (team A wins with proba 60% and does not win with proba 70%, no draw)

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Bets

Approach: Use willingness to pay to elicit belief.

$(\Omega, \mathcal{F})$ outcome space with (σ-) algebra

$A \in \mathcal{F}$, $M > 0$

$b(M, A)$ is the bet $\begin{cases} A & \text{pays } M \\ A^c & \text{pays } 0 \end{cases}$

Behavioural definition of **your** probability of A:

$m(M, A)$ is the maximum you would pay for entering the bet $b(M, A)$.

Comments:

- Can do this for other event, too. $A, B \in \mathcal{F}$ with $m(M,A) = m(M,B)$ means they are equally likely.
- Assume willingness to bet.

Assume this has been normalised, e.g. use

$$\tilde{m}(M,A) = \frac{m(M,A)}{m(M,\Omega)}$$

and then define subjective probability $P_{you}(A) = \tilde{m}(M, A)$

- Note that r.h.s. depends on M. For this to make sense we need to assume that (through normalisation) M is irrelevant. However, it's a psych. ass!