Properties of binary relations

$(B, \succ)$ B set $\succ$ relation   (or $\prec, \sim, \preceq, \succeq$)

$\wedge$ means AND
$\vee$ means OR
$\neg$ means NOT (for entire expression)

(C) Completeness:
$\forall x, y \in B \quad x \succ y \vee y \succ x \vee x \sim y$

(A) Asymmetry:
$\forall x, y \in B \quad x \succ y \Rightarrow \neg\, y \succ x$

(T) Transitivity:
$\forall x, y, z \in B \quad x \succ y \wedge y \succ z \Rightarrow x \succ z$

(NT) Negative transitivity:
$\forall x, y, z \in B \quad \neg\, x \succ y \wedge \neg\, y \succ z \Rightarrow \neg\, x \succ z$

Lemma: (A) & (NT) $\Rightarrow$ (T)

Proof (indirect): Let $x \succ y \wedge y \succ z$. Show $x \succ z$.
Assume the latter is wrong: $\neg\, x \succ z$.
Because of (A), we also have $\neg\, z \succ y$.
Hence by (NT) it is $\neg\, x \succ y$. Contradiction. $\square$

Properties of preferences as binary relations and discussion

(c) Completeness:

- Forces people to "make up their minds". Often appropriate, but sometimes not realistic.
  e.g. choose between
  x = "be stupid and satisfied"
  y = "be smart and dissatisfied"
  (Original example by Savage is more dramatic: Choose whether you are hung until your health suffers or your reputation is damaged.)
  The choices in these examples are
  __incommensurable__.
  An assumption necessary to achieve completeness is that the actions in $\mathcal{A}$ are sufficiently __commensurable__.

  See www.merriam-webster.com/dictionary/commensurable
  (Can also demonstrate how to pronounce it!)
  Check also WiKi pages for this term under economics, philosophy of science and mathematics.

(C) (ct)

- Another assumption on $\succsim$ is that the choices are <u>available</u> at the same time in the same place.

  E.g. $x =$ "eat foie gras" $y =$ "eat hotdog"

  Surely, there will be a restaurant in NYC that offers both French haute cuisine and streetfood, but that is unusual!

(T) Transitivity:

- What do you want in your coffee?

  People tend to be indifferent between small differences, e.g.

  no sugar $\sim$ 1 grain sugar $\sim$ 2 grains sugar $\sim$ ... $\sim$ 1 spoon sugar

  But this does not imply

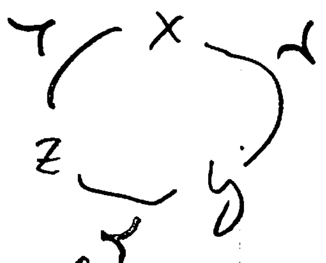
  no sugar $\sim$ 1 spoon sugar

  (Similar issue to the fact that the world is locally flat, but still not globally flat.)

(T) (ct)

- Transitivity means "no cycles".

A cycle would be this:



How realistic is the no-cycles-assumption?

Dutch Book type argument says that a rational person should not have cycles in their preferences, because otherwise one could exploit this by constructing a money pump as follows:

Assume Julia's preferences are

$$x \succ y \wedge y \succ z \wedge z \succ x$$

Then she would be happy to pay some amount $\alpha_1 > 0$ to swap z for x. She would also be willing to pay then $\alpha_2 > 0$ to swap z for y, and then $\alpha_3 > 0$ to swap y for x. So she is back to x and has lost $\alpha_1 + \alpha_2 + \alpha_3$!

Continue this and she will end up loosing any arbitrary amount of money.

(T) (ct)

An example for how this can be exploited by sellers of whatever goods is this:

Purchase of a smart phone

Evaluate preferences using ranks (1 best)

	Design	Function	Price
iphone	1	2	3
jphone	2	3	1
kphone	3	1	2

and say $x \succ y$ iff majority of criteria in x is better than in y. This yields

$$i \succ j \;\wedge\; j \succ k \;\wedge\; k \succ i$$

which is cyclic!

Hence, smart phone preferences may not be transitive.

This explains why (smart) people buying smartphones can be exploited using money pump techniques explained above.

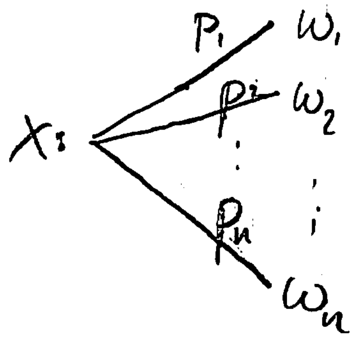
Review/extension of definitions:

Ω outcome space

$\mathcal{A}$ action space (e.g. bets, lotteries)

For Ω finite: $x \in \mathcal{A}$ has form $b(\omega_1, p_1; \dots; \omega_n, p_n)$

$(p_i)_{i=1,\dots,n}$ PMF



For Ω countable: similar

$(p_i)_{i \in \mathbb{N}}$ PMF, $b(\omega_i, p_i; i \in \mathbb{N})$

Essentially, action x is a prob. measure on Ω, so $\mathcal{A}$ can be identified with a set of probability measures on Ω (with some algebra $\mathcal{F}$).

For Ω continuous: $(\Omega, \mathcal{F})$ measurable space,
(not examinable)
i.e. Ω outcome space with σ-algebra $\mathcal{F}$.

$\mathcal{A}$ is a set of probability measures on $(\Omega, \mathcal{F})$.