A model-based selling propensity: Prospect Theory, multiple agents and the disposition effect*

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Abstract

We develop a model-based, state-dependent selling propensity, $\mathtt{D}(\cdot)$, in an optimal asset sale framework where the price P follows a time-homogeneous diffusion. $\mathtt{D}(\cdot)$ is defined for a class of selling strategies, but it only depends upon the optimal target distribution and price dynamics. We extend this to a heterogeneous-agent economy by aggregating individual target distributions. We specialise to a Prospect theory agent and give examples that aim to match empirical hazard rates from the literature, which exhibit the disposition effect. Using the Bertoin-Le Jan embedding construction, we formalize the interpretation of $\mathtt{D}(\cdot)$ as a stopping intensity. If the agent sells at a level-dependent rate per unit time given by $\mathtt{D}(\cdot)$, the optimal distribution is attained.

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1 Introduction

The disposition effect (DE) can be described as the observation that investors tend to sell assets that have increased in value while holding on to assets that have declined. In other words, holding losers for "too long" and selling winners "too soon", see [20]. Odean's well known empirical study, [17], compares the frequency with which investors sell winners and losers relative to opportunities to sell each and finds gains are realised at a rate around 50% higher than losses in his dataset.[1] Similar estimates were found by [8].

Prospect theory (PT) provides a leading candidate explanation of the disposition effect. A number of optimal stopping models of asset sale timing employing the S-shaped utility of PT, have linked their results to the DE, including [2], [9], [11] and [14]. The optimal prospect takes the form of a one or two-sided threshold strategy. [9] proposes a continuous time model-analog to the Odean definition (the ratio of the propensity to sell at a gain to the propensity to sell at a loss) where the propensity to sell at a gain is the probability that the stopped process stops at the gain threshold divided by the expected time that the process spends above the reference level up to the stopping time.[2] [10] (see also [22]) includes probability weighting and solves for the optimal prospect (the optimal distribution of the stopped process) under a class of preference structures and price

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[1]The PGR (PLR) is the number of times a gain (loss) is realised as a fraction of the total number of times a gain (loss) could have been realised. Odean reports a PGR=0.148 and a PLR=0.098, giving a disposition ratio of 1.51, or equivalently, investors realise gains at a 50% higher rate than losses in his dataset.

[2]In fact, although it is possible to match Odean's estimate, because the investor realizes gains when they are relatively small but waits until losses are relatively large, losses are realized rarely, and the disposition effect obtained

dynamics. The optimal distribution involves a point mass in the loss regime (loss threshold), a possible point mass at some level a in the gains regime together with a continuous distribution on (a, ∞). When probability weighting is used in tandem with the other ingredients of PT, it moderates the level of the DE predicted by the model (using the measure of [9]) to give values much closer to empirical estimates. However, both the empirical study of [17], and the models discussed so far, are concerned with a single-number estimate of the DE, and are not considering the magnitude of the gains or losses.

More recent empirical studies of the DE investigate how the rate of sale depends on the magnitude of gains and losses. [3] documents the probability of selling as a function of the size of the gains/losses whilst allowing for different prior holding periods. Other researchers estimate proportional hazards models to derive the hazard rate for the sale of stock conditional on return since purchase (see [1]). There is agreement that the estimated hazard rate or probability of sale as a function of returns since purchase is asymmetric, higher on gains than on losses. This supports a DE. Excluding data for very short holding periods, researchers find that the hazard or probability of selling is fairly flat on losses ([3], [13])[3].

The aim of this work is to derive a model based state-dependent measure $\mathtt{D}(\cdot)$ for the selling propensity in an optimal asset sale framework where the price P follows a time-homogeneous diffusion. $\mathtt{D}(\cdot)$ depends upon the optimal target distribution and price dynamics. We show in Section 2.3 how the target distributions for single agents can be aggregated to provide a measure of the selling propensity for the population of investors. In Section 2.4, we show that the selling propensity can be interpreted as a stopping intensity, under some regularity assumptions. In other words, if the agent sells at a level-dependent rate per unit time dictated by the selling intensity, the target distribution is attained. The argument exploits the Bertoin-Le Jan embedding construction presented in [4] and further discussed in [7]. Despite being similar in structure, our construction is different from the one presented in [6] in its aim and specification. In [6], given a target distribution μ, the goal is to construct a diffusion process via its *speed measure* that embeds μ at an independent and exponentially distributed random time. In our setting, both the diffusion process and the target distribution are given as inputs, and we want to construct a stopping time that solves the Skorokhod embedding problem (SEP) of μ into the process used to model the price dynamics. On one hand, this slightly complicates the derivation of the embedding for the truncated version of the target distribution (see Lemma 2.12). On the other hand, the different aim affects the way this truncation is relaxed (see Lemma 2.14).

We specialise to a Prospect theory setting with the price process following a geometric Brownian motion in Section 3. The target distribution represents the unique optimal prospect of an agent solving the stopping problem in [10]. We numerically compute $\mathtt{D}(\cdot)$ and demonstrate the asymmetric treatment between gains and losses for a single agent. In order to compute $\mathtt{D}(\cdot)$ for multiple agents, we derive a scaling property that allows us to specify the optimal distribution of a PT agent for different levels of the loss aversion coefficient. We show the aggregate target distribution admits a density and we give a numerical example. Finally, Section 3.3.3 shows that the model based framework proposed is flexible enough to accommodate a selling propensity that is flat on losses, which is consistent with the findings of the empirical literature on the disposition effect.

is overly pronounced. [11] extends [9] to a realisation utility setting with reinvestment, and again shows it is difficult to match the Odean statistic. To counter this, they ultimately mix their optimal threshold strategy in a 50-50 mix with random Poisson traders to match numerically with Odean's figure.

[3]There is less consensus for the overall shape, with findings depending upon the length of the holding period. Over short holding periods, [3] demonstrates an asymmetric V-shaped pattern in their probability of sale graph. However, when holding periods are aggregated, the selling intensity function may be much flatter (but higher on gains than losses) or even show a slightly inverted V shape (e.g., [17], [13]). Since our modelling does not distinguish between holding periods, our focus is on the empirical findings that have been computed at the aggregate level.

2 State-dependent selling propensity

2.1 General framework

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space under usual conditions such that we can define a Brownian motion $B = (B_t)$ on it. Assume that the price process $P = (P_t)$ is a one-dimensional time-homogeneous Itô diffusion process such that

$$dP_t = \mu(P_t)dt + \sigma(P_t)dB_t \quad \text{with} \quad P_0 = p_0 \tag{2.1}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are continuous functions and such that a weak solution to the SDE above exists. Let $I_P \subseteq \mathbb{R}$ be the state space of P having endpoints $-\infty \leq \ell_p < u_p \leq \infty$, the process is assumed to be regular in I_P, and to have natural boundaries. We want to transform P into a local martingale through its scale function $s(\cdot)$. In particular, the process (X_t) defined as $X_t = s(P_t)$ is a local martingale with state space $I_X \subseteq \mathbb{R}$ (see [19] Section VII.3).

Assumption 2.1. Let ℓ and u be the endpoints of I_X, assume X is bounded below by $\ell > -\infty$.

By Fatou's lemma, X is a supermartingale. Doob's supermartingale convergence theorem implies that, as t goes to ∞, X converges $\mathbb{P}$-a.s. to a $\mathcal{F}_\infty$-measurable and integrable random variable, say X_∞. Later we will assume that $X_\infty = \ell$ $\mathbb{P}$-a.s..

If $\mu(\cdot)$ and $\sigma(\cdot)$ are continuous with $\sigma(\cdot)$ not vanishing in I_P, then, for an arbitrary point c in the interior of the state space I_P, $s(\cdot)$ is such that

$$s(x) = \int_c^x \exp\left(-2\int_c^y \frac{\mu(z)}{\sigma^2(z)}dz\right)dy. \tag{2.2}$$

Then, $s(\cdot)$ is twice continuously differentiable, and by Itô's formula $dX_t = \xi(X_t)dB_t$ for $\xi(\cdot) = ((s'\sigma) \circ s^{-1})(\cdot) = s'(s^{-1}(\cdot))\sigma(s^{-1}(\cdot))$ continuous function. This gives the quadratic variation of X as $d\langle X\rangle_t = \xi^2(X_t)dt = [s'(s^{-1}(X_t))\sigma(s^{-1}(X_t))]^2dt$.

The utility framework we consider in Section 3 is such that utility comes from realized gains and losses, which are defined with respect to a reference level r. If τ is a liquidation strategy, then $P_\tau \geq r$ represents the *gain* region, similarly $P_\tau < r$ specifies the *loss* region. Scale functions are unique up to *positive affine transformation*; therefore, we can shift $s(\cdot)$ such that whenever $X_t \geq 0$, $P_t \geq r$ (*gain region*) and similarly for $X_t < 0$ (*loss region*). This holds if we use as a scale function defined as $x \mapsto s(x) - s(r)$ and note if $r = P_0$, then $X_0 = 0$.

2.2 Derivation of the selling propensity

Let τ be a $\mathbb{P}$-a.s. finite stopping time representing a stopping rule that the agent (or the market itself, in the multi-agent framework) follows to decide when to liquidate her position, and assume that $X_\tau \sim \nu$. On the one hand, ν is a target distribution exogenously determined with support included in the interior of I_X[4]. This requires finding a stopping time τ such that $X_\tau \sim \nu$, which, under additional regularity assumptions, can be identified through Skorokhod embedding theory. On the other hand, we can provide a stopping time as input to the model and work out the distribution of X_τ, which will be endogenously derived; no embedding theory is needed in that case.

We aim to derive a state-dependent quantity that measures the agent's *selling propensity*. We extend [9] (see also [10]), where a measure of the *disposition effect* is derived as the ratio between a notion of selling propensity on *gains* and selling propensity on *losses*

$$\mathtt{D} = \frac{\mathbb{P}(X_\tau > 0)}{E[\int_0^\tau \mathbb{1}_{(X_t>0)}dt]} \cdot \frac{E[\int_0^\tau \mathbb{1}_{(X_t<0)}dt]}{\mathbb{P}(X_\tau < 0)}. \tag{2.3}$$

[4] As specified later, we will work with $\mathbb{P}$-a.s. finite stopping times. This implies that, since ℓ can not be reached by the diffusion X in finite time; the target distribution ν can not assign positive mass to ℓ.

Let $\text{supp}(\nu) = \{x \in \mathbb{R} : \text{for any open } B \in \mathscr{B}(\mathbb{R}) \text{ such that } x \in B,\ \nu(B) > 0\}$ and let I_ν be the smallest interval such that $\text{supp}(\nu) \subseteq I_\nu$. A natural approach would be to fix an interior state $x \in \text{int}(I_\nu)$, $\varepsilon > 0$, define $B(x,\varepsilon)$ as the open ball with center x and radius ε, and consider

$$\mathtt{D}(x) = \lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))}{E[\int_0^\tau \mathbb{1}_{B(x,\varepsilon)}(X_t)dt]} = \lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))/m(B(x,\varepsilon))}{E[\int_0^\tau \mathbb{1}_{B(x,\varepsilon)}(X_t)dt]/m(B(x,\varepsilon))} \tag{2.4}$$

with m denoting the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. By focusing on the numerator, we have

$$\lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))}{m(B(x,\varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))}{2\varepsilon} \tag{2.5}$$

Since ν is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, and probability measures are Radon measures, we can rely on the notion of derivative for Radon measure with respect to the Lebesgue measure m, Whenever the extended limit exists, denote the resulting density at x by $D_m\nu(x)$. The limit in (2.5) then represents the derivative of ν with respect to m, *m-a.e.* in $\mathbb{R}$. We can decompose ν into its *singular* and *absolutely continuous* components, ν_s and ν_c respectively. This implies that

$$\lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))}{m(B(x,\varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{\nu_c(B(x,\varepsilon))}{m(B(x,\varepsilon))} + \lim_{\varepsilon \to 0^+} \frac{\nu_s(B(x,\varepsilon))}{m(B(x,\varepsilon))} = D_m\nu_c(x) + D_m\nu_s(x) \quad m\text{-a.e.} \tag{2.6}$$

where $D_m\nu_s(x) = 0$ and $D_m\nu_c(x) = D_m\nu(x)$ *m-a.e.* in $\mathbb{R}$

Remark 2.2. These results guarantee that $D_m\nu(x)$ is well defined *m-a.e.*. This does not exclude the existence of exceptional points at which the limit is infinite or does not exist. Both sets of exceptional points are m-null.

Assumption 2.3. ν_s is defined through at most countably many point masses. In other words, the singular component derived in the Lebesgue decomposition is purely atomic:

$$\nu_s(da) = \sum_{n \geq 1} p_n \delta_{c_n}(da) \tag{2.7}$$

where (c_n) is the sequence of jump locations characterizing the cdf of X_τ (i.e., $F(x) = \nu((-\infty, x])$), and (p_n) is the sequence of probability masses assigned to each jump location. If we call $f(x)$ the density of $\nu_c \ll m$, then we can write

$$\nu(da) = \sum_{n \geq 1} p_n \delta_{c_n}(da) + f(a)da = \nu_s(da) + \nu_c(da) \tag{2.8}$$

Computations show that $D_m\nu(c_n) = \infty$, for any $n \geq 1$. This does not contradict the fact that $D_m\nu_s(x) = 0$ *m-a.e.* since $\{c_n : n \geq 1\} = \cup_{n \geq 1}\{c_n\}$, which is a countable union of sets having Lebesgue measure 0.

For $x \in \text{int}(I_\nu)$ consider the limit

$$\lim_{\varepsilon \to 0^+} \frac{1}{m(B(x,\varepsilon))} E\left[\int_0^\tau \mathbb{1}_{(X_t \in B(x,\varepsilon))}dt\right] = \lim_{\varepsilon \to 0^+} E\left[\frac{1}{2\varepsilon}\int_0^\tau \mathbb{1}_{(X_t \in B(x,\varepsilon))}dt\right] \tag{2.9}$$

Let $(L_t^a(X))$ be the bi-continuous modification of the family of local times for X (see [19] chapter VI). Moreover, if $\sigma(\cdot) > 0$, the fact that $s(\cdot)$ is strictly increasing implies $\xi(\cdot) > 0$. The occupation time formula then gives:

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} E\left[\frac{1}{2\varepsilon}\int_0^\tau \mathbb{1}_{B(x,\varepsilon)}(X_s)ds\right] = \lim_{\varepsilon \to 0^+} E\left[\frac{1}{2\varepsilon}\int_0^\tau \mathbb{1}_{B(x,\varepsilon)}(X_s)\frac{1}{\xi^2(X_s)}d\langle X\rangle_s\right] \\ &= \lim_{\varepsilon \to 0^+} E\left[\frac{1}{2\varepsilon}\int_\ell^u \mathbb{1}_{B(x,\varepsilon)}(a)\frac{L_\tau^a(X)}{\xi^2(a)}da\right] = \lim_{\varepsilon \to 0^+} \frac{1}{m(B(x,\varepsilon))}\int_{B(x,\varepsilon)} \frac{E\left[L_\tau^a(X)\right]}{\xi^2(a)}da \end{aligned} \tag{2.10}$$

Proposition 2.4. Let τ be a $\mathbb{P}$-a.s. finite stopping time such that $X_\tau \sim \nu$, with ν being an integrable measure with mean X_0 (i.e., $E[X_\tau] = X_0$). If $E[\langle X \rangle_\tau] < \infty$, then

$$E\left[L_\tau^x(X)\right] = E\left[|X_\tau - x|\right] - E\left[|X_0 - x|\right] = \int |y-x|\nu(dy) - |X_0 - x| = U_\nu(x) - U_{\delta_{X_0}}(x) \quad (2.11)$$

$U_\nu(\cdot)$ is the potential of the measure ν (see [5], [16]). Properties of potentials show that for any integrable distribution μ with mean $\bar{\mu}$, $\{x : U_\mu(x) > |x - \bar{\mu}|\} = \text{int}(I_\mu)$, while $U_\mu(x) = |x - \bar{\mu}|$ elsewhere. This implies that $E\left[L_\tau^x(X)\right] = U_\nu(x) - U_{\delta_{X_0}}(x)$ is strictly positive for $x \in \text{int}(I_\nu)$ and 0 otherwise.

This representation, jointly with properties of potentials, shows that $a \mapsto E[L_\tau^a(X)]$ is a continuous function. Then, under these modeling assumptions,

$$\lim_{\varepsilon \to 0^+} \frac{1}{m(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \frac{E\left[L_\tau^a(X)\right]}{\xi^2(a)} da = \frac{E\left[L_\tau^x(X)\right]}{\xi^2(x)} \qquad \text{for any } x \in \text{int}(I_\nu). \quad (2.12)$$

Remark 2.5. If X is a continuous local martingale, $E[X_\tau] = X_0$ and $E[X_\tau^2] < \infty$ (ν is a squared integrable distribution with mean X_0), then we can rely on a class of $\mathbb{P}$-a.s. finite embeddings such that $E[\langle X \rangle_\tau] = E[X_\tau^2] - X_0^2$. This last property directly verifies the assumption of integrability for $\langle X \rangle_\tau$, to obtain the expansion of $E[L_\tau^\cdot(X)]$ as in (2.11). In the literature of Skorokhod embeddings (see [16] for a survey) many constructions are such that (X_t^τ) is a uniformly integrable (UI) martingale. Actually, UI of the stopped process is a requirement in the "classical" formulation of the SEP (it expresses the idea of τ being "small"). This is closely related with the representation $E[\langle X \rangle_\tau] = E[X_\tau^2] - X_0^2$, in case of square integrable target distributions. Indeed, one shows that they are equivalent. Uniform integrability of the stopped process comes with interesting implications in Skorokhod embedding theory[5]. As argued in [15] and [16], (X_t^τ) UI martingale is equivalent to say that τ is minimal. This means that for any embedding $\sigma \leq \tau$ such that $X_\sigma \sim X_\tau \sim \nu$, then $\tau = \sigma$ $\mathbb{P}$-a.s.. This class of embeddings that make (X_t^τ) UI martingale is rather rich, Azéma-Yor and Chacon-Walsh constructions are some examples, others can be found in [16].

The results in this section are summarized by the following proposition:

Proposition 2.6. Under the assumptions of Proposition 2.4, for every $x \in \text{int}(I_\nu)$ at which $D_m\nu(x)$ exists as an extended limit, the selling propensity is given by

$$\mathtt{D}(x) = \lim_{\varepsilon \to 0^+} \frac{\nu(B(x,\varepsilon))}{E[\int_0^\tau \mathbb{1}_{(X_t \in B(x,\varepsilon))} dt]} = \frac{D_m\nu(x)\xi^2(x)}{\int |y-x|\,\nu(dy) - |X_0 - x|} \in [0, \infty]. \quad (2.13)$$

Under the assumptions of Proposition 2.4, $0 < E[L_\tau^x(X)] < \infty$ for any $x \in \text{int}(I_\nu)$. This means that, where the $D_m\nu(\cdot)$ exists pointwise, $\mathtt{D}(x) = \infty$ if and only if $D_m\nu(x) = \infty$ (e.g., when x is an atom), while $\mathtt{D}(x) = 0$ if and only if $D_m\nu(x) = 0$ (e.g., when x is in $\text{int}(I_\nu)$ but not in $\text{supp}(\nu)$).

To conclude, we adopt the convention that $\mathtt{D}(x) = \infty$ for $x \notin \text{int}(I_\nu)$. This reflects the fact that no expected local time is accumulated at such states at liquidation. At a boundary state, the convention represents immediate liquidation rather than a finite selling intensity. In particular,

$$\mathtt{D}(x) = \begin{cases} \frac{D_m\nu(x)\xi^2(x)}{\int |y-x|\nu(dy) - |X_0 - x|} & \text{if} \quad x \in \text{int}(I_\nu) \\ \infty & \text{if} \quad x \notin \text{int}(I_\nu). \end{cases} \quad (2.14)$$

Recall that ν represents the target distribution such that $X_\tau \sim \nu$. Since X is the transformed price process $s(P)$, $\mathtt{D}(\cdot)$ above represents the selling propensity expressed in terms of X. From a conceptual point of view, we want to return to the original price process P. This can be done by computing $p \mapsto \mathtt{D}(s(p))$ for $p \in I_\mu$, where I_μ is the smallest interval containing $\text{supp}(\mu)$; μ is such that $\nu(A) = \mu(s^{-1}(A))$ for any $A \in \mathscr{B}(\mathbb{R})$ and $P_\tau \sim \mu$.

Remark 2.7. The selling propensity derived does not depend on a specific stopping strategy. Let τ_1, τ_2 be stopping times such that $X_{\tau_1} \sim X_{\tau_2} \sim \nu$ and $\max(E[\langle X \rangle_{\tau_1}], E[\langle X \rangle_{\tau_2}]) < \infty$, then $\mathtt{D}(\cdot)$ as defined in Proposition 2.6, will coincide for both the stopping strategies τ_1 and τ_2.

[5] [16] also provides a characterization of uniform integrability of the stopped process X^τ in terms of potentials of the target distribution and initial distribution.

2.3 Multi-agent framework

We previously assumed that the target distribution ν admits the following decomposition

$$\nu(da) = \sum_{n\geq 1} p_n \delta_{c_n}(da) + f(a)da = \nu_s(da) + \nu_c(da). \tag{2.15}$$

Computations show that $D_m\nu(c_n) = \infty$ for any $n \geq 1$, given $p_n > 0$. This comes as a feature of the model to produce "spikes" in the selling propensity $\mathtt{D}(\cdot)$ when computed in the jump locations. We propose here an approach that can be used to smooth out these spikes in $D_m\nu(\cdot)$ by averaging over uncountably many model parameterisations.

Let us assume that the target distribution ν depends on a set of parameters $\vartheta \in \Theta \subseteq \mathbb{R}^d$ open. We consider ϑ as a random variable taking values in Θ with distribution such that it admits a density $\varphi(\cdot)$. This approach will lead to the derivation of an "average" distribution that describes the market-level mixture of liquidation distributions. Since ϑ describes some aspects that affect the agents' optimal liquidation strategies (e.g., risk preferences), by assuming a "reasonable" distribution on Θ, we can average out the behavior of each agent across the entire market. Later, in the PT setting of Section 3, we will use ϑ to model the risk preference parameters of the S-shaped utility function and the distortion parameters in the probability weighting functions.

For a given realization of ϑ, let ν^ϑ be the target distribution, assume that $\vartheta \mapsto \nu^\vartheta(A)$ is a measurable function for any Borel set A. Define $\hat{\nu} : \mathscr{B}(\mathbb{R}) \to [0,1]$ as

$$\hat{\nu}(da) = \int_\Theta \nu^\vartheta(da)\varphi(\vartheta)d\vartheta. \tag{2.16}$$

It follows that $\hat{\nu}$ above is a probability measure defined on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, which writes as

$$\hat{\nu}(A) = \int_\Theta \nu_s^\vartheta(A)\varphi(\vartheta)d\vartheta + \int_\Theta \nu_c^\vartheta(A)\varphi(\vartheta)d\vartheta = \hat{\nu}_s(A) + \hat{\nu}_c(A). \tag{2.17}$$

Notice that $\hat{\nu} = \hat{\nu}_s + \hat{\nu}_c$ does not necessarily coincide with the Lebesgue decomposition of $\hat{\nu}$. On the contrary, we will show that, under additional assumptions, $\hat{\nu} \ll m$. On the one hand, $\nu_c^\vartheta \ll m$ directly implies $\hat{\nu}_c \ll m$. Indeed, let $A \in \mathscr{B}(\mathbb{R})$ such that $m(A) = 0$

$$\hat{\nu}_c(A) = \int_\Theta \nu_c^\vartheta(A)\varphi(\vartheta)d\vartheta = \int_\Theta 0\varphi(\vartheta)d\vartheta = 0. \tag{2.18}$$

The density of $\hat{\nu}_c$ with respect to m can be obtained by exploiting $\nu_c^\theta \ll m$ and its integral representation. Let $A \in \mathscr{B}(\mathbb{R})$, Fubini shows

$$\hat{\nu}_c(A) = \int_\Theta \nu_c^\vartheta(A)\varphi(\vartheta)d\vartheta = \int_\Theta \left(\int_A f^\vartheta(x)dx\right)\varphi(\vartheta)d\vartheta = \int_A \left(\int_\Theta f^\vartheta(x)\varphi(\vartheta)d\vartheta\right)dx. \tag{2.19}$$

Let us now focus on first component $\hat{\nu}_s$, since ν_s is purely atomic, we can express $\hat{\nu}_s$ as follows

$$\hat{\nu}_s(A) = \int_\Theta \left(\sum_{n\geq 1} p_n^\vartheta \delta_{c_n^\vartheta}(A)\right)\varphi(\vartheta)d\vartheta = \sum_{n\geq 1}\int_\Theta p_n^\vartheta \delta_{c_n^\vartheta}(A)\varphi(\vartheta)d\vartheta \quad \text{for } A \in \mathscr{B}(\mathbb{R}). \tag{2.20}$$

We want to show $\hat{\nu}_s \ll m$. We can approach this problem by looking for a non-negative integrable function f_s defined on $\mathbb{R}$ that allows us to derive the following integral representation for $\hat{\nu}_s$.

$$\hat{\nu}_s(A) = \int_A f_s(x)dx. \tag{2.21}$$

Proposition 2.8. Let ν be an admissible target distribution satisfying the decomposition in equation (2.15), let Θ be an open subset of $\mathbb{R}^d$ and φ a density defined on it. Assume that, for any $n \geq 1$, the relation that assigns to every ϑ the *n-th* jump location is a continuously differentiable

function $\vartheta \mapsto c_n(\vartheta)$ such that $|\nabla c_n(\vartheta)| > 0$[6]. Assume for simplicity that the number of point masses is independent of ϑ[7]. Then $\hat{\nu}_s \ll m$ with density defined as below

$$\hat{\nu}_s(A) = \int_A \left(\sum_{n\geq 1} \int_{\{\vartheta : c_n(\vartheta)=x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) \right) dx = \int_A f_s(x)dx \tag{2.22}$$

where $\mathscr{H}_k$ is the k-dimensional Hausdorff measure.

Therefore, under the assumptions stated in Proposition 2.8, we have that $\hat{\nu} \ll m$ with density

$$f(x) = D_m\hat{\nu}(x) = \sum_{n\geq 1} \int_{\{\vartheta : c_n(\vartheta)=x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) + \int_\Theta f^\vartheta(x)\varphi(\vartheta)d\vartheta = f_s(x) + f_c(x). \tag{2.23}$$

This procedure, under certain assumptions, allows us to derive a new distribution that is absolutely continuous with respect to the Lebesgue measure. It also lets us smooth out the spikes that would otherwise be present in $D_m\nu_s^\vartheta(\cdot)$ at the jump locations for the single-agent problem.

Remark 2.9. Since we want to identify the stopping strategy that delivers the target distribution also in the multi-agent framework (to be consistent with the derivation of $\mathtt{D}(\cdot)$), we need to be careful in the way we specify the set Θ. In particular, we want to average over Θ in such a way that the distribution $\hat{\nu}$ can be embedded into X; for example, we need $\text{supp}(\hat{\nu}) \subseteq \text{int}(I_X)$ (see [6]).

2.4 $\mathtt{D}(\cdot)$ as stopping intensity and a related embedding

This section focuses on the construction of a specific embedding that uses the selling propensity $\mathtt{D}(\cdot)$ as a state-dependent stopping intensity. The main idea is to use the construction of embeddings presented in [4] and further discussed in [7] to build a stopping time $\tau^\star$ that embeds the target distribution ν into the local martingale X. In fact $\tau^\star$ can be interpreted as a "*default time*" represented by the first jump of a Cox process whose intensity is defined by using $\mathtt{D}(\cdot)$.

Without loss of generality, set $X_0 = 0$[8], and consider the time change below expressed as the right-continuous inverse of $t \mapsto \langle X\rangle_t$

$$T(t) = \inf\{s \geq 0 : \langle X\rangle_s > t\} \tag{2.24}$$

with $\langle X\rangle_\infty = \infty$ $\mathbb{P}$-a.s., the clock $T(t) < \infty$ $\mathbb{P}$-a.s. for any $t \geq 0$. In our case, $\mathbb{P}$-a.s. convergence of X implies $\langle X\rangle_\infty < \infty$ $\mathbb{P}$-a.s. (see [19], Chapter IV, Section 1). Therefore, if $t \geq \langle X\rangle_\infty$, $t \mapsto \langle X\rangle_t$ being increasing implies that $\{s \geq 0 : \langle X\rangle_s > t\} = \varnothing$. Therefore $T(t) = \infty$ for any $t \geq \langle X\rangle_\infty$. Therefore, by time changing the local martingale X, we obtain a *stopped* Brownian motion defined with respect to the time-changed filtration $\hat{\mathbb{F}}$ (see DDS theorem, [19], Chapter V). In particular,

$$W_t = \begin{cases} X_{T(t)} & \text{if } \langle X\rangle_\infty > t \\ X_\infty & \text{if } \langle X\rangle_\infty \leq t. \end{cases} \tag{2.25}$$

If X_∞ is $\mathbb{P}$-a.s. equal to a state not accessible in finite time, as later specified, $\langle X\rangle_\infty$ can be equivalently written as $\inf\{t \geq 0 : W_t = X_\infty\}$, while $\hat{\mathbb{F}}$ is such that $\hat{\mathcal{F}}_t = \mathcal{F}_{T(t)}$ (with the convention of $\hat{\mathcal{F}}_t = \mathcal{F}_\infty$ on $\{T(t) = \infty\}$).

The aim is to show that the SEP for the continuous local martingale X can be solved by looking at the embedding of the same target distribution in the stopped Brownian motion W (i.e., finding $\hat{\tau}$ such that $W_{\hat{\tau}} \sim \nu$), and solve the latter under the assumption of ν integrable with mean less than $X_0 = 0$. Provided that, if $\hat{\tau} < \langle X\rangle_\infty$ is a stopping time with respect to $\hat{\mathbb{F}}$, then $T(\hat{\tau})$ is a stopping time with respect to $\mathbb{F}$, and $\hat{\tau}$ is such that $W_{\hat{\tau}} \sim \nu$, then $X_{T(\hat{\tau})} \sim \nu$. $T(\hat{\tau})$ would then be solution to the original SEP of ν into X (see [18] and [6] for more details).

[6] $|\cdot|$ represents the *d-dimensional* euclidean norm.

[7] This assumption makes continuity of of $c_n(\cdot)$ easier to verify.

[8] We can focus on an equivalent formulation of the problem where the target distribution is $\tilde{\nu}$ such that $\tilde{\nu}(A) = \nu(A + X_0)$ for any $A \in \mathscr{B}(\mathbb{R})$, while the local martingale of interest $(\tilde{X}_t)$ is defined as $\tilde{X}_t = X_t - X_0$. It is easy to see that, once we find $\tilde{X}_\tau \sim \tilde{\nu}$, then $X_\tau \sim \nu$.

2.4.1 Construction of $\tau^\star$

Assumption 2.10. The limiting random variable $X_\infty = \ell < 0$ $\mathbb{P}$-a.s..

Assumption 2.11. Assume that the underlying probability $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is such that $\mathbb{F}$ is such that $\mathcal{F}_0$ is large enough to support a random variable that is exponentially distributed with parameter 1 and independent of W.[9]

In this framework, the stopped Brownian motion W is defined as follow

$$W_t = \begin{cases} X_{T(t)} & \text{if} \quad t < \langle X \rangle_\infty \\ \ell & \text{if} \quad t \geq \langle X \rangle_\infty \end{cases} \qquad \text{with} \qquad T(t) = \inf\{s \geq 0 : \langle X \rangle_s > t\}. \tag{2.26}$$

W can be seen as a Markov process starting in 0, with 0 being a non-recurrent state having $\mathbb{P}$-a.s. finite lifetime $\langle X \rangle_\infty$ (or equivalently $H_\ell = \inf\{t \geq 0 : W_t = \ell\}$) and with absorption state given by ℓ. This would allow us to match the setup discussed in [4] in Section 2, where the main result is stated for transient Markov processes. Let us further stop the process W. In particular, for some $N > 0$, let $H_N = \inf\{t \geq 0 : W_t = N\}$ (from now on we will define the hitting time for W at state w as H_w), and define a new process $\tilde{W}$ as

$$\tilde{W}_t = \begin{cases} W_t & \text{if} \quad t < H_N \\ \ell & \text{if} \quad t \geq H_N. \end{cases} \tag{2.27}$$

By doing so, we construct a new transient Markov process $\tilde{W}$ with $\mathbb{P}$-a.s. finite lifetime $\zeta = \inf\{t \geq 0 : W_t \notin (\ell, N)\} = \min(H_\ell, H_N)$ and absorption state ℓ[10]. Before ζ, $\tilde{W}$ behaves like Brownian motion.

The arguments discussed in [4] require working with Y being the *recurrent extension* of $\tilde{W}$. Y can be constructed as in [7]. In particular, let $\dot{\mathbb{P}}^y$ and $\mathbb{P}^x$ be the law of the processes Y starting in y and $\tilde{W}$ starting in x respectively, with $y \in (\ell, N) \cup \{\ell\}$ and $x \in (\ell, N)$ (expectations under $\dot{\mathbb{P}}^y$ and $\mathbb{P}^x$ will be denoted with $\dot{E}^y[\cdot]$ and $E^x[\cdot]$ respectively). Y can be seen as an Itô excursion process taking values in $(\ell, N) \cup \{\ell\}$, with ℓ being a holding point with parameter 1, and corresponding excursion measure $n = \mathbb{P}^0$. If not specified otherwise, Y starts in ℓ; on exit from ℓ at time ρ, the process behaves as $\tilde{W}$ under the measure $\mathbb{P}^0$ until it exits from (ℓ, N), when Y jumps back to ℓ[11]. In other words, $Y_t = \tilde{W}_{t-\rho}$ for any $\rho \leq t < \tau = \inf\{t > \rho : Y_t = \ell\}$, with τ representing the first time Y reaches again ℓ after ρ. Given ζ being the killing time for $\tilde{W}$, one shows that

$$\begin{aligned} \tau = \inf\{t > \rho : Y_t = \ell\} &= \inf\{s + \rho > \rho : Y_{s+\rho} = \tilde{W}_s = \ell\} \\ &= \rho + \inf\{s > 0 : \tilde{W}_s = \ell\} = \rho + \zeta. \end{aligned} \tag{2.28}$$

Let $(L^a_t(Y))$ and $(L^a_t(\tilde{W}))$ be the families of local times for Y and $\tilde{W}$ respectively. Let (L^a_t) be the family of Brownian local times, one has that $L^\ell_\rho(Y)$ is independent of $\tilde{W}$ and exponentially distributed with scale parameter 1. Moreover, throughout the first excursion, we have

$$L^a_t(Y) = L^a_{t-\rho}(\tilde{W}) = L^a_{t-\rho} \qquad \text{for} \quad \ell < a < N \quad \text{and} \quad \rho \leq t < \tau. \tag{2.29}$$

Lemma 2.12. Let ν be a centered distribution and let I_ν the smallest interval containing $\text{supp}(\nu)$, such that $\underline{\nu} = \inf(I_\nu)$, $\overline{\nu} = \sup(I_\nu)$, with $\ell < \underline{\nu}$ and $\overline{\nu} < N$, for some $N > 0$. Then, $W_{\tau_{BLJ}} \sim \nu$,

[9] For example, let $E \sim \exp(1)$, and let $\sigma(E)$ be the smallest σ-algebra that makes E measurable. If $\mathbb{F}^X$ is the original filtration, then $\mathbb{F}$ could be defined such that $\mathcal{F}_t = \sigma\left(\mathcal{F}^X_t \cup \sigma(E)\right)$ for any $t \geq 0$. Under this assumption, we have that E is also $\hat{\mathcal{F}}_0$-measurable, since $\hat{\mathcal{F}}_0 = \mathcal{F}_{T(0)} = \mathcal{F}_0$, which means that $\hat{\mathbb{F}}$ is large enough to support an exponential random variable independent of W.

[10] This means that we are "killing" the starting $(\mathbb{P}, \hat{\mathbb{F}})$-Brownian motion not only when it reaches ℓ, but also when it reaches N; we keep the absorption state ℓ unchanged.

[11] The recurrent extension can be though as a regenerative extension obtained by: holding at ℓ, jumping from ℓ to 0 ($n = \mathbb{P}^0$ implies that the process Y will be always resurrected in 0 when it leaves the holding point), evolving according to the law of $\tilde{W}$ started from 0, returning to ℓ at the lifetime ζ, and repeating the procedure independently.

with τ_{BLJ} such that $\tau_{BLJ} < \zeta = \min(H_\ell, H_N)$ $\mathbb{P}$-a.s. and

$$\tau_{BLJ} = \inf\left\{\zeta > t \geq 0 : \int_{I_\nu} \frac{L_t^a(\tilde{W})\nu(da)}{2(P_\nu(a) - P_{\delta_0}(a))} > L_\rho^\ell(Y)\right\} \tag{2.30}$$

where $P_\nu(a) = \int (a-x)^+ \nu(dx)$.

Remark 2.13. This result holds unchanged if $\ell = \underline{\nu} = \inf(I_\nu)$ under the assumption of $\nu(\{\ell\}) = 0$.

We use Lemma 2.12 as an instrumental result to perform the embedding for a centered distribution with support unbounded above. The intuition behind the argument is as follows: given ν target distribution with unbounded support, for any M, we can derive a truncated version ν_M which is centered and with bounded support. We can use Lemma 2.12 to derive τ_{BLJ}^M and embed ν_M into W. Sending M to ∞, we can relax the truncation and derive an embedding of ν into X.

Lemma 2.14. Let ν be a centered distribution with $\text{supp}(\nu)$ such that $\ell < \underline{\nu} < 0 < \overline{\nu} = \infty$ (unbounded above). Assume moreover that $\nu \ll m$ with density $f_\nu(\cdot)$. Then, $W_{\tau_{BLJ}} \sim \nu$ such that $\tau_{BLJ} < H_\ell$ and

$$\tau_{BLJ} = \inf\left\{H_\ell > s \geq 0 : \int_{I_\nu} \frac{L_{s\wedge H_\ell}^a f_\nu(a)}{U_\nu(a) - U_{\delta_0}(a)} da > E\right\} \tag{2.31}$$

where $E \sim \exp(1) \perp\!\!\!\perp W$, $\mathcal{F}_0$-measurable.

The last step of the argument now consists of inverting the clock back to express the embedding in the time scale of the starting diffusion X.

Proposition 2.15. Let ν be a target distribution satisfying the condition stated in Lemma 2.14, $E \sim \exp(1)$ and independent of X, and let $\mathtt{D}(\cdot)$ be exogenously given. Then $\tau^\star$ below solves the SEP of ν into X

$$\tau^\star = \inf\left\{q \geq 0 : \int_0^q \mathtt{D}(X_z)dz > E\right\}. \tag{2.32}$$

These arguments show that, if $\mathtt{D}(\cdot)$ is exogenously given, then we can use it to construct a stopping time $\tau^\star$ which solves the SEP for a centered target ν into X, under some regularity assumptions. $\tau^\star$ is a default time expressed as the first jump of a Cox process with intensity given by $(\mathtt{D}(X_t))$. The agent "defaults" into the liquidation of the asset at a rate given by $\mathtt{D}(\cdot)$. $\mathtt{D}(\cdot)$ can then be validly interpreted as a stopping intensity.

Remark 2.16. Most of the results in Section 2 are stated for the transformed price process X. However, we can easily convert back to the original price process P. $\tau^\star$ defined above can be seen as the first jump of a Cox process with intensity process given by $\mathtt{D}(s(P_t))$ solves the SEP of μ into P, with μ such that $\nu(A) = \mu(s^{-1}(A))$, for any $A \in \mathscr{B}(\mathbb{R})$. In other words, if the agent sells at a level-dependent rate per unit time given by $\mathtt{D}(s(\cdot))$, the distribution μ is attained.

The interpretation of $\tau^\star$ as the first jump of a Cox process allows us to derive the distribution of $\tau^\star$ using standard results for this class of processes. In particular, if $F_{\tau^\star}(\cdot)$ represents the cdf of $\tau^\star$ and $f_{\tau^\star}(\cdot)$ its density, we can write

$$\begin{aligned} F_{\tau^\star}(t) &= \mathbb{P}(\tau^\star \leq t) = 1 - E\left[\exp\left(-\int_0^t \mathtt{D}(X_s)ds\right)\right] \\ f_{\tau^\star}(t) &= E\left[\exp\left(-\int_0^t \mathtt{D}(X_s)ds\right)\mathtt{D}(X_t)\right] \quad m\text{-a.e. in } \mathbb{R}_+ \end{aligned} \tag{2.33}$$

Since the embedding τ_{BLJ} for W is strictly lower than $\langle X\rangle_\infty$, $\tau^\star$ in the original clock will be finite $\mathbb{P}$-a.s.. This means that, for any $q > \tau^\star$, the integrated $\mathtt{D}(X_\cdot)$ will be above E, hence integrating $\mathtt{D}(X_\cdot)$ over the positive real line will provide a quantity above E $\mathbb{P}$-a.s.. Independence of E from X allows us to conclude that

$$0 = \mathbb{P}\left(\int_0^\infty \mathtt{D}(X_s)ds \leq E\right) = E\left[\exp\left(-\int_0^\infty \mathtt{D}(X_s)ds\right)\right] \longrightarrow \int_0^\infty \mathtt{D}(X_s)ds = \infty \ \mathbb{P}\text{-a.s.} \tag{2.34}$$

3 An Application to Prospect Theory

We now specialize to Prospect theory. Our aim is to compute the selling propensity $\mathtt{D}(\cdot)$ for an agent with PT preferences and probability distortion. We will derive a scaling property allowing us to scale the optimal distribution for different levels of the loss aversion parameter. This enables us to extend to multiple agents by proposing a density describing the way loss aversion distributes across a continuum of investors. Our aim is to extend the 'single number' measure of the disposition effect considered in models such as [10] to a price level dependent measure.

Our continuous-time optimal stopping model of an asset sale with PT preferences under precommitment will follow the work of [10]. This paper built upon an existing literature of trading models based on S-shaped utility functions ([2], [9], [11]). The novelty in this work consists of the inclusion of probability weighting implemented through distortion functions $w(\cdot)$ (continuous, strictly increasing, differentiable, and such that $w(0) = 0$ and $w(1) = 1$). Probability weighting was responsible for the optimal prospect taking the form of a stop-loss threshold and a non-trivial distribution over gains.

3.1 The PT optimization problem and the optimal distribution

As in Section 2.2, let (P_t) be the one-dimensional regular diffusion process, and let (X_t) be the local martingale defined as $X = s(P)$, where $s(\cdot)$ is the scale function. Assume X has lower bound given by ℓ.

Let $U(\cdot)$ be the PT utility function (continuous, increasing, twice differentiable away from zero, S-shaped function ie. convex over losses, concave over gains) with $U(0) = 0$. Let $w_\pm(\cdot)$ be a pair of admissible distortion functions (continuous, strictly increasing, differentiable, and such that $w_\pm(0) = 0$ and $w_\pm(1) = 1$). Overweighting of small probabilities on extreme events suggests $w_\pm(\cdot)$ should be inverse S-shaped functions, in particular, there exist $q_\pm$ such that $w_\pm$ is concave on $[0, q_\pm]$ and convex on $[q_\pm, 1]$. Note $w_+(\cdot)$ is the distortion for gains and $w_-(\cdot)$ for losses.

Later, we will specialize to the popular piecewise power S-shaped utility and TK distortion functions introduced in [12]. The piecewise power S-shaped utility function is defined as

$$U(x) = \begin{cases} x^{\alpha_+} & \text{if } \ x \geq 0 \\ -k(-x)^{\alpha_-} & \text{if } \ x < 0 \end{cases} \tag{3.1}$$

where parameters $k, \alpha_+, \alpha_- > 0$. The TK distortion function is given by

$$w_\pm(x) = \frac{x^{\delta_\pm}}{(x^{\delta_\pm} + (1-x)^{\delta_\pm})^{1/\delta_\pm}} \qquad \text{with} \qquad \delta_\pm \in (0.28, 1). \tag{3.2}$$

The PT optimization problem in [10] is given by:

$$\sup_{\tau \in \mathscr{T}} \left[\int_0^{+\infty} w_+ \left(\mathbb{P} \left(v(X_\tau) > y \right) \right) dy - \int_{-\infty}^0 w_- \left(\mathbb{P} \left(v(X_\tau) < y \right) \right) dy \right] = \sup_{\tau \in \mathscr{T}} \mathscr{E} \left[v(X_\tau) \right] \tag{3.3}$$

where $\mathscr{T}$ represents the collection of stopping times with respect to $\mathbb{F}$, while $v(\cdot) = U(s^{-1}(\cdot) - r)$. Let $u > \ell > -\infty$ be the endpoints of the diffusion interval for X. $X_\tau \sim \nu$, while $F_\nu(\cdot)$ and $G_\nu(\cdot)$ are the corresponding distribution and quantile functions.

The optimization problem can be equivalently expressed in its quantile ((3.5)) and distribution ((3.4)) formulation (see [22]):

$$\sup_{\nu \in \mathscr{A}} \left[\int_0^{v(u)} w_+ \left(1 - F_\nu(v^{-1}(y)) \right) dy - \int_{v(\ell)}^0 w_- \left(F_\nu(v^{-1}(y)) \right) dy \right] = \sup_{\nu \in \mathscr{A}} \mathscr{E} \left[v(X_\tau) \right] \tag{3.4}$$

$$\sup_{\nu \in \mathscr{A}} \left[\int_{\nu((-\infty,0])}^1 v(G_\nu(u)) w'_+(1-u) du + \int_0^{\nu((-\infty,0))} v(G_\nu(u)) w'_-(u) du \right] = \sup_{\nu \in \mathscr{A}} \mathscr{E} \left[v(X_\tau) \right] \tag{3.5}$$

$$\mathscr{A} = \left\{ \nu \text{ prob. measure on } (\mathbb{R}, \mathscr{B}(\mathbb{R})) : \int_{\mathbb{R}} y d\nu(y) \leq X_0 \right\}. \tag{3.6}$$

The constraint in the set of admissible distributions states that the target ν has to be such that $E[X_\tau] \leq X_0$, which comes from the fact that X is a supermartingale. Notice that the optimization problem in its quantile/distribution formulation can be equivalently written by replacing $\mathscr{A}$ with $\mathscr{G}/\mathscr{F}$ representing the set of corresponding quantile/distribution functions that reflect the constraint on the mean of ν specified in $\mathscr{A}$.

Remark 3.1. One can show that set of admissible distributions can be shrunk as follows

$$\mathscr{A} = \left\{ \nu \text{ prob. measure on } (\mathbb{R}, \mathscr{B}(\mathbb{R})) : \int_{\mathbb{R}} y d\nu(y) = X_0 \right\}. \tag{3.7}$$

This means that we can solve the optimization problem above in its distribution/quantile formulation by looking at those probability measures with mean X_0. This automatically satisfies one of the conditions stated in Proposition 2.4 (i.e., $E[X_\tau] = X_0$) and it implies the sufficient and necessary condition needed for ν to be embedded into X (see [18]).

Theorem 2 in [10] states that, if $v(\cdot) = U(s^{-1}(\cdot) - r)$ is concave on $[0, \infty)$, convex on $[\ell, 0]$ with $v'(0+) = \infty$ and $v'(\infty) = 0$ and if particular elasticity assumptions hold, then the optimal prospect (in terms of P) has a distribution which consists of a point mass $s^{-1}(c_1)$ in the loss regime and a point mass at some point $s^{-1}(c_2)$ in the gains regime, together with a continuous distribution on the unbounded interval $(s^{-1}(c_2), \infty)$.

Proposition 9 of [10] shows how the optimal distribution can be computed, assuming $w_\pm(\cdot)$ being inverse-S-shaped with $q_\pm$ denoting the point where convexity changes (e.g., TK distortion functions). The optimization problem in (3.5) is defined over an infinite-dimensional space of quantile functions such that $E[X_\tau] = X_0$; this result shows that the optimal quantile function belongs to a class of parametric non-decreasing, left-continuous functions defined by

$$G(u) = \begin{cases} -\frac{1}{1-\phi}\left[\int_0^\phi (v')^{-1}\left(\frac{\lambda}{w'(\phi \wedge y)}\right) dy - X_0\right] = c_1 & \text{if} \quad 0 < u \leq 1-\phi \\ (v')^{-1}\left(\frac{\lambda}{w'(\psi)}\right) = c_2 & \text{if} \quad 1-\phi < u \leq 1-\psi \\ (v')^{-1}\left(\frac{\lambda}{w'(1-u)}\right) & \text{if} \quad 1-\psi < u \leq 1 \end{cases} \tag{3.8}$$

where $\phi \in (0,1)$ represents the probability mass on gains, $\lambda > 0$ is a Lagrange multiplier, while $\psi \leq \min(\phi, q_+)$ is a parameter that comes from the optimization over the gain region. c_1 and c_2 are the jump locations on losses and gains, respectively. The jump size on losses is $p_1 = F(c_1) - F(c_1^-) = 1 - \phi$, while the analogous quantity for gains is $p_2 = F(c_2) - F(c_2^-) = \phi - \psi$.

This allows us to focus on the following constrained optimization problem, which is defined over a finite-dimensional parametric space. In particular, using $G(\cdot)$ the parametrised quantile function presented in (3.8), the optimization problem in (3.5) becomes

$$\max \left(\int_{1-\phi}^1 v(G(u)) w_+'(1-u) du + \int_0^{1-\phi} v(G(u)) w_-'(u) du \right) \quad \text{subj. to} \tag{3.9}$$

$$\lambda > 0;\ \psi \leq \min(\phi, q_+);\ \phi \in (0,1);\ X_0^+ \leq \int_0^\phi (v')^{-1}\left(\frac{\lambda}{w_+'(\min(\psi, y))}\right) dy \leq X_0 + |\ell|(1-\phi).$$

Now, for a given choice of inverse-S-shaped distortion functions $w_\pm(\cdot)$, a given specification of the agent's utility function, using the expression for $G(\cdot)$ in (3.8), we can represent the target of the maximization problem as a function of the parameters (λ, ϕ, ψ).

We now specialize to the situation where the price process (P_t) is a geometric Brownian motion (GBM) solving

$$dP_t = P_t(\mu dt + \sigma dB_t) \quad \text{with} \quad P_0 = p_0 > 0 \text{ and } \mu \in \mathbb{R},\ \sigma > 0 \tag{3.10}$$

Define $\beta = 1 - 2\mu/\sigma^2$, which involves the return-for-risk-per-unit-variance μ/σ^2 and thus reflects the expected performance of the asset. Assume $\beta \in (0, 1]$, which rules out the case where the agent waits indefinitely with some positive probability.

Define the transformed process X by $X_t = s(P_t) = P_t^\beta - r^\beta$. Note also that [10] moreover requires $\alpha_+ < \beta\delta_+$ for the problem to be well-posed. This assumption implies (strict) concavity of $v(\cdot)$ on gains (i.e., $v''(\cdot) < 0$ on $(0, \infty)$), which holds as long as $\alpha_+ < \beta$. On the other hand, to get (strict) convexity on losses (i.e., $v''(\cdot) > 0$ in $(-r^\beta, 0)$) we need to require $\alpha_- < \beta$. Under the assumption $\beta \in (0, 1]$, the transformed price process $X = s(P)$ converges $\mathbb{P}$-a.s. to ℓ. Moreover, it admits the integral representation below with $x \mapsto \xi(x) = ((s'\sigma) \circ s^{-1})(x) = \beta\sigma(x + r^\beta)$

$$dX_t = \xi(X_t)dB_t = \beta\sigma\left(X_t + r^\beta\right) dB_t \tag{3.11}$$

Proposition 9 in [10] can be used to derive the optimal distribution that solves the optimization problem in its distribution/quantile formulation in the PT/GBM framework.

3.2 A useful scaling property in k

This section derives a scaling property for ν in the loss aversion parameter k, given PT preferences in the GBM framework. This scaling property proves very useful when we study the multi-agent framework. Under the assumption $\beta = 1$, the function v becomes $v(x) = U(x)$. Let $\mathscr{G}$ be the set of quantile functions with mean X_0

$$\sup_{G\in\mathscr{G}} \left[\int_{1-\mathbb{P}(X>0)}^{1} v_k(G(u))w'_+(1-u)du + \int_0^{\mathbb{P}(X<0)} v_k(G(u))w'_-(u)du \right] = \sup_{G\in\mathscr{G}} \mathscr{E}[v_k(G)]. \tag{3.12}$$

Then we have the following scaling result:

Proposition 3.2. Assume $X_0 = 0$, $\beta = 1$ and $\alpha_- > \alpha_+$, and let $\overline{G}(\cdot)/\overline{F}(\cdot)$ be the quantile/distribution function in optimality for $k = 1$. Then we can identify the optimal quantile/distribution function ($G_k(\cdot)/F_k(\cdot)$ respectively) for any $k \in \mathcal{K} \subseteq (k_{\min}, \infty)$, with $k_{\min} = (r/|\bar{c}_1|)^{\alpha_+ - \alpha_-}$ as follows

$$G_k(u) = g(k)\overline{G}(u) \quad \text{and} \quad F_k(x) = \overline{F}\left(\frac{x}{g(k)}\right) \quad \text{with} \quad g(k) = k^{-\frac{1}{\alpha_- - \alpha_+}}. \tag{3.13}$$

The inequality $\alpha_- > \alpha_+$ means that the measure of risk-seeking on losses is lower than the measure of risk aversion on gains, which is consistent with the parameterization in [10] and [11]. It implies that $g(\cdot)$ is decreasing.

Remark 3.3. In a PT framework, one typically assumes that loss aversion $k \in [1, \infty)$ for economic reasons. However, we can have $0 < k_{\min} < 1$ from a modelling perspective since the assumptions in Proposition 9 of [10] are verified for $k > 0$.

3.3 Selling propensity in PT/GBM framework

In this section, we will analyze the main example of application of the state-dependent selling propensity $\mathtt{D}(\cdot)$, by considering PT preferences, TK distortion functions and by assuming P to be a geometric Brownian motion with $\beta = 1 - 2\mu/\sigma^2$ in $(0, 1]$, $\alpha_+ < \beta\delta_+$, and $\alpha_- < \beta$. Under these assumptions, we can derive the optimal distribution ν by applying the results presented in [10], as in Section 3.1. We can then embed ν (centered in X_0) into X by some Skorokhod embedding such that $X_\tau \sim \nu$ with τ $\mathbb{P}$-a.s. finite. We will assume that the specific embedding used belong to the class of stopping times such that $E[\langle X\rangle_\tau] = E[X_\tau^2] - X_0^2$ (e.g., Azéma-Yor, and Chacon-Walsh embeddings). This makes it easier to check the integrability of $\langle X\rangle_\tau$ by focusing on the square integrability of the target distribution ν.

Proposition 3.4. The distribution ν solving the optimization problem presented in [10] is square integrable, if $\alpha_+ < \beta(2\delta_+ - 1)$[12].

[12]Since $\delta_+ < 1$, this condition is stricter than $\alpha_+ < \beta\delta_+$ and guarantees a sufficiently fast convergence of the density of the continuous component of ν to get square integrability of the whole distribution.

Since ν has mean zero and is square integrable, then, for any $\mathbb{P}$-a.s. finite stopping time that embeds ν into X with $E[\langle X\rangle_\tau] = E[X_\tau^2] - X_0^2$, the conditions in Proposition 2.4 are satisfied. This allows us to express $\mathtt{D}(\cdot)$ as

$$\mathtt{D}(x) = \frac{D_m\nu(x)\xi^2(x)}{\int |y-x|\,\nu(dy) - |X_0 - x|} \qquad \text{for } x \in I_\nu \quad \text{and } \infty \text{ otherwise.} \tag{3.14}$$

3.3.1 Single-agent framework

Let us first consider an application to the single-agent framework. Figure 1 shows that, for a given parameterisation of the agent's preferences and the price process, the optimal distribution consists of a point mass over losses and a continuous distribution over gains. This means that the agent adopts a stop-loss strategy and sells at a gain at a price greater than $s^{-1}(c_2)$. For this specific parameterisation, the agent finds it optimal to sell in gains at a price that is quite close to $s^{-1}(c_2)$, indeed 75% of the mass in gains is contained in the interval $(1.06268, 1.12264)$. With this parameterisation, in optimality, there is no jump over gains (i.e., $p_2 = \phi - \psi = 0$). Figure 1 clearly illustrates the asymmetric treatment between gains and losses. Just by comparing $|s^{-1}(c_1) - r|$ and $|s^{-1}(c_2) - r|$, one can see that the investor tends to hold on losses longer than she does on gains, which is the core idea of the *disposition effect*.

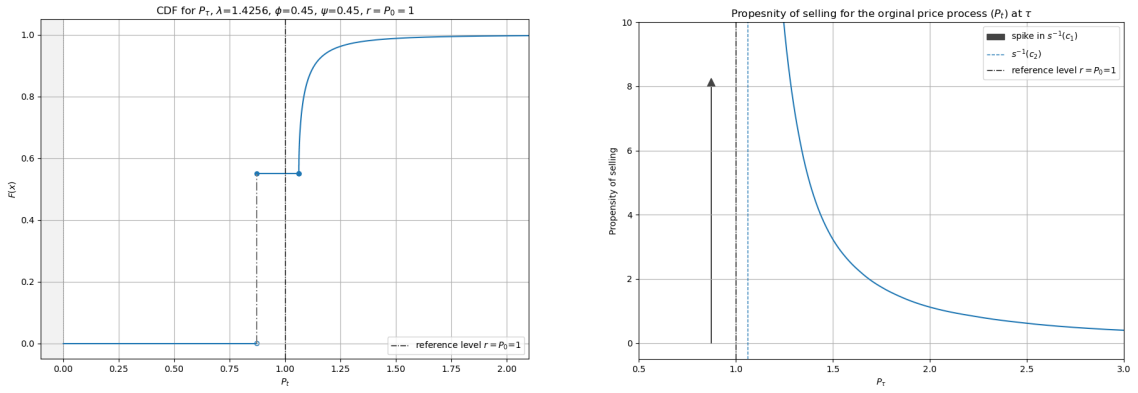


Figure 1: Single agent framework: These figures represent, respectively, the distribution function that we get by solving the optimization problem in [10] and the corresponding selling propensity, D. We assume $w_+(\cdot) = w_-(\cdot)$ equal to TK distortion function with parameter $\delta_+ = \delta_- = 0.7$. $P \sim$ GBM where $\mu = 1/40$, $\sigma = 1$, $P_0 = r = 1$. Agent's preferences are specified through PT utility function with parameters $\alpha_+ = 0.35$, $\alpha_- = 0.9$, and $k = 1.25$. Jump location on losses in $s^{-1}(c_1) \approx 0.87385$, while the infimum of the support on gains is $s^{-1}(c_2) \approx 1.06268$.

3.3.2 Multi-agent framework

Let us now consider the multi-agent framework. The idea is to look for a distribution that, by averaging across multiple agents' optimal selling behaviors, describes how the market liquidates the position in the asset P. As in Section 2.3, the target distribution that we obtained by applying the results in [10] depends on a set of parameters representing the investor's preferences and probability weighting. In other words, $\nu \equiv \nu^\vartheta$ with $\vartheta = (\alpha_+, \alpha_-, k, \delta_+, \delta_-)$.

Assumption 3.5. $(P_t) \sim$ GBM such that $\beta = 1$, $X_0 = 0$ (reference level is the purchase price, $r = P_0$) and $\alpha_- > \alpha_+$. We moreover assume that $\alpha_+ < \beta(2\delta_+ - 1)$ for square integrability of the optimal distribution ν^ϑ.

We can use the scaling property in k in Proposition 3.2 to solve the optimization problem for any $k \in \mathcal{K} \subseteq (k_{\min}, \infty)$, by transforming the solution for $k = 1$, *ceteris paribus*. In other words, we can apply Section 2.3, by averaging across *uncountably many* investors with preferences that differ just in terms of the loss-aversion coefficient k (i.e., $\vartheta = k$). We specify a density $\varphi(\cdot)$ on $\Theta \equiv \mathcal{K} = (\underline{k}, \overline{k}) \subseteq (k_{\min}, \infty)$ one-dimensional open parametric space such that $0 < k_{\min} < \underline{k} < \overline{k} \leq \infty$. We implicitly assume that $\varphi(\cdot)$ is null outside of $\mathcal{K}$ and strictly positive in $\mathcal{K}$. We obtain the following result:

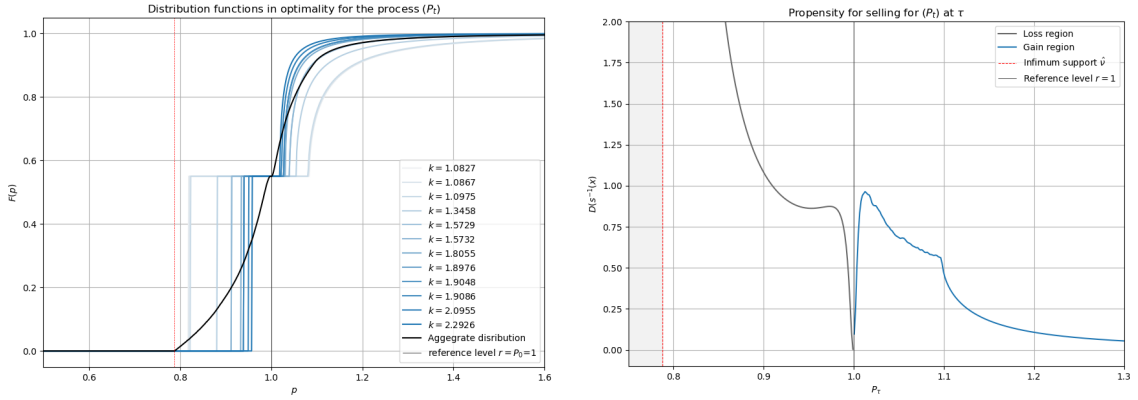


Figure 2: Multiple agent framework: These figures represent, respectively, the distribution function that we get through the averaging procedure and the corresponding selling propensity. The mixing distribution used is a translated exponential with support $(1, \infty)$ and parameterized in such a way that the expected loss aversion coincides with the point estimate provided in the seminal work of [21], where $\hat{k} \approx 2.25$. Notice that $w_+(\cdot) = w_-(\cdot)$ are TK distortion function with parameter $\delta_+ = \delta_- = 0.7$. $P \sim$ GBM where $\mu = 0$, $\sigma = 0.1$ are such that $\beta = 1$, while $P_0 = r = 1$. PT utility function is such that $\alpha_+ = 0.39$, $\alpha_- = 0.9$.

Proposition 3.6. The distribution $\hat{\nu}$ defined in (3.15) is such that $\hat{\nu}(da) = f(a)da$

$$\hat{\nu}(A) = \int_{\mathcal{K}} \nu_k(A)\varphi(k)dk \tag{3.15}$$

$$f(x) = \sum_{i\in\{1,2\}} \frac{\overline{p}_i \varphi(c_i^{-1}(x))}{|\nabla c_i(c_i^{-1}(x))|} \mathbb{1}_{\{x\in \mathrm{Img}(c_i)\}} + \left(\int_{\mathcal{K}} f_k(x)\varphi(k)dk\right) \mathbb{1}_{\{x\in(0,\infty)\}} \tag{3.16}$$

$$\text{where} \quad f_k(x) = -\frac{d}{dx}\left((w'_+)^{-1}\left(\frac{\lambda}{v'(x/g(k))}\right)\right)\mathbb{1}_{\{x\geq c_2(k)\}} = \eta_k(x)\mathbb{1}_{\{x>c_2(k)\}}.$$

For $i \in \{1,2\}$, $c_i(k) = g(k)\overline{c}_i$ assigns to every $k \in \mathcal{K}$ the jump locations in optimality, with $g(k) = k^{-\frac{1}{\alpha_- - \alpha_+}}$. $\overline{c}_i$ represents the jump locations over losses/gains in optimality under $k = 1$, while $\overline{p}_i$ refers to the corresponding jump sizes. The distribution $\hat{\nu}$ specified through the mixing procedure has positive density on $(c_1(\underline{k}), c_1(\overline{k})) \cup (c_2(\overline{k}), \infty)$.

Corollary 3.7. $\hat{\nu}$ defined as in Proposition 3.6 is centered 0 and square integrable under the condition stated in Assumption 3.5. Moreover, $\hat{\nu}$ preserves the mass allocation on losses and gains.

Remark 3.8. Let $\overline{k} = \infty$ (from a modeling perspective, we allow $\mathcal{K}$ to be unbounded above), then $c_1(\overline{k}) = c_2(\overline{k}) = 0$, since $\lim_{k\to\infty} g(k) = 0$. This implies that $\mathrm{supp}(\hat{\nu}) = (c_1(\underline{k}), 0) \cup (0, \infty)$. This means that, for any $\varepsilon > 0$, $\hat{\nu}(B(0,\varepsilon)) > 0$, which would allow us to extend the support of $\hat{\nu}$ as $\mathrm{supp}(\hat{\nu}) = (c_1(\underline{k}), \infty)$. This can be easily seen from the left panel in Figure 2 where $\varphi(\cdot)$ is the density of a properly parameterized translated exponential distribution. The left panel shows how this averaging procedure allows us to smooth out point masses and rule out the related spikes that would appear in the propensity at the jump locations for the single-agent problem.

Remark 3.9. We could alternatively say that the continuum of agents (differing just in terms of loss aversion) is described by an atomless probability space $(\mathcal{K}, \mathscr{K}, \Phi)$, with $\Phi(dk) = \varphi(k)dk$ and where each $k \in \mathcal{K}$ represents a different agent. In this work, we assumed that the agents solve the stopping problem independently, and then we averaged those distributions with respect to the measure Φ to derive a new distribution $\hat{\nu}$ representing the aggregate behavior.

3.3.3 A new approach towards empirical evidence

This section aims to derive a density $\varphi(\cdot)$ that gives a selling propensity that better matches empirical observations in the work of [17], [13], [3], and [1]. There is less consensus for the overall shape, with findings depending upon the length of the holding period; however, there is agreement

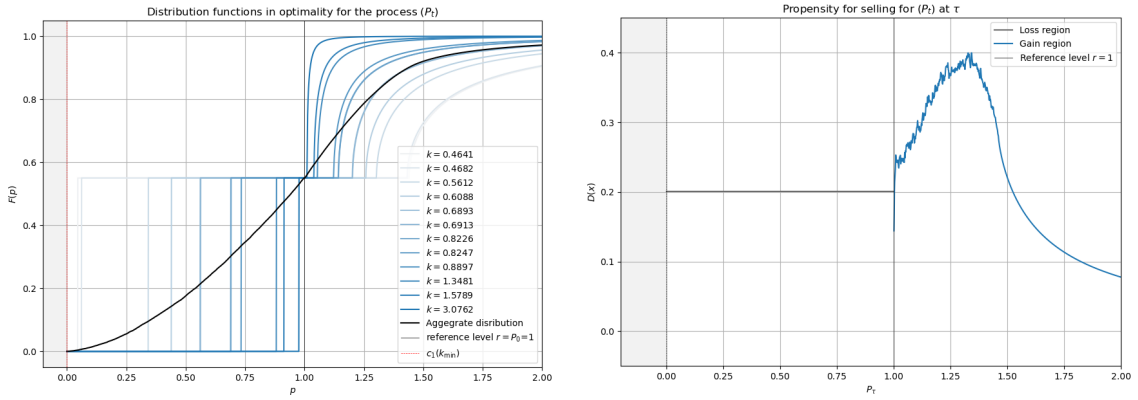


Figure 3: These figures represent, respectively, the distribution function that we get through this specific averaging procedure and the corresponding selling propensity. As in Proposition 3.10, we can attain a constant selling propensity on losses for an arbitrary level λ (here set to 0.2). $w_+(\cdot) = w_-(\cdot)$ are TK distortion function with parameter $\delta_+ = \delta_- = 0.7$. $P \sim$ GBM where $\mu = 0$, $\sigma = 0.3$ are such that $\beta = 1$, while $P_0 = r = 1$. PT utility function is such that $\alpha_+ = 0.39$, $\alpha_- = 0.9$.

that the estimated hazard rate or probability of sale as a function of returns since purchase is asymmetric, higher on gains than on losses. This supports a disposition effect. Excluding data for very short holding periods, researchers find that the hazard or probability of selling is fairly flat on losses ([3], [13]).

For any fixed value $\lambda > 0$, we would like to derive a density $\varphi(\cdot)$ describing the distribution of the risk-aversion parameter such that $\mathtt{D}(x) = \lambda$ for any $x \in \mathrm{supp}(\nu) \cap (-\infty, 0)$, loss region expressed in terms of the transformed process X. We want to find a distribution for k that delivers a constant selling propensity *on losses* to better match this feature of the empirical data.

Proposition 3.10. Let $\lambda > 0$ arbitrary, let $k_{\min} = (r/|\bar{c}_1|)^{\alpha_+ - \alpha_-}$ and consider $\varphi(\cdot)$ power-law density defined on $\mathcal{K} = (k_{\min}, \infty)$ such that

$$\varphi(k) = \frac{|\bar{c}_1|\,\vartheta}{r^\vartheta(\alpha_- - \alpha_+)}(c_1(k) + r)^{\vartheta - 1}\frac{g(k)}{k} \tag{3.17}$$

$$\text{where} \quad c_1(k) = g(k)\bar{c}_1 = k^{-\frac{1}{\alpha_- - \alpha_+}}\bar{c}_1 \quad \text{and} \quad \vartheta = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{8\lambda}{\sigma^2}}\right).$$

Then, the aggregate distribution $\hat{\nu}$ obtained by averaging over k according to $\varphi(\cdot)$ above is such that $\hat{\nu} \ll m$ and its support coincides with the diffusion interval of the underlying local martingale X, $(\ell, \infty) = (-r, \infty)$. Moreover, for any $x \in (-r, 0)$ (loss region) one has that

$$\hat{\nu}((-\infty, x)) = F(x) = \frac{1 - \phi}{r^\vartheta}(x + r)^\vartheta \qquad \text{and} \qquad \mathtt{D}(x) = \lambda. \tag{3.18}$$

Remark 3.11. So, by properly extending the support of the target measure $\hat{\nu}$ to the whole state space of the underlying diffusion X, we can construct the mixing density $\varphi(\cdot)$ that allows us to recover an arbitrary constant level $\lambda > 0$ for the selling propensity on losses and preserving the probability mass assigned to that region. In Figure 3 we show an example of the distribution function that we get through this specific averaging procedure and the corresponding selling propensity. One can easily see that the support of the average distribution $\hat{\nu}$ coincides with the state space of the local martingale X. The right hand panel demonstrates that we can attain a constant selling propensity on losses, here given by $\lambda = 0.2$.

By comparing Figure 2 with these results, it is quite clear how differently the selling propensity on losses behaves for these two different modeling choices. On the other hand, the general shape of the selling propensity for gains seems to be preserved.

The following corollary shows that, if we want to derive a constant selling propensity on losses for any arbitrary level $\lambda > 0$, then we cannot have control over $\underline{\nu}$, the lower bound of the support of the measure $\hat{\nu}$.

Proposition 3.12. For any arbitrary level $\lambda > 0$ there exist $\hat{\nu}$ probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ derived through the usual mixing procedure such that $x \mapsto F(x) = \hat{\nu}((-\infty, x])$ satisfies $F(\underline{\nu}) = 0$, $F(0) = 1 - \phi$ and $\mathtt{D}(x) = \lambda$ for any $x \in (\underline{\nu}, 0)$ if and only if $\underline{\nu} = -r$.

This section tells us that a constant market-level selling propensity on losses does not arise because individual investors liquidate at a constant rate. Each individual places the loss mass at a single threshold $c_1(k)$, while heterogeneity in loss aversion smooths those thresholds into the power-law density above.

4 Conclusions

This work provides a continuous-time framework linking individual PT preferences to aggregate market trading patterns, specifically focusing on the DE. Existing PT optimal stopping models (e.g., [9] without probability weighting and [10] with weighting) typically produce optimal distributions with point masses, which conflict with empirically observed smooth hazard rates. To address this, we aggregate a continuum of investors with heterogeneous utility parameters, which smooths atomic individual strategies into an absolutely continuous market distribution. We show that a constant hazard rate on losses, an established empirical regularity, arises endogenously in a PT setting when market loss aversion follows a specific, explicitly derived power-law distribution and the target measure's support spans the full loss domain. Finally, using the Bertoin-Le Jan embedding, we show that under regularity conditions on the target distribution, this selling propensity can be interpreted as a state-dependent selling intensity.

5 Appendix

Lemma 5.1. Let τ be a $\mathbb{P}$-a.s. finite stopping time and $H \in L^2_{loc}(W)$ (a predictable process that is locally integrable with respect to Brownian motion), then, if $E\left[\int_0^\tau H_u^2 du\right] < \infty$ (assumption $(\star)$)

$$E\left[\int_0^\tau H_u dW_u\right] = 0 \qquad \text{and} \qquad E\left[\left(\int_0^\tau H_u dW_u\right)^2\right] = E\left[\int_0^\tau H_u^2 du\right] \tag{5.1}$$

Proof of Lemma 5.1. Since it is a standard result, we provide here just a sketch of the proof for this lemma. Let τ be a $\mathbb{P}$-a.s. finite stopping time. Under the previous assumptions one has that $\mathbb{1}_{[0,\tau]} H \in L^2_{loc}(B)$. Therefore, for any stopping times τ_n in a localizing sequence (τ_n)

$$E\left[\int_0^{\tau\wedge\tau_n} H_u dW_u\right] = 0 \qquad \text{and} \qquad E\left[\left(\int_0^{\tau\wedge\tau_n} H_u dW_u\right)^2\right] = E\left[\int_0^{\tau\wedge\tau_n} H_u^2 du\right] \tag{5.2}$$

We now want to relax the localization. Let $(Z^{(n)})$ be a sequence of processes where $Z_t^{(n)}$ is the stochastic integral $(\mathbb{1}_{[0,\tau\wedge\tau_n]} H \bullet B)_t$. $Z^{(n)}$ is an L^2-bounded uniformly integrable martingale and converges $\mathbb{P}$-a.s. and in L^2 to an integrable random variable $\tilde{Z}_n := Z_\infty^{(n)} = (H \bullet B)_{\tau\wedge\tau_n}$. One then shows that the sequence of limiting random variables $(\tilde{Z}_n) \subseteq L^2$ converges $\mathbb{P}$-a.s. to $(H \bullet B)_\tau$. Therefore, if $(\tilde{Z}_n)$ converges in L^2, then the limiting random variable must be $(H \bullet B)_\tau$. Under assumption $(\star)$, DCT can be used to show that $(\tilde{Z}_n)$ is a Cauchy sequence. L^2 convergence of $(\tilde{Z}_n)$ to $(H \bullet B)_\tau$ then implies the result. □

Proof of Proposition 2.4. Let us set $H = \operatorname{sgn}(X_\cdot - x)\xi(X_\cdot)$, $H \in L^2_{loc}(B)$. Let τ be an $\mathbb{P}$-a.s. finite stopping time embedding ν into X. Assumption $(\star)$ from Lemma (5.1) in this framework reads as

$$E\left[\int_0^\tau H_u^2 du\right] = E\left[\int_0^\tau \operatorname{sgn}^2(X_u - x)\xi^2(X_u) du\right] = E\left[\int_0^\tau \xi^2(X_u) du\right] = E\left[\langle X\rangle_\tau\right] < \infty \tag{5.3}$$

By using Tanaka formula, we have

$$E\left[L_\tau^x(X)\right] = E\left[|X_\tau - x|\right] - E\left[|X_0 - x|\right] - E\left[\int_0^\tau \operatorname{sgn}(X_s - x) dX_s\right] \tag{5.4}$$

If $E[\langle X\rangle_\tau] < \infty$, then Lemma (5.1) applies and we can conclude

$$E\left[L_\tau^x(X)\right] = E\left[|X_\tau - x|\right] - E\left[|X_0 - x|\right] = E\left[|X_\tau - x|\right] - |X_0 - x| \leq E[|X_\tau|] + |x| \tag{5.5}$$

Since we assume that ν is an integrable distribution, $E[|X_\tau|] < \infty$, confirming that this expectation is well-defined and finite. □

Proof of Proposition 2.8. Under the assumption stated, coarea formula can be applied. In particular, let $F : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function (or alternatively in $\mathcal{C}^1$) and assume that $|\nabla F| > 0$ on the integration set (level set, $F^{-1}(x) = \{\vartheta : F(\vartheta) = x\}$). Let $g : \mathbb{R}^d \to \mathbb{R}$ be an integrable function and $\Theta \subseteq \mathbb{R}^d$ be open, then coarea formula implies

$$\int_\Theta g(\vartheta) d\vartheta = \int_{\mathbb{R}} \left(\int_{F^{-1}(x)} \frac{g(\vartheta)}{|\nabla F(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) \right) dx \tag{5.6}$$

where $\mathscr{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure

$$\mathscr{H}_{d-1}(A) = \liminf_{\delta \to 0^+} \left\{ \sum_i (\operatorname{diam}(S_i))^{d-1} : A \subseteq \cup_i S_i \quad \text{with} \quad \operatorname{diam}(S_i) \leq \delta \right\}. \tag{5.7}$$

For any $n \geq 1$ and $A \in \mathscr{B}(\mathbb{R})$, let $\vartheta \mapsto g(\vartheta) = p_n^\vartheta \delta_{c_n^\vartheta}(A)\varphi(\vartheta)$. This function is integrable over Θ, since $g(\vartheta) \leq \varphi(\vartheta)$, which integrates to 1 over $\Theta \subseteq \mathbb{R}^d$. For any $n \geq 1$, we can set $F(\vartheta) = c_n(\vartheta)$, by assumption this function is continuously differentiable and such that $|\nabla c_n(\vartheta)| > 0$. This allows us to use the coarea formula stated above

$$\int_\Theta p_n^\vartheta \delta_{c_n(\vartheta)}(A)\varphi(\vartheta) d\vartheta = \int_{\mathbb{R}} \left(\int_{\{\vartheta : c_n(\vartheta) = x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \delta_{c_n(\vartheta)}(A) \mathscr{H}_{d-1}(d\vartheta) \right) dx \tag{5.8}$$

$$= \int_{\mathbb{R}} \left(\int_{\{\vartheta : c_n(\vartheta) = x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \delta_x(A) \mathscr{H}_{d-1}(d\vartheta) \right) dx = \int_A \left(\int_{\{\vartheta : c_n(\vartheta) = x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) \right) dx$$

$$\hat{\nu}_s(A) = \sum_{n \geq 1} \int_A \left(\int_{\{\vartheta : c_n(\vartheta) = x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) \right) dx \tag{5.9}$$

For any $n \geq 1$, the term identified in equation (5.8) is non-negative, Fubini's theorem implies

$$\hat{\nu}_s(A) = \int_A \left(\sum_{n \geq 1} \int_{\{\vartheta : c_n(\vartheta) = x\}} \frac{p_n^\vartheta \varphi(\vartheta)}{|\nabla c_n(\vartheta)|} \mathscr{H}_{d-1}(d\vartheta) \right) dx = \int_A f_s(x) dx. \tag{5.10}$$

The function f_s derived above is therefore non-negative and integrable (the integral over $\mathbb{R}$ is bounded above by 1 since $\nu_s(\mathbb{R}) \leq \nu(\mathbb{R}) = 1$). □

Proof of Lemma 2.12. The arguments below extend the ones in the proof of Theorem 3 in [7] to excursions taking values in possibly asymmetric intervals. Upon noticing that ζ is a Brownian first exit time from (ℓ, N), by properties of Brownian local times (see Chapter VI [19]), one has

$$E^x[L_\zeta^a] = 2\frac{(\min(a,x) - \ell)(N - \max(a,x))}{N - \ell} \quad \text{for} \quad \ell < \min(a,x) \leq \max(a,x) < N. \tag{5.11}$$

We can use this result to compute expected local times for Y at the exit time τ,

$$\begin{aligned} \dot{E}^\ell[L_\tau^a(Y)] &= E^0[L_\zeta^a(\tilde{W})] = E^0[L_\zeta^a] \\ \dot{E}^x[L_\tau^a(Y)] &= E^x[L_\zeta^a(\tilde{W})] = E^x[L_\zeta^a] \quad \text{for} \quad x \in (\ell, N) \end{aligned} \tag{5.12}$$

Y constructed as above is a recurrent Markov process, we can then use the result in equation (1) from [4] to represent its invariant measure, let's say π. In particular, let $n = \mathbb{P}^0$ be the excursion measure, then for any test function $f(\cdot)$

$$\int f(z)\pi(dz) = \int \left(\int_\rho^\tau f(Y_s)ds \right) dn + f(\ell) = E^0 \left[\int_\rho^\tau f(Y_s)ds \right] + f(\ell) = E^0 \left[\int_0^\zeta f(\tilde{W}_s)ds \right] + f(\ell)$$
$$= \int f(a)E^0[L_\zeta^a]da + f(\ell). \tag{5.13}$$

The result follows by the occupation time formula for Brownian motion. This would imply that the invariant measure for Y can be decomposed as $\pi(da) = E[L_\zeta^a]da + \delta_\ell(da)$[13].

Let η be a positive measure on (ℓ, N). Consider the PCAF $t \mapsto F_t = \int_{(\ell,N)} L_t^a(Y)\eta(da)$[14]. For π invariant measure of Y, we can compute the Revuz measure associated with the PCAF $t \mapsto F_t$ say χ. From [19] (Chapter X), for any test function $f(\cdot)$

$$\int_{(\ell,N)} f(a)\chi(da) = \frac{1}{t}\dot{E}^\pi \left[\int_0^t f(Y_s)dF_s \right] = \frac{1}{t}\dot{E}^\pi \left[\int_0^t f(Y_s) \left(\int dL_s^a(Y)\eta(da) \right) \right] \tag{5.15}$$
$$= \frac{1}{t}\dot{E}^\pi \left[\int \int_0^t f(Y_s)dL_s^a(Y)\eta(da) \right] = \frac{1}{t}\dot{E}^\pi \left[\int f(a) \int_0^t dL_s^a(Y)\eta(da) \right] = \frac{1}{t} \int_{(\ell,N)} f(a)\dot{E}^\pi[L_t^a(Y)]\eta(da)$$

For a fixed, the measure $dL_t^a(Y)$ is $\mathbb{P}$-a.s. carried by the set $\{t : Y_t = a\}$ (see [19] Proposition 1.3 Chapter VI). Y defined in this way is a semimartingale, so by occupation time formula

$$\int_0^t g(Y_s)d\langle Y \rangle_s^c = \int_0^t g(Y_s)(\mathbb{1}_{(\ell,N)}(Y_s) + \mathbb{1}_{\{\ell\}}(Y_s))d\langle Y \rangle_s^c \tag{5.16}$$
$$= \int_0^t g(Y_s)\mathbb{1}_{(\ell,N)}(Y_s)ds = \int_\ell^N g(a)L_t^a(Y)da.$$

The result above holds upon realizing that the quadratic variation of Y is 0 when the process is constant at its holding point and equal to quadratic variation of Brownian motion when it is in (ℓ, N). Apply $\dot{E}^\pi[\cdot]$ on both sides, since $Y_0 \sim \pi$, with π being the invariant measure for Y, we can say that $Y_s \sim \pi$ (up to normalization of π). We can then rewrite the LHS as

$$\int_0^t \dot{E}^\pi \left[g(Y_s)\mathbb{1}_{(\ell,N)}(Y_s) \right] ds = \int_0^t \left(\int_{[\ell,N)} g(a)\mathbb{1}_{(\ell,N)}(a)\pi(da) \right) ds = t \int_\ell^N g(a)E^0[L_\zeta^a]da. \tag{5.17}$$

We have that $tE^0[L_\zeta^a] = \dot{E}^\pi[L_t^a(Y)]$ a.e. on (ℓ, N), since for any test function $g(\cdot)$, we have that

$$t \int_\ell^N g(a)E^0[L_\zeta^a]da = \int_\ell^N g(a)\dot{E}^\pi[L_t^a(Y)]da. \tag{5.18}$$

So the Revuz measure χ associated with the PCAF $t \mapsto F$ is such that

$$\int_{(\ell,N)} f(z)\chi(dz) = \int_{(\ell,N)} f(a)E^0[L_\zeta^a]\eta(da) \tag{5.19}$$

[13] If Y was a Brownian motion, then $\pi(da) = da$; however, we need to correct for the fact that Y is defined through excursions from a holding point ℓ of a Brownian motion started in 0 and killed at ζ. This would explain the atom at ℓ and the continuous component with density defined as $E^0[L_\zeta^\cdot]$. π is a finite measure on $[\ell, N)$; by occupation time formula for Brownian motion

$$\int_\ell^N \pi(da) = \int_\ell^N E^0[L_\zeta^a]da + 1 = \int_\ell^N 2\frac{(\min(a,0) - \ell)(N - \max(a,0))}{N - \ell}da + 1 < \infty \tag{5.14}$$

[14] We are averaging out the space variable from $L_t^a(Y)$ through the measure η.

This implies $\chi(da) = E^0[L^a_\zeta]\eta(da)$ for any arbitrary admissible positive measure η defining a PCAF. The result in [4] requires a PCAF whose Revuz measure χ is the target distribution ν. Since $\chi(da) = E^0[L^a_\zeta]\eta(da)$, this requires setting $\eta(da) = \nu(da)/E^0[L^a_\zeta]$. This implies that

$$t \mapsto F_t = \int_{(\ell,N)} \frac{L^a_t(Y)}{E^0[L^a_\zeta]}\nu(da) = \int_{I_\nu} \frac{L^a_t(Y)}{E^0[L^a_\zeta]}\nu(da). \tag{5.20}$$

$E^0[L^a_\zeta] > 0$ for any $a \in I_\nu$. Indeed, we are integrating over I_ν strictly included in (ℓ, N) and ζ represents the first exit time from (ℓ, N) for W with $\ell < \underline{\nu} < 0 < \overline{\nu} < N$. For any $a \in I_\nu$, L^a_ζ is $\mathbb{P}$-a.s. non-negative and strictly positive with strictly positive probability.

As in [7], let us compute $\hat{V}_\nu$ and V_χ as defined in [4]. In particular, we want to compute $\hat{V}_\nu$ such that $\int \hat{V}_\nu(a)\chi(da) = \int V_\chi(x)\nu(dx)$, with $V_\chi(x) = \dot{E}^x\left[\int_\rho^\tau dF_s\right]$ and $x \in (\ell, N)$. We then aim to determine $\lambda = \|\hat{V}_\nu\|_\infty$. Let us first compute V_χ defined as above

$$V_\chi(x) = \dot{E}^x\left[\int_\rho^\tau dF_s\right] = \dot{E}^x\left[\int_\rho^\tau \int_{(\ell,N)} dL^a_s(Y)\eta(da)\right] = E^x\left[\int_0^\zeta \int_{(\ell,N)} dL^a_s(\tilde{W})\eta(da)\right] \tag{5.21}$$

$$= E^x\left[\int_{(\ell,N)} L^a_\zeta(\tilde{W})\eta(da)\right] = \int_{(\ell,N)} E^x[L^a_\zeta]\eta(da) = \int_{(\ell,N)} 2\frac{(\min(a,x)-\ell)(N-\max(a,x))}{N-\ell}\eta(da).$$

Let us integrate V_χ against the target measure ν

$$\int_{I_\nu} V_\chi(x)\nu(dx) = \int_{I_\nu}\left(\int_{(\ell,N)} E^x[L^a_\zeta]\eta(da)\right)\nu(dx) = \int_{(\ell,N)}\left(\int_{I_\nu} E^x[L^a_\zeta]\nu(dx)\right)\eta(da) \tag{5.22}$$

$$= \int_{I_\nu}\left(\int_{I_\nu} \frac{E^x[L^a_\zeta]}{E^0[L^a_\zeta]}\nu(dx)\right)\nu(da) = \int_{I_\nu} \hat{V}_\nu(a)\nu(da).$$

Let us check the boundedness condition for $\hat{V}_\nu(\cdot)$ and compute its sharpest bound λ.

$$\hat{V}_\nu(a) = \int_{I_\nu} \frac{E^x[L^a_\zeta]}{E^0[L^a_\zeta]}\nu(dx) = \int_{I_\nu} \frac{(\min(a,x)-\ell)(N-\max(a,x))}{(\min(a,0)-\ell)(N-\max(a,0))}\nu(dx) \tag{5.23}$$

$$= \frac{\int(\min(a,x)-\ell)(N-\max(a,x))\nu(dx)}{\int(\min(a,x)-\ell)(N-\max(a,x))\delta_0(dx)}.$$

Let μ be a distribution on (ℓ, N) with mean 0, then it is true that[15]

$$\int_{I_\mu} (\min(a,x)-\ell)(N-\max(a,x))\mu(dx) = (N-a)(a-\ell) - (N-a)P_\mu(a) - (a-\ell)C_\mu(a).$$

Since both ν and δ_0 are centered at 0, the ratio in equation (5.23) becomes

$$\hat{V}_\nu(a) = \frac{(N-a)(a-\ell) - (N-a)P_\nu(a) - (a-\ell)C_\nu(a)}{(N-a)(a-\ell) - (N-a)P_{\delta_0}(a) - (a-\ell)C_{\delta_0}(a)} \quad \text{for} \quad a \in I_\nu \tag{5.25}$$

$\hat{V}_\nu(\cdot)$ is non negative since it is defined as an integral of a non negative function. By Jensen's inequality one shows that $P_\nu(a) \geq P_{\delta_0}(a)$ for any $a \in \mathbb{R}$ (similarly, $C_\nu(a) \geq C_{\delta_0}(a)$). This would imply that $|\hat{V}_\nu(a)| = \hat{V}_\nu(a) \leq 1$ for any $a \in (\ell, N)$. We want to show that 1 is the sharpest bound (i.e., $\|\hat{V}_\nu\|_\infty = 1$). Let $a \in \text{int}(I_\nu) \cap (0, \infty)$, then we know that $C_{\delta_0}(a) = (-a)^+ = 0$, while $P_{\delta_0}(a) = (a)^+ = a$, therefore

$$\hat{V}_\nu(a) = \frac{(N-a)(a-\ell) - (N-a)P_\nu(a) - (a-\ell)C_\nu(a)}{(N-a)(a-\ell) - (N-a)a} \tag{5.26}$$

[15] Let μ be an integrable distribution, $P_\mu(\cdot)$ and $C_\mu(\cdot)$ (*put* and *call* functions of μ, respectively) are

$$P_\mu(a) = \int (a-x)^+\mu(dx) \quad \text{and} \quad C_\mu(a) = \int (x-a)^+\mu(dx). \tag{5.24}$$

Notice that $P_\mu(a) + C_\mu(a) = U_\mu(a)$, while $P_\mu(a) - C_\mu(a) = a - \int y\mu(dy)$.

$$= \left(1 - \frac{(N-a)P_\nu(a) + (a-\ell)C_\nu(a)}{(N-a)(a-\ell)}\right)\frac{a-\ell}{|\ell|}.$$

Let us consider $a \uparrow \sup(I_\nu) = \overline{\nu} < N$, by continuity of $P_\nu(\cdot)$ and $C_\nu(\cdot)$

$$\lim_{a\uparrow\overline{\nu}} \hat{V}_\nu(a) = \left(1 - \frac{(N-\overline{\nu})P_\nu(\overline{\nu}) + (\overline{\nu}-\ell)C_\nu(\overline{\nu})}{(N-\overline{\nu})(\overline{\nu}-\ell)}\right)\frac{\overline{\nu}-\ell}{|\ell|}. \tag{5.27}$$

Properties of potentials (see [5]) show $(\underline{\nu},\overline{\nu}) = \{x : P_\nu(x) > \max(x,0)\} = \{x : C_\nu(x) > \max(-x,0)\}$. Jointly with $P_\nu(a) \geq P_{\delta_0}(a)$ and $C_\nu(a) \geq C_{\delta_0}(a)$, this implies that $P_\nu(\overline{\nu}) = \overline{\nu}$ and $C_\nu(\overline{\nu}) = 0$. Then

$$\lim_{a\uparrow\overline{\nu}} \hat{V}_\nu(a) = \left(1 - \frac{\overline{\nu}}{\overline{\nu}-\ell}\right)\frac{\overline{\nu}-\ell}{|\ell|} = \frac{-\ell}{|\ell|} = 1 \quad \text{since} \quad \ell < 0. \tag{5.28}$$

The crucial assumption for this argument is $\overline{\nu} < N$. We previously showed that 1 is a global upper bound on (ℓ, N), here we show that it can be attained when $a \uparrow \bar{\nu}$, hence $\|\hat{V}_\nu\|_\infty = \lambda = 1$. By using the result in [4], we can now construct T_{BLJ} that solves the SEP for Y. In particular,

$$\begin{aligned} T_{BLJ} &= \inf\left\{t \geq 0 : \int_0^t (\lambda - \hat{V}_\nu(Y_s))^{-1} dF_s > L_\rho^\ell(Y)\right\} \\ &= \inf\left\{t \geq 0 : \int_0^t (\lambda - \hat{V}_\nu(Y_s))^{-1} \left(\int_{I_\nu} \frac{dL_s^a(Y)}{E^0[L_\zeta^a]}\nu(da)\right) > L_\rho^\ell(Y)\right\} \\ &= \inf\left\{t \geq 0 : \int_{I_\nu} \frac{(\lambda - \hat{V}_\nu(a))^{-1}}{E^0[L_\zeta^a]} L_t^a(Y)\nu(da) > L_\rho^\ell(Y)\right\} \end{aligned} \tag{5.29}$$

Let $\lambda = 1$ and, for $f(a) = (N-a)(a-\ell)$, consider

$$\lambda - \hat{V}_\nu(a) = 2\frac{(N-a)(P_\nu(a) - P_{\delta_0}(a)) + (a-\ell)(C_\nu(a) - C_{\delta_0}(a))}{E^0[L_\zeta^a](N-\ell)} \tag{5.30}$$

This holds by applying the result in equation (5.25) backwards jointly with the representation of $E^0[L_\zeta^a]$ in equation (5.11). Therefore

$$\frac{(1-\hat{V}_\nu(a))^{-1}}{E^0[L_\zeta^a]} = \frac{1}{2}\frac{(1-\ell/N)}{(P_\nu(a) - P_{\delta_0}(a)) - (\ell/N)(C_\nu(a) - C_{\delta_0}(a))}. \tag{5.31}$$

Since ν and δ_0 have both mean zero, $P_\nu(a) - C_\nu(a) = P_{\delta_0}(a) - C_{\delta_0}(a) = a$, which implies that $C_\nu(a) - C_{\delta_0}(a) = P_\nu(a) - P_{\delta_0}(a)$, this allows to further simplify the expression above to get

$$\frac{(1-\hat{V}_\nu(a))^{-1}}{E^0[L_\zeta^a]} = \frac{1}{2}\frac{(1-\ell/N)}{(1-\ell/N)(P_\nu(a) - P_{\delta_0}(a))} = \frac{1}{2(P_\nu(a) - P_{\delta_0}(a))}. \tag{5.32}$$

The embedding T_{BLJ} can be expressed as

$$T_{BLJ} = \inf\left\{t \geq 0 : \int_{I_\nu} \frac{L_t^a(Y)}{2(P_\nu(a) - P_{\delta_0}(a))}\nu(da) > L_\rho^\ell(Y)\right\}. \tag{5.33}$$

It is straightforward to realize that $\rho \leq T_{BLJ}$. Indeed, the process Y stays in ℓ in $[0,\rho)$, which makes it impossible to find a solution to the SEP for ν into Y in that interval. On the other hand, the arguments presented in [4] guarantee that $T_{BLJ} < \tau = \inf\{t > \rho : Y_t = \ell\}$ (the first time Y goes back to the state ℓ after ρ, the embedding happens in the first excursion)[16]. This allows us

[16]We previously said that $\rho \leq T_{BLJ}$, by properties of local times $L_{T_{BLJ}}^\ell(Y) - L_\rho^\ell(Y) \geq 0$ $\mathbb{P}$-a.s. This implies that

$$\dot{E}^\ell[L_{T_{BLJ}}^\ell(Y) - L_\rho^\ell(Y)] = \dot{E}^\ell[L_{T_{BLJ}}^\ell(Y)] - \dot{E}[L_\rho^\ell(Y)] = 1 - 1 = 0. \tag{5.34}$$

The first expectation is 1 by the corollary at page 540 in [4], the second is 1 by the construction of Itô excursion processes. This would mean that Y does not "accumulate local time" at the boundary ℓ between ρ and T_{BLJ}, which proves that T_{BLJ} is bounded by the first time Y returns to ℓ after ρ. Moreover, since the target distribution is such that $\nu(\{\ell\}) = 0$, then $T_{BLJ} < \tau$.

to rewrite T_{BLJ} as

$$T_{BLJ} = \rho + \inf\left\{ s \geq 0 : \int_{I_\nu} \frac{L^a_{s+\rho}(Y)}{2(P_\nu(a) - P_{\delta_0}(a))}\nu(da) > L^\ell_\rho(Y) \right\} \tag{5.35}$$
$$= \rho + \inf\left\{ \zeta > s \geq 0 : \int_{I_\nu} \frac{L^a_s(\tilde{W})}{2(P_\nu(a) - P_{\delta_0}(a))}\nu(da) > L^\ell_\rho(Y) \right\} = \rho + \tau_{BLJ}.$$

Indeed, we already said that, by construction of Y, $L^a_t(Y) = L^a_{t-\rho}(\tilde{W}) = L^a_{t-\rho}$ for $\ell < a < N$ and $\rho \leq t < \tau$. Both those conditions are verified since $\rho \leq T_{BLJ} < \tau = \rho + \zeta$ and I_ν is strictly included in (ℓ, N). Remember that $\tilde{W}$ is a $(\mathbb{P}, \hat{\mathbb{F}})$-Brownian motion killed at the first exit time from (ℓ, N) (i.e. ζ) and sent to the state ℓ.

Remark 5.2. Assume now that ν places mass at one or both extremes of its support; for example, $\nu(\{\overline{\nu}\}) > 0$. In such a case, the integral used in the definition of τ_{BLJ} has a contribution that is proportional to the integrand computed in $\overline{\nu}$ (due to the Dirac mass there). For $t \leq H_{\overline{\nu}}$, $L^{\overline{\nu}}_t = 0$, while $L^{\overline{\nu}}_t > 0$ for $t > H_{\overline{\nu}}$. This means that this contribution will be 0 for any $t \leq H_{\overline{\nu}}$ and ∞ otherwise, which implies that the crossing condition is satisfied for any $t > H_{\overline{\nu}}$. In other words, the contribution coming from $\bar{\nu}$ is infinite at times arbitrarily close to and strictly larger than $H_{\bar{\nu}}$. Consequently, the infimum of the times at which the threshold is crossed equals $H_{\bar{\nu}}$. This reads as the condition that triggers the embedding of the point mass placed at $\overline{\nu}$, on the set $\{H_{\bar{\nu}} < \zeta\}$.

The fact that $Y_{T_{BLJ}} \sim \nu$ and $\rho \leq T_{BLJ} < \tau = \rho + \zeta$ imply that $0 \leq \tau_{BLJ} < \zeta$ and $Y_{T_{BLJ}} = \tilde{W}_{\tau_{BLJ}} = W_{\tau_{BLJ}} \sim \nu$. We previously said that the lifetime of $\tilde{W}$ is defined as $\zeta = \inf\{t \geq 0 : W_t \notin (\ell, N)\} = \min(H_\ell, H_N) \leq H_\ell = \langle X \rangle_\infty$. This implies that $\tau_{BLJ} < \langle X \rangle_\infty$. □

Proof of Proposition 2.14. Let $U_\nu(\cdot)$ be the potential of ν (i.e., $U_\nu(a) = \int |x - a|\nu(dx) = P_\nu(a) + C_\nu(a)$) and consider M sufficiently large. Here we argue as in [7] to identify a truncated version of ν, in this case however we only need truncation for the right tail, since the support of the target measure is already bounded below by $\underline{\nu} > \ell$. Define $g(\cdot)$ such that

$$g(a) = \begin{cases} U_\nu(a) & \text{for} \quad a \in [\ell, q_M) \\ (a - q_M)\frac{M - U_\nu(q_M)}{M - q_M} + U_\nu(q_M) & \text{for} \quad a \in [q_M, M) \\ |a| & \text{for} \quad a \in (-\infty, \ell) \cup [M, \infty). \end{cases} \tag{5.36}$$

We want to pick q_M such that $g(\cdot)$ is not only continuous, but also a convex, this is achieved for q_M such that the tangent of $U_\nu(\cdot)$ at q_M passes through the point (M, M) (i.e., smooth pasting of the line passing through (M, M) and $U_\nu(q_M)$ for some level $0 < q_M < M$). This means that q_M has to verify the equation $U'_\nu(q_M) = (M - U_\nu(q_M))/(M - q_M)$[17]. By using the fact that $U_\nu(a) = x(1 - 2\nu([x, \infty))) + 2\int_{[x,\infty)} u\nu(dy)$ (see [16]), we can express the smoothing condition as $M = E[X|X \geq q_M]$, where $X \sim \nu$.

Given the correspondence between convex functions and probability measures, we can derive a distribution ν_M such that $\nu_M((-\infty, a]) := (1 + g'_+(a))/2$[18]. One can show that $U_{\nu_M} = g$ with U_{ν_M} being the potential of ν_M. By properties of potentials (see [5]), ν_M and ν agree on $(-\infty, q_M]$[19], and has mean zero. Thus, for any $M > 0$, ν_M places mass on (a subset of) $(\underline{\nu}, q_M) \cup \{M\}$, this is a combination of the previous properties and the fact that potentials grow linearly in regions of zero mass (in this case, $[q_M, M)$), and they coincide with $a \mapsto |a|$ outside the interior of the smallest interval that contains the support of the corresponding measure (i.e., $U_\nu(a) = |a|$ for $a \notin \text{int}(I_\nu)$). $U_{\nu_M}(a) > a$ in $[q_M, M)$ and $U_{\nu_M}(a) = a$ jointly isolate an atom in M. As M goes to ∞, $q_M \uparrow \infty$ and $U_{\nu_M} \uparrow U_\nu$ pointwise, this implies that ν_M converges *weakly* to ν (see [5], and [7]).

[17] $U_\nu(\cdot)$ is differentiable under the assumption of $\nu \ll m$.

[18] $g'_+(\cdot)$ is the right derivative of $g(\cdot)$. Convexity of $g(\cdot)$ guarantees that the distribution function $a \mapsto \nu_M((-\infty, a])$ is non decreasing. Potential jumps in $a \mapsto \nu_M((-\infty, a])$ happen at the kinks of $g(\cdot)$, their magnitudes coincides with the jump size of $g'_+(\cdot)$, up to a linear transformation.

[19] In general, $\nu(\{q_M\}) \geq \nu_M(\{q_M\})$, but this implies that $\nu_M(\{q_M\}) = 0$ since ν is atomless.

For M sufficiently large, we will refer to ν_M as the truncated version of ν. The previous arguments justify the following decomposition

$$\begin{aligned}\nu_M(A) &= \nu(A \cap (\underline{\nu}, q_M)) + \kappa\delta_M(A) \qquad \text{for} \quad A \in \mathscr{B}(\mathbb{R}) \\ \nu_M(A) &= \int_A \mathbb{1}_{(\underline{\nu},q_M)}(a) f_\nu(a)da + \kappa\delta_M(A)\end{aligned} \tag{5.37}$$

where $\kappa = 1 - \nu((\ell, q_M))$ represents the mass that the original measure places on $[q_M, \infty)$. We derived a distribution ν_M with bounded support such that $\ell < \underline{\nu} = \underline{\nu}_M = \inf(I_{\nu_M})$ and $\overline{\nu}_M = \sup(I_{\nu_M}) = M$. Let $\varepsilon > 0$, consider $N = M + \varepsilon$ and apply the result in Proposition 2.12 to derive τ^M_{BLJ} such that $W_{\tau^M_{BLJ}} \sim \nu_M$, with

$$\tau^M_{BLJ} = \inf\left\{\zeta > s \geq 0 : \int_{(\underline{\nu},M]} \frac{L^a_s(\tilde{W})\nu_M(da)}{2(P_{\nu_M}(a) - P_{\delta_0}(a))} > L^\ell_\rho(Y)\right\}. \tag{5.38}$$

So, by using the decomposition in equation (5.37), the integral has a singular contribution that makes the functional jump to ∞ for any $t > H_M$ and that is zero otherwise (see Remark 5.2, H_M is the trigger that allows the embedding of the atom at M). Since the crossing condition is always verified for $t > H_M$, we can write

$$\begin{aligned}\tau^M_{BLJ} &= \min\left(\inf\left\{s \geq 0 : \int_{\underline{\nu}}^{q_M} \frac{L^a_s(\tilde{W})\nu_M(da)}{2(P_{\nu_M}(a) - P_{\delta_0}(a))} > L^\ell_\rho(Y)\right\}, H_M\right) \\ \tau^M_{BLJ} &= \min\left(\sigma_M, H_M\right) \\ \text{with} \quad \sigma_M &= \inf\left\{s \geq 0 : \int_{\underline{\nu}}^{q_M} \frac{L^a_s(\tilde{W})f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))}da > L^\ell_\rho(Y)\right\}.\end{aligned} \tag{5.39}$$

Remember that W is a stopped Brownian motion that we obtain through time-changing the local martingale X, while $\tilde{W}$ is obtained by further stopping W at the first hitting time of a state $N > 0$, so it evolves like a Brownian motion before $\zeta = \min(H_\ell, H_N)$, at this first exit time it is sent to ℓ. This means that, up to ζ, the trajectories of W and $\tilde{W}$ must coincide.

Local times depend on the path of the process up to some time t. So, if two Brownian motions coincide up to t, they will have the same local time[20]. This implies that $L^a_t(\tilde{W}) = L^a_t(W)$ for any $t \leq \zeta$. Since $\zeta \leq H_\ell$, consider now what happens *pathwise* between ζ and H_ℓ. If $\zeta < H_\ell$, then the process $\tilde{W}$ stays in the state ℓ, while W evolves until it hits ℓ. Since $t \mapsto L^a_t(\tilde{W})$ increases on $\{u : \tilde{W}_u = a\}$, it "freezes" when computed for $a > \ell$ and $\zeta \leq t < H_\ell$. If $\zeta = H_\ell$, then we stop $\tilde{W}$ and W at the same time. If (L^a_t) is the collection of Brownian local times, we can write

$$\begin{aligned}L^a_s(\tilde{W}) &= L^a_{s\wedge\zeta}(W) = L^a_{s\wedge\zeta} \qquad \text{for any} \quad s \in [0, H_\ell) \quad \text{and} \quad a \in (\ell, N) \\ \sigma_M &= \inf\left\{s \geq 0 : \int_{\underline{\nu}}^{q_M} \frac{L^a_{s\wedge\zeta} f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))}da > L^\ell_\rho(Y)\right\}.\end{aligned} \tag{5.40}$$

We aim to relax the truncation and send M to ∞. Since $H_M \uparrow \infty$ $\mathbb{P}$-a.s., to study the convergence of τ^M_{BLJ} we can focus on σ_M. To this end, let M be sufficiently large and write the integral in the crossing condition for σ_M as

$$\int_{\underline{\nu}}^{\infty} \frac{\mathbb{1}_{(\underline{\nu},q_M)}(a) L^a_{s\wedge\zeta} f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))}da. \tag{5.41}$$

Given $a \in (\ell, \infty)$, define the functions $t \mapsto f_M(a,t)$, and $t \mapsto f(a,t)$ as

$$f_M(a,t) = \frac{\mathbb{1}_{(\underline{\nu},q_M)}(a) L^a_{t\wedge\zeta} f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))}$$

[20]If the Brownian motions X and Y have the same trajectory up to time t, then $\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{B(a,\varepsilon)}(X_u)du = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{B(a,\varepsilon)}(Y_u)du$.

$$f(a,t) = \frac{L^a_{t\wedge H_\ell} f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))}.$$

For any M, properties of Brownian local times show that $s \mapsto L^a_s$ is $\mathbb{P}$-a.s continuous, non-decreasing, non-negative, 0 at $s = 0$. Moreover, for $M_0 < M_1$ we have that $\zeta_0 = \min(H_\ell, H_{M_0+\varepsilon}) \leq \min(H_\ell, H_{M_1+\varepsilon}) = \zeta_1$, while $q_{M_0} \leq q_{M_1}$. Since local times are non decreasing in time, this means that $f_{M_0}(a,t) \leq f_{M_1}(a,t)$ for any $t \geq 0$ and $a \in (\underline{\nu}, \infty)$. (f_M) is a non-decreasing sequence of non negative functions. Moreover, as M increases, $H_N = H_{M+\varepsilon}$ diverges to ∞, implying that ζ converges to H_ℓ; similarly we have that $q_M \uparrow \infty$. Continuity of local times imply that the sequence (f_M) pointwise converges to the limit f. We can use MCT to show that

$$\lim_{M\uparrow\infty} F_M(t) = \lim_{M\uparrow\infty} \int_{\underline{\nu}}^{\infty} f_M(a,t)da = \int_{\underline{\nu}}^{\infty} f(a,t)da = F(t). \tag{5.42}$$

This shows that the sequence of functionals (F_M) (used to derive the crossing condition in σ_M) pointwise converges to the functional F, we can similarly argue that $F_M \leq F$ for any M. One also shows that monotonicity of local times imply monotonicity for $t \mapsto F_M(t)$ and $t \mapsto F(t)$.

We can show that (F_M) does not converge only pointwise to F, but it does uniformly on compact sets. In particular, for M sufficiently large, and $k > 0$, consider

$$\begin{aligned}\sup_{0\leq t\leq k} |F_M(t) - F(t)| &= \sup_{0\leq t\leq k} \left(\int_{\underline{\nu}}^{\infty} \frac{\left(L^a_{t\wedge H_\ell} - \mathbb{1}_{(\underline{\nu},q_M)}(a) L^a_{t\wedge\zeta}\right) f_\nu(a)da}{2(P_\nu(a) - P_{\delta_0}(a))} \right) \\ &\leq \sup_{0\leq t\leq k} \left(\int_{\underline{\nu}}^{q_M} \frac{\left(L^a_{t\wedge H_\ell} - L^a_{t\wedge\zeta}\right) f_\nu(a)da}{2(P_\nu(a) - P_{\delta_0}(a))} \right) + \int_{q_M}^{\infty} \frac{\sup_{0\leq t\leq k}\left(L^a_{t\wedge H_\ell}\right) f_\nu(a)da}{2(P_\nu(a) - P_{\delta_0}(a))}.\end{aligned} \tag{5.43}$$

The second term converges to 0 as M goes to ∞. This means that we can focus on the first term. Since the integrand is non negative, let us bound it by extending the integration interval to $(\underline{\nu}, \infty)$ and bounding the difference of local times by its supremum over the compact set $[0, k]$. Since $\zeta \uparrow H_\ell$ as M increases, the difference $(L^a_{t\wedge H_\ell} - L^a_{t\wedge\zeta})$ forms a non-increasing sequence that converges to 0. This implies that $(L^a_{t\wedge H_\ell} - L^a_{t\wedge\zeta})$ converges to 0 uniformly on $[0, k]$. On the other hand, the supremum of the difference between local times can be bounded above by $L^a_{k\wedge H_\ell}$, under integrability of the ratio $(L^a_{k\wedge H_\ell} f_\nu(a))/2(P_\nu(a) - P_{\delta_0}(a))$, DCT applies and

$$\lim_{M\uparrow\infty} |F_M(t) - F(t)| \leq \lim_{M\uparrow\infty} \int_{\underline{\nu}}^{\infty} \frac{\sup_{0\leq t\leq k}(L^a_{t\wedge H_\ell} - L^a_{t\wedge\zeta}) f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))} da = 0. \tag{5.44}$$

The functional $F(\cdot)$ is non-decreasing on $[0, H_\ell)$, and it increases strictly over an interval $(s,t) \subseteq [0, H_\ell)$ if and only if W spends a strictly positive amount of time, in the Lebesgue sense, in the region where $f_\nu(\cdot)$ is positive. Indeed, given $t > s$

$$\begin{aligned}F(t) - F(s) &= \int_{\underline{\nu}}^{\infty} \frac{L^a_t f_\nu(a)da}{2(P_\nu(a) - P_{\delta_0}(a))} - \int_{\underline{\nu}}^{\infty} \frac{L^a_s f_\nu(a)da}{2(P_\nu(a) - P_{\delta_0}(a))} \\ &= \int_s^t \frac{\mathbb{1}_{(\underline{\nu},\infty)}(W_u) f_\nu(W_u)du}{2(P_\nu(W_u) - P_{\delta_0}(W_u))}.\end{aligned} \tag{5.45}$$

The last integral is strictly positive if Brownian motion spends an amount of time of positive Lebesgue measure where $f_\nu(\cdot) > 0$ in the interval (s,t). $F(\cdot)$ is flat for any $t \geq H_\ell$. Indeed, for $s \geq H_\ell$ and $t > s$, $F(t) = F(s) = F(H_\ell)$ since the local time used is evaluated at $t \wedge H_\ell$[21]. Moreover, since $L^\ell_\rho(Y) \sim \exp(1)$ is independent from W (this comes from the construction of Y, recurrent extension of $\tilde{W}$), then we can argue that $\mathbb{P}(F(t) = L^\ell_\rho(Y)) = 0$ for any $t \geq 0$.

Consider now the following arguments. We know that $\tau^M_{BLJ} = \min(\sigma_M, H_M)$ and $\tau^M_{BLJ} < \zeta$

[21] If $\underline{\nu} = \ell$ and $\text{supp}(\nu) = I_\nu$, then $F(\cdot)$ is strictly increasing on $[0, H_\ell)$ and flat otherwise.

$\mathbb{P}$-a.s.. Assume that $\tau^M_{BLJ} = H_M$, this is equivalent to saying that $\sigma_M \geq H_M$. If we look at the definition of σ_M, then we realize that $\sigma_M \geq H_M$ is equivalent to $F_M(s) \leq L^\ell_\rho(Y)$ for any $s \in [0, H_M)$. We also know that if $W_{\tau^M_{BLJ}} \sim \nu_M$, then $\tau^M_{BLJ} = H_M$ if and only if $W_{\tau^M_{BLJ}} = M$[22]. Therefore

$$\begin{aligned}\mathbb{P}(\tau^M_{BLJ} = H_M) &= \mathbb{P}\left(F_M(s) \leq L^\ell_\rho(Y) \quad \text{for any } s \in [0, H_M)\right) \\ \mathbb{P}(\tau^M_{BLJ} = H_M) &= \mathbb{P}\left(W_{\tau^M_{BLJ}} = M\right) = \nu_M(\{M\}) = 1 - \nu((\underline{\nu}, q_M)).\end{aligned} \tag{5.46}$$

This implies that $\mathbb{P}(A_M) = \nu((\underline{\nu}, q_M))$, where $A_M = \{F_M(s) > L^\ell_\rho(Y) \text{ for some } s \in [0, H_M)\}$. Since (F_M) is a non-decreasing sequence of functions, if $F_{M_0}(s) > L^\ell_\rho(Y)$ for some $s \in [0, H_{M_0})$, then we have $F_{M_1}(s) > L^\ell_\rho(Y)$ for some $s \in [0, H_{M_1})$, for $M_0 < M_1$. This means that (A_M) is a non decreasing sequence of events, therefore

$$\mathbb{P}\left(\bigcup_M A_M\right) = \lim_{M \uparrow \infty} \mathbb{P}(A_M) = \lim_{M \uparrow \infty} \nu((\underline{\nu}, q_M)) = 1. \tag{5.47}$$

Consider now the event $A = \{F(s) > L^\ell_\rho(Y) \text{ for some } s \in [0, \infty)\}$. Since F is an upper bound for the sequence (F_M), then we have that, if for some M it holds that $F_M(s) > L^\ell_\rho(Y)$ for some $s \in [0, H_M)$, the it is true that $F(s) > L^\ell_\rho(Y)$ for some $s \in [0, \infty)$. This means that $P(A) = 1$; the functional F crosses $L^\ell_\rho(Y)$ at some $t \in [0, \infty)$, and remains above it from that time onward. Since we said that on $[H_\ell, \infty)$ F is constantly equal to $F(H_\ell)$, then the crossing must happen before (or at) H_ℓ. We can now discuss whether the crossing is *strict*[23] or not.

From the previous arguments, it follows that $F(\cdot)$ is constant in $(s, t) \subseteq [0, H_\ell)$ whenever the underlying Brownian motion does not spend a positive amount of time, in the Lebesgue sense, in a region of the state space where $f_\nu(\cdot) > 0$. In other words $F(t) = F(s)$ if and only if $m(\{u \in (s, t) : f_\nu(W_u) > 0\}) = 0$. Let us call *plateau* the maximum interval where $F(\cdot)$ is constantly equal to a given level. Plateaus are pairwise disjoint intervals, since $F(\cdot)$ is non-decreasing, distinct maximal plateaus correspond to distinct levels. Moreover, one shows that there are at most countably many plateau and plateau levels[24]. The crossing does not happen at the plateaus $\mathbb{P}$-a.s. since we have at most countably many of them before H_ℓ and $L^\ell_\rho(Y)$ is an atomless random variable independent of W (i.e., $\mathbb{P}(L^\ell_\rho(Y)$ is at a plateau level for $F(\cdot)) = 0$). In other words, $F(\cdot)$ strictly crosses $L^\ell_\rho(Y)$ $\mathbb{P}$-a.s.. Heuristically, the independence between $L^\ell_\rho(Y)$ and W takes care of the strict crossing.

Let $\mathscr{D}$ be the space of non-decreasing continuous functions $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$. We endow $\mathscr{D}$ with a metric $d(\cdot, \cdot)$ that induces the topology of uniform convergence on compact sets. Given $x > 0$, define the first passage time functional $\Phi : \mathscr{D} \to [0, \infty]$

$$\Phi(f) = \inf\{s \geq 0 : f(s) > x\}. \tag{5.48}$$

Let $\mathscr{C}_\Phi \subseteq \mathscr{D}$ be the continuity set of $\Phi(\cdot)$. This means that for a sequence (f_n) that converges to f in $d(\cdot, \cdot)$, we have that the real sequence $(\Phi(f_n))$ converges to $\Phi(f)$. It is possible to show that f belongs to the continuity set of the operator Φ if f strictly crosses the threshold x.

(F_M) and F can be seen as random elements defined on the metric space $(\mathscr{D}, d)$. We can now argue pathwise. In other words, we fix an ω such that the needed $\mathbb{P}$-a.s. properties invoked so far hold. To keep notation as light as possible, we will avoid specifying the dependence on ω, which lies in the background. One notices that

$$\sigma_M = \Phi(F_M) = \inf\{s \geq 0 : F_M(s) > L^\ell_\rho(Y)\}. \tag{5.49}$$

[22]The implication $\tau^M_{BLJ} = H_M$ to $W_{\tau^M_{BLJ}} = M$ is immediate. Let $W_{\tau^M_{BLJ}} = M$ and assume by contradiction that $\tau^M_{BLJ} \neq H_M$. Since $\tau^M_{BLJ} = \min(\sigma_M, H_M) \leq H_M$, this means assuming that $\tau^M_{BLJ} < H_M$, which would imply that $W_{\tau^M_{BLJ}} < M$. This generates a contradiction.

[23]The infima of the sets $\{t \geq 0 : F(t) \geq L^\ell_\rho(Y)\}$ and $\{t \geq 0 : F(t) > L^\ell_\rho(Y)\}$ will coincide.

[24]Let $\mathcal{O} = \{t \in (0, H_\ell) : F(\cdot)$ is constant in an open ball of $t\}$, $\mathcal{O}$ is open and can be written as a countable union of pairwise disjoint open sets, which will then represents the interiors of each plateaus.

If F belongs to the continuity set for the operator $\Phi(\cdot)$ (i.e., $\mathscr{C}_\Phi$), then we conclude that $\Phi(F_M) \to \Phi(F)$ as M goes to ∞. So, the fact that F crosses strictly $L^\ell_\rho(Y)$ before H_ℓ $\mathbb{P}$-a.s. is sufficient to show that $\Phi(\cdot)$ is continuous at F, which would imply convergence of $\Phi(F_M)$ to $\Phi(F)$. In other words, this result allows us to conclude that

$$\begin{aligned}&\lim_{M\to\infty} \tau^M_{BLJ} = \lim_{M\to\infty} \min(\sigma_M, H_M) = \lim_{M\to\infty} \sigma_M = \lim_{M\to\infty} \Phi(F_M) = \tau_{BLJ} \quad \mathbb{P}\text{-a.s.} \\ &\tau_{BLJ} = \Phi(F) = \inf\left\{H_\ell > s \geq 0 : F(s) > L^\ell_\rho(Y)\right\}.\end{aligned} \tag{5.50}$$

Since the crossing happens before H_ℓ, we can consider the infimum over the time indices less than H_ℓ. Continuity of the sample paths for W and $\tau^M_{BLJ} \to \tau_{BLJ}$ $\mathbb{P}$-a.s. imply that $W_{\tau^M_{BLJ}} \to W_{\tau_{BLJ}}$ $\mathbb{P}$-a.s., which implies $W_{\tau^M_{BLJ}} \to W_{\tau_{BLJ}}$ in distribution. this is equivalent to saying that ν_M weakly converges to μ, where μ represents the distribution of $W_{\tau_{BLJ}}$. We previously said the convergence of U_{ν_M} to U_ν implies weak convergence of the corresponding distributions. To sum up, for any continuous and bounded test functions $f(\cdot)$, we have

$$\begin{aligned}\int f(x)\nu_M(dx) &\longrightarrow \int f(x)\mu(dx) \qquad \text{as} \quad M \to \infty \\ \int f(x)\nu_M(dx) &\longrightarrow \int f(x)\nu(dx) \qquad \text{as} \quad M \to \infty.\end{aligned} \tag{5.51}$$

By uniqueness of the limit, we have that, for any continuous and bounded test function $f(\cdot)$, $\int f(x)\nu(dx) = \int f(x)\mu(dx)$. This implies that $\nu = \mu$ and therefore $W_{\tau_{BLJ}} \sim \nu$ with

$$\tau_{BLJ} = \inf\left\{H_\ell > s \geq 0 : \int_{I_\nu} \frac{L^a_{s\wedge H_\ell} f_\nu(a)}{2(P_\nu(a) - P_{\delta_0}(a))} da > L^\ell_\rho(Y)\right\}. \tag{5.52}$$

As discussed in the description of Y, this process can be seen as an Itô excursion process with ℓ being a holding point with parameter 1. This implies that $L^\ell_\rho(Y) \sim \exp(1)$ and independent of the W. So, let us replace $L^\ell_\rho(Y)$ with E. In the proof of Lemma 2.12, we argued that $P_\nu(a) - P_{\delta_0}(a) = C_\nu(a) - C_{\delta_0}(a)$, this can be used to show that $U_\nu(a) - |a| = 2(P_\nu(a) - P_{\delta_0}(a))$[25]. With a slight abuse of notation we have

$$\tau_{BLJ} = \inf\left\{H_\ell > s \geq 0 : \int_{I_\nu} \frac{L^a_{s\wedge H_\ell} f_\nu(a)}{U_\nu(a) - |a|} da > E\right\}. \tag{5.53}$$

This concludes the proof, $W_{\tau_{BLJ}} \sim \nu$. □

Proof of Proposition 2.15. By applying Lemma 2.14 we can derive the embedding τ_{BLJ} as written in Equation (2.31).

$$\begin{aligned}\tau_{BLJ} &= \inf\left\{H_\ell > s \geq 0 : \int_{I_\nu} L^a_{s\wedge H_\ell} \frac{f_\nu(a)}{U_\nu(a) - U_{\delta_0}(a)} \frac{\xi^2(a)}{\xi^2(a)} da > E\right\} \\ &= \inf\left\{H_\ell > s \geq 0 : \int_{I_\nu} L^a_{s\wedge H_\ell} \frac{\mathtt{D}(a)}{\xi^2(a)} da > E\right\} = \inf\left\{H_\ell > s \geq 0 : \int_0^s \frac{\mathtt{D}(W_t)}{\xi^2(W_t)} dt > E\right\}\end{aligned} \tag{5.54}$$

τ_{BLJ} described in equation (5.54) is such that $\tau_{BLJ} < H_\ell = \langle X\rangle_\infty$ $\mathbb{P}$-a.s.. As discussed in Section 2.4, this would allow us to conclude $X_{T(\tau_{BLJ})} \sim \nu$, with $T(\tau_{BLJ}) = \inf\{s \geq 0 : \langle X\rangle_s > \tau_{BLJ}\}$ ($T(\cdot)$ indicates the time change used to derive the stopped Brownian motion W from the local martingale X).

We want now to show that $T(\tau_{BLJ})$ can be equivalently written as the first jump of a Cox process with intensity given by $(\mathtt{D}(X_s))$, which would allow us to interpret $\mathtt{D}(\cdot)$ as a stopping intensity. For any $0 < t \leq s < H_\ell = \langle X\rangle_\infty$, we can express $W_t = X_{T(t)}$ and then set $z = T(t)$. For $0 < t \leq s < H_\ell = \langle X\rangle_\infty$, given $\xi(\cdot) > 0$, the quadratic variation of X is continuous and increasing.

[25] One can equivalently prove that the expected local time at an arbitrary embedding τ such that $\langle X\rangle_\tau$ is integrable can be represented as $U_\nu(a) - |a|$, as $2(P_\nu(a) - P_{\delta_0}(a))$ or as $2(C_\nu(a) - C_{\delta_0}(a))$.

Therefore $\langle X\rangle_z = \langle X\rangle_{T(t)} = t$, since the time change $t \mapsto T(t)$ is the right-continuous inverse of $t \mapsto \langle X\rangle_t$. The fact that we can express $d\langle X\rangle_z = \xi^2(X_z)dz$ would imply

$$\tau_{BLJ} = \inf\left\{H_\ell > s \geq 0 : \int_0^s \frac{\mathtt{D}(X_{T(t)})}{\xi^2(X_{T(t)})}dt > E\right\} = \inf\left\{H_\ell > s \geq 0 : \int_0^{T(s)} \frac{\mathtt{D}(X_z)}{\xi^2(X_z)}d\langle X\rangle_z > E\right\}$$
$$= \inf\left\{H_\ell > \langle X\rangle_q \geq 0 : \int_0^{T(\langle X\rangle_q)} \mathtt{D}(X_z)dz > E\right\} = \inf\left\{H_\ell > \langle X\rangle_q \geq 0 : \int_0^q \mathtt{D}(X_z)dz > E\right\}.$$

This holds upon realizing that, under the change of variable $\langle X\rangle_q = s < H_\ell$, $T(s) = T(\langle X\rangle_q) = q$. Although the integrated intensity is expressed in the original time of X, the stopping time τ_{BLJ} is defined in the quadratic variation clock of W; it is indeed the solution of the SEP of ν into W. Consider now $T(\tau_{BLJ}) = \inf\{s \geq 0 : \langle X\rangle_s > \tau_{BLJ}\}$, we know that this represents a solution of the SEP of ν into X, it is therefore expressed in the "original clock".

Since $\xi(\cdot) > 0$, $t \mapsto T(t)$ is continuous and strictly increasing for $t < \langle X\rangle_\infty$. Moreover, we know that τ_{BLJ} is by construction less than $H_\ell = \langle X\rangle_\infty$. If A is the continuity set of $T(\cdot)$, we can use $T(\inf(A)) = \inf(T(A)) = \inf\{T(a) : a \in A\}$[26], which delivers the final result

$$\tau^\star = T(\tau_{BLJ}) = T\left(\inf\left\{H_\ell > \langle X\rangle_q \geq 0 : \int_0^q \mathtt{D}(X_z)dz > E\right\}\right) \tag{5.55}$$
$$= \inf\left\{T(H_\ell) > T(\langle X\rangle_q) \geq 0 : \int_0^q \mathtt{D}(X_z)dz > E\right\} = \inf\left\{q \geq 0 : \int_0^q \mathtt{D}(X_z)dz > E\right\}.$$

□

Lemma 5.3. For $k \geq k_{\min}$, let $v_k(\cdot)$ be PT utility function with loss aversion parameter k, let G be a quantile function with mean $X_0 = 0$, and let $g(k) = k^{-\frac{1}{\alpha_- - \alpha_+}}$, it holds that

$$\mathscr{E}\left[v_k(g(k)G)\right] = \mathscr{E}\left[v_1(G)\right] g(k) k^{\frac{1-\alpha_+}{\alpha_- - \alpha_+}}. \tag{5.56}$$

Proof of Lemma 5.3. Let $h(k) = g(k)k^{\frac{1-\alpha_+}{\alpha_- - \alpha_+}}$ and consider $\mathscr{E}\left[v_1(G)\right] h(k)$. Let X have distribution with quantile function G, then the RHS of equation (5.56) is

$$\int_{1-\mathbb{P}(X>0)}^1 h(k)v_1(G(u))w'_+(1-u)du + \int_0^{\mathbb{P}(X<0)} h(k)v_1(G(u))w'_-(u)du \tag{5.57}$$

$$h(k)v_1(x) = \begin{cases} h(k)(x)^{\alpha_+} = k^{-\frac{\alpha_+}{\alpha_- - \alpha_+}}(x)^{\alpha_+} & \text{if} \quad x \geq 0 \\ -h(k)(-x)^{\alpha_-} = -k^{-\frac{\alpha_+}{\alpha_- - \alpha_+}}(-x)^{\alpha_-} & \text{if} \quad x < 0 \end{cases} \tag{5.58}$$

Let us focus on the LHS now. Let us define the random variable $X' = g(k)X$ and let G/F and G'/F' be the quantile/distribution function of X and X', respectively. Since $g(\cdot) > 0$, it holds

$$F'(x) = \mathbb{P}(X' \leq x) = \mathbb{P}(X \leq x/g(k)) = F(x/g(k)) \tag{5.59}$$
$$G'(u) = \inf\{x : F'(x) \geq u\} = \inf\{x : F(x/g(k)) \geq u\} = \inf\{z : F(z) \geq u\}g(k) = G(u)g(k)$$
$$\mathscr{E}\left[v_k(g(k)G)\right] = \int_{1-\mathbb{P}(X'>0)}^1 v_k(G'(u))w'_+(1-u)du + \int_0^{\mathbb{P}(X'<0)} v_k(G'(u))w'_-(u)du. \tag{5.60}$$

Since $F'(x) = F(x/g(k))$, the mass placed by X' on the positive and negative real line coincide with the mass placed by X there. Consider now

$$v_k(g(k)x) = \begin{cases} (g(k)x)^{\alpha_+} = k^{-\frac{\alpha_+}{\alpha_- - \alpha_+}}(x)^{\alpha_+} & \text{if} \quad x \geq 0 \\ -k(-g(k)x)^{\alpha_-} = -k^{1-\frac{\alpha_-}{\alpha_- - \alpha_+}}(-x)^{\alpha_-} = -k^{-\frac{\alpha_+}{\alpha_- - \alpha_+}}(-x)^{\alpha_-} & \text{if} \quad x < 0. \end{cases} \tag{5.61}$$

[26]Let $h(\cdot)$ be increasing and continuous and let $A \subseteq \mathbb{R}$ be non empty. Let $\underline{a} = \inf(A)$. Then, monotonicity of $h(\cdot)$ implies that, for any $a \in A$, $h(\underline{a}) \leq h(a)$, which implies that $h(\inf(A)) = h(\underline{a}) \leq \inf\{h(a) : a \in A\} = \inf(h(A))$. To prove the inverse inequality, let $a_n \downarrow \underline{a}$ (such a sequence exists by definition of inf), and consider $\inf(h(A)) = \inf\{h(a) : a \in A\} \leq h(a_n)$. If we apply limit on both sides, then by continuity of $h(\cdot)$, we have that $\inf(h(A)) \leq h(\inf(A))$.

Equations (5.58) and (5.61) show that $h(k)v_1(x) = v_k(g(k)x)$. If we use these results, we derive the non-linear expectation described in equation (5.57) from the one presented in equation (5.60), and vice versa. This is enough to show the claim and conclude that

$$\mathscr{E}\left[v_k(g(k)G)\right] = \mathscr{E}\left[v_1(G)\right] h(k). \tag{5.62}$$

This shows that Lemma 5.3 hinges on positive homogeneity of the branches defining $\nu_k(\cdot)$. □

Proof of Proposition 3.2. From the result in Lemma 5.3

$$\sup_{G\in\mathscr{G}} \mathscr{E}\left[v_k(g(k)G)\right] = h(k) \sup_{G\in\mathscr{G}} \mathscr{E}\left[v_1(G)\right] \quad \text{with} \quad \mathscr{G} = \left\{G : \int_0^1 G(u)du = 1\right\}. \tag{5.63}$$

Let $\overline{G}(\cdot)$ be the optimizer for the problem with $k = 1$, Lemma 5.3 shows that

$$\sup_{G\in\mathscr{G}} \mathscr{E}\left[v_1(G)\right] = \mathscr{E}\left[v_1(\overline{G})\right] \frac{h(k)}{h(k)} = \mathscr{E}[v_k(g(k)\bar{G})]\frac{1}{h(k)}. \tag{5.64}$$

If G corresponds to a distribution with mean 0, then $g(k)G$ will have mean 0 as well, implying that $g(k)G \in \mathscr{G}$. By combining the results in equation (5.63) and (5.64), we can conclude that

$$\sup_{H\in\mathscr{G}} \mathscr{E}[v_k(H)] = \sup_{G\in\mathscr{G}} \mathscr{E}\left[v_k(g(k)G)\right] = \mathscr{E}[v_k(g(k)\bar{G})] \tag{5.65}$$

This implies that the optimizer for $k \geq k_{\min}$ can be obtained by solving the optimization problem for $k = 1$ and scaling it by $g(k)$. From [10], we know that $-r < \inf(\text{supp}(\nu_1)) = \bar{c_1} < 0$, the fact that we impose $k \geq k_{\min}$ will imply that $G_k = g(k)\bar{G} > -r$. Therefore $k \geq k_{\min}$ can be seen as a condition that guarantees the support of the optimal distribution for a given admissible k to be bounded below by $-r$. □

Proof of Proposition 3.4. As discussed in Section 3.1, the distribution ν that solves the optimisation problem in [10] is such that its quantile function belongs to the parametric family of left continuous functions defined as in equation (3.8). Let $G(\cdot;\phi,\psi,\lambda) = G(\cdot)$ be an arbitrary quantile function in this parametric collection, if $G(\cdot)$ is square integrable, then the claim is verified. In other words, let $U \sim \text{Unif}(0,1)$ and define $Y = G(U)$ (by construction, the quantile function characterizing the distribution of Y is $G(\cdot)$), we want to show that $E[Y^2] < \infty$. We can exploit the shape of the quantile function to write

$$\begin{aligned} E[Y^2] &= \int_0^1 G^2(u)du = \int_0^1 \left(c_1 \mathbb{1}(u)_{(0,1-\phi]} + c_2 \mathbb{1}(u)_{(1-\phi,1-\psi]} + G(u)\mathbb{1}_{(1-\psi,1]}\right)^2 du \\ &= (1-\phi)c_1^2 + (\phi-\psi)c_2^2 + \int_{1-\psi}^1 G^2(u)du = \zeta + \int_0^{\psi} \left((v')^{-1}\left(\frac{\lambda}{w'(u)}\right)\right)^2 du. \end{aligned} \tag{5.66}$$

The last integral presents a singularity near 0, indeed $w'(0+) = \infty$, $v'(0+) = \infty$, and $v'(\infty) = 0$. We want to find conditions under which we have integrability at 0. Given the modeling choices (price process and distortion function), we can use some asymptotic approximations for $w_+(\cdot)$ and $(v')^{-1}$ near the singularity by considering

$$w_+(u) = \frac{u^{\delta_+}}{\left(u^{\delta_+} + (1-u)^{\delta_+}\right)^{\frac{1}{\delta_+}}} \approx u_+^{\delta} \qquad \text{and} \qquad v(u) = \left((u + r^{\beta})^{\frac{1}{\beta}} - r\right)^{\alpha_+} \approx u^{\frac{\alpha_+}{\beta}}. \tag{5.67}$$

So, near 0, we have the following approximation

$$(v')^{-1}\left(\frac{\lambda}{w'(u)}\right) \approx \left(\frac{\beta\lambda}{\alpha_+\delta_+ u^{\delta_+-1}}\right)^{\frac{\beta}{\alpha_+-\beta}} = K u^{\frac{(1-\delta_+)\beta}{\alpha_+-\beta}} = K u^{\xi}. \tag{5.68}$$

This means that in a right-neighborhood of 0, we can asymptotically approximate the integrand appearing in the second term of $E[Y^2]$ as $Cu^{2\xi}$. Therefore, in order to get square integrability of the distribution associated with $G(\cdot)$, the exponent $2\xi > -1$. This is equivalent to require

$\beta(2\delta_+ - 1) > \alpha_+$ since we assumed that $\alpha_+ < \beta$.

So, given the stock dynamic parametrization defining β, and the agent's preferences parameterization $(\alpha_\pm, k, \delta_\pm)$, any distribution that corresponds to the parametric class of quantile functions used in the optimization problem is square integrable, therefore the optimal distribution is square integrable as well. □

Proof of Proposition 3.6. Proving $\hat{\nu} \ll m$ consists of checking the assumptions of Proposition 2.8. $\mathcal{K} \subseteq (k_{\min}, \infty)$ is open by assumption, so we only need to check if $c_i(\cdot)$ (i.e., the function that assigns to every $k \in \mathcal{K}$ the jump location in optimality) is continuously differentiable and such that $|\nabla c_i(\cdot)| > 0$, for $i \in \{1, 2\}$. Remember that

$$c_i(k) = g(k)\overline{c}_i \qquad \text{with} \qquad g(k) = k^{-\frac{1}{\alpha_- - \alpha_+}} \tag{5.69}$$

with $\overline{c}_i \neq 0$ since in [10] it is proved that there is no probability mass in 0. Therefore, those conditions completely depend on the properties of $g(\cdot)$. In particular, computation shows

$$g'(k) = -\frac{1}{\alpha_- - \alpha_+} k^{\frac{\alpha_- - \alpha_+ - 1}{\alpha_- - \alpha_+}} \quad \text{is continuous and} \quad |g'(k)| > 0 \ \text{ since } \ k > 0. \tag{5.70}$$

The conditions stated in Proposition 2.8 are satisfied, which implies that $\hat{\nu} \ll m$. If p_i^k represents the jump size on losses/gains, then we also have the following representation for $D_m\hat{\nu}_s(\cdot)$

$$\int_{\{k:c_1(k)=x\}} \frac{p_1^k \varphi(k)}{|\nabla c_1(k)|} \mathscr{H}_0(dk) + \int_{\{k:c_2(k)=x\}} \frac{p_2^k \varphi(k)}{|\nabla c_2(k)|} \mathscr{H}_0(dk). \tag{5.71}$$

Working with a one-dimensional parametric space simplifies computation since the Hausdorff measure collapses to a counting measure. This allows us to rewrite equation (5.71) as

$$D_m\hat{\nu}_s(x) = \sum_{i\in 1,2} \left(\sum_{k_j \in \{k:c_i(k)=x\}} \frac{p_i^{k_j} \varphi(k_j)}{|\nabla c_i(k_j)|} \right) = \sum_{i\in\{1,2\}} \frac{\overline{p}_i \varphi\left(c_i^{-1}(x)\right)}{\left|\nabla c_i\left(c_i^{-1}(x)\right)\right|} \mathbb{1}_{\{x\in \mathrm{Img}(c_i)\}}. \tag{5.72}$$

This equality is justified upon noticing that invertibility of $g(\cdot)$ (with $g^{-1}(\cdot) = (\cdot/\overline{c}_i)^{-\alpha_- + \alpha_+}$) provides invertibility of $c_i(\cdot)$. This implies that $\{k : c_i(k) = x\} = \{c_i^{-1}(x)\}$. Moreover, one can notice that the jump sizes are not affected by the loss-version coefficient k. Indeed, by using the scale property expressed in terms of distribution function in equation (3.13), one shows that $p_1^k = F_k(c_1(k)) - F_k(c_1(k)^-) = 1 - \phi = \overline{p}_1$ and $p_2^k = F_k(c_2(k)) - F_k(c_2(k)^-) = \phi - \psi = \overline{p}_2$.[27].

We can use again the scaling property for the distribution function to derive the mapping $k \mapsto f_k(\cdot)$, where $f_k(\cdot)$ represents the density of ν_c^k with respect to the Lebesgue measure (i.e., $D_m\nu_c^k(\cdot)$). In this framework, ν_c^k is such that $\nu_c^k((-\infty, x]) = F_k(x)\mathbb{1}_{(x>c_2(k))} = \overline{F}(x/g(k))\mathbb{1}_{(x>c_2(k))}$ (as usual, $\overline{F}$ is the distribution function in optimality for $k = 1$). One has

$$D_m\nu_c^k(x) = f^k(x) = \frac{d}{dx}\overline{F}\left(\frac{x}{g(k)}\right)\Big|_{x>c_2(k)} = -\frac{d}{dx}\left((w'_+)^{-1}\left(\frac{\lambda}{v'(x/g(k))}\right)\right)\Big|_{x>c_2(k)}. \tag{5.73}$$

To show that $f_{\hat{\nu}}(\cdot) > 0$ on $(c_1(\underline{k}), c_1(\overline{k})) \cup (c_2(\overline{k}), \infty)$ we can consider separately what happens on gains $(x > 0)$ and losses $(x \leq 0)$. Let $\overline{p}_1 > 0$ (i.e., jump size on losses strictly positive for the $k = 1$)[28]. Since $c_1(\cdot)$ identifies the optimal jump location on losses, then $\mathrm{Img}(c_1) \subseteq (-\infty, 0)$. Similarly, one argues that $\mathrm{Img}(c_2) \subseteq (0, \infty)$. On the other hand, for an arbitrary $k \in \mathcal{K}$, $f_k(x) > 0$ for any $x \geq c_2(k) > 0$, otherwise it is 0. Given these premises, the decomposition of the density of

[27] Coarea formula in this case can be replaced by a change of variable formula in the one dimensional integral $\int_{\mathcal{K}} p_i^k \delta_{c_i(k)}(A)\varphi(k)dk$ by setting $c_i(k) = y$ and exploiting invertibility of $c_i(\cdot)$. The two approaches lead to the same representation of $D_m\hat{\nu}(\cdot)$ in such a context.

[28] Notice that, in optimality, this has to be the case. On one hand, we previously discussed that ν_k, the distribution related to the solution(s) of the optimal stopping problem for the single agent presented in [10], is centered in $X_0 = 0$. On the other hand, $\bar{p}_1$ represents the unique way ν_k assigns mass on (ℓ, X_0). Having $\overline{p}_1 = 0$ would contradict one of those statements.

$\hat{\nu}$ in equation (3.16), suggests that for any $x \leq 0$ we have only one contribution coming from the ratio involving $c_1^{-1}(\cdot)$

$$f(x) = D_m\hat{\nu}(x) = \frac{\overline{p}_1\varphi\left(c_1^{-1}(x)\right)}{\left|\nabla c_1\left(c_1^{-1}(x)\right)\right|}\mathbb{1}_{\{x\in \text{Img}(c_1)\}}. \tag{5.74}$$

We previously said that $\varphi(k) = 0$ for any $k \notin \mathcal{K}$. This implies that

$$\varphi(c_1^{-1}(x)) > 0 \quad \text{for any} \quad \underline{k} < c_1^{-1}(x) < \overline{k}, \quad \text{or equivalently for any} \quad c_1(\underline{k}) < x < c_1(\overline{k}). \tag{5.75}$$

The last claim holds since $c_1(\cdot)$ is an increasing function on $\mathcal{K}$. On the other hand, computations show that the denominator in equation (5.74) is finite for any $x \in \text{Img}(c_1)$ (we already showed that it is strictly positive). Let $x \in \text{Img}(c_1)$, then the density on losses is strictly positive as long as the mapping $\text{Img}(c_1) \ni x \mapsto \varphi(c_1^{-1}(x))$ is strictly positive, which happens on the interval $(c_1(\underline{k}), c_1(\overline{k})) = \text{Img}(c_1)$. In other words, $\text{int}(\text{supp}(\hat{\nu})) \cap (-\infty, 0] = (c_1(\underline{k}), c_1(\overline{k}))$.

We can use the same remarks discussed above to argue that the density of $\hat{\nu}$ has two contributions on gains, one coming from the jump component of the distribution of ν_k for $k \in \mathcal{K}$, the other coming from its continuous component[29]. In particular, for $x \geq 0$

$$f(x) = \frac{\overline{p}_2\varphi(c_2^{-1}(x))}{|\nabla c_2(c_2^{-1}(x))|}\mathbb{1}_{\{x\in \text{Img}(c_2)\}} + \int_{\mathcal{K}} f_k(x)\varphi(k)dk. \tag{5.76}$$

By using the same arguments discussed above for the loss component, one can show that the first term in equation (5.76) is strictly positive for $x \in (c_2(\overline{k}), c_2(\underline{k})) = \text{Img}(c_2)$ ($\bar{c}_2 > 0$ makes $c_2(\cdot)$ strictly decreasing). For any $k \in \mathcal{K}$, we previously represented $f_k(x) = \eta(x)\mathbb{1}_{\{x\geq c_2(k)\}}$ where $\eta(\cdot) > 0$ on $\{x \geq 0 : x \geq c_2(k)\}$, or equivalently on $\{x \geq 0 : c_2^{-1}(x) \leq k\}$ due $c_2(\cdot)$ being invertible and strictly decreasing. This would allow us to write

$$\int_{\mathcal{K}} f_k(x)\varphi(k)dk = \int_{c_2^{-1}(x)}^{\overline{k}} \eta_k(x)\varphi(k)dk > 0 \qquad \text{as long as} \quad c_2^{-1}(x) < \overline{k}. \tag{5.77}$$

This allows us to conclude that the second component in equation (5.76) is strictly positive on $(c_2(\overline{k}), \infty)$. By combining these results on gains, one has that $\text{int}(\text{supp}(\hat{\nu})) \cap [0, \infty) = (c_2(\overline{k}), \infty)$. This concludes the argument. □

Proof of Corollary 3.7. The mean of the distribution $\hat{\nu}$ is $X_0 = 0$, while mass on losses is $1 - \phi$

$$\begin{aligned} \int y\hat{\nu}(dy) &= \int_0^1 G_k(u)du = g(k)\int_0^1 \bar{G}(u)du = 0 \\ \hat{\nu}((-\infty, 0)) &= \int_{\mathcal{K}} \nu_k((-\infty, 0))\varphi(k)dk = (1-\phi)\int_{\mathcal{K}} \varphi(k)dk = 1 - \phi. \end{aligned} \tag{5.78}$$

We know from Proposition 3.4 that, under the assumption $\alpha_+ < \beta(2\delta_+ - 1)$ the distribution ν_k is square integrable. In particular,

$$\int_{I_{\nu_k}} y^2\nu_k(dy) = \int_0^1 (G_k(u))^2\, du = (g(k))^2\int_0^1 (\overline{G}(u))^2 du < \infty \tag{5.79}$$

The last equation holds due to the scaling property in Proposition 3.2.

$$\begin{aligned} \int y^2\hat{\nu}(dy) &= \int y^2 \int_{\mathcal{K}} \nu_k(dy)\varphi(k)dk = \int_{\mathcal{K}} \left(\int y^2\nu_k(dy)\right)\varphi(k)dk \\ &= \int_{\mathcal{K}} (g(k))^2 \left(\int_0^1 (\bar{G}(u))^2 du\right)\varphi(k)dk = C\int_{\mathcal{K}} g(k)^2\varphi(k)dk \leq Cg(k_{\min})^2 \end{aligned}$$

The bound above holds since $g(\cdot)$ is positive, decreasing and $\mathcal{K} \subseteq (k_{\min}, \infty)$. □

[29] This would allow for $\overline{p}_2$ (jump size on gains for any $k \in \mathcal{K}$) to be 0 without contradicting the necessary condition $\int y\nu_k(dy) = X_0$. In other words, single-agent optimal distribution may be purely continuous on gains.

Lemma 5.4. Let $\hat{\nu}$ be the measure obtained by averaging through $\varphi(\cdot)$, let $\hat{\nu}$ have density $f(\cdot)$. Let $\underline{\nu} = \inf(I_{\hat{\nu}})$, and τ be such that $X_\tau \sim \hat{\nu}$, and $E[\langle X \rangle_\tau] < \infty$. If $X_0 = 0$, on losses, we have

$$E[L_\tau^x(X)] = 2\int_{\underline{\nu}}^{x} (x-y)f(y)dy \qquad \text{for} \qquad x \in (\underline{\nu}, 0). \tag{5.80}$$

Proof of Lemma 5.4. Under the assumption $E[\langle X \rangle_\tau] < \infty$, the expected local time can be represented as difference between potentials (see Proposition 2.4)

$$\begin{aligned} E[L_\tau^x(X)] &= \int_{I_{\hat{\nu}}} |y-x|\hat{\nu}(dy) - |X_0 - x| = \int_{I_{\hat{\nu}}} |y-x| f(y)dy - |x| \\ &= 2\int_{\underline{\nu}}^{x} (x-y)f(y)dy + \int_{I_{\underline{\nu}}} (y-x)f(y)dy - |x| \\ &= 2\int_{\underline{\nu}}^{x} (x-y)f(y)dy + E[X_\tau] - x - |x| = 2\int_{\underline{\nu}}^{x} (x-y)f(y)dy. \end{aligned} \tag{5.81}$$

The last equation holds since $x \in (\underline{\nu}, 0)$ (loss region expressed in terms of the transformed price process X) and $E[X_\tau] = X_0 = 0$ as discussed in Corollary 3.7. □

Proof of Proposition 3.10. Let $\varphi(\cdot)$ an arbitrary density on $\mathcal{K} \subseteq (k_{\min}, \infty)$ such that the aggregate distribution $\hat{\nu}$ admits a density $f(\cdot)$. Under the assumption $\beta = 1$, the process X writes as a simple parallel shift of P (i.e., $X = P - r$, with $r = P_0$ because of the assumption $X_0 = 0$), this allows us to write $\xi(\cdot) = \sigma(\cdot + r)$. Let $\lambda > 0$, and let $\underline{\nu} \geq -r$ be the lowerbound of the support of the target measure $\hat{\nu}$. Lemma 5.4 allows us to equivalently formulate $\mathtt{D}(x) = \lambda$ for any $x \in (\underline{\nu}, 0)$ (i.e., selling propensity constantly equal to λ on losses) as

$$\begin{aligned} \lambda &= \frac{f(x)\sigma^2(x+r)^2}{2\int_{\underline{\nu}}^{x}(x-y)f(y)dy} \qquad \text{for any} \qquad x \in (\underline{\nu}, 0) \\ f(x) &= \frac{2\lambda}{\sigma^2}\int_{\underline{\nu}}^{x} \frac{x-y}{(x+r)^2} f(y)dy \qquad \text{for any} \qquad x \in (\underline{\nu}, 0). \end{aligned} \tag{5.82}$$

So, given the market parametrization, for any $\lambda > 0$, we want to find $f(\cdot)$ solving the integral equation described above. In doing so, we need to remember that $f(\cdot)$ is the density of the distribution that we get by averaging over the market participants solving the optimization problem in [10]. This means that, for any k, we know how ν_k will look; therefore, we can modify the density of $\hat{\nu}$ only through $\varphi(\cdot)$, density on $\mathcal{K}$.

Moreover, we need to find $f(\cdot)$ that preserves the mass on losses. Indeed, as discussed in Corollary 3.7, the probability mass that the average distribution $\hat{\nu}$ places on losses is $1 - \phi$.

Remark 5.5. At this stage, it seems that we have complete control over the lower bound of the support of the target distribution $\underline{\nu}$. Equivalently, it seems that, for any $\lambda > 0$, we are able to find $f(\cdot)$ solving the integral equation in (5.82) and integrating to $1-\phi$ on losses, for any arbitrary level of $\underline{\nu} \in [-r, 0)$. Actually, we will see that if we want to be free to choose any $\lambda > 0$ and preserve mass on losses, we are forced to choose a specific value for $\underline{\nu}$.

Remark 5.6. The outline of the proof is the following: we derive the ODE with specific boundary conditions by assuming that $f(\cdot)$ solves the integral equation above. The ODE will be used to derive an guess for the solution of the integral equation. We then do a verification step to check that our guess indeed solves equation (5.82).

Assume that, for an arbitrary $\lambda > 0$, $\mathtt{D}(x) = \lambda > 0$ for any $x \in (\underline{\nu}, 0)$. Equivalently, we are assuming that $f(\cdot)$ solves equation (5.82). By rearranging terms, one has that, for $x \in (\underline{\nu}, 0)$

$$\frac{\sigma^2}{2\lambda} f(x)(x+r)^2 = \int_{\underline{\nu}}^{x} (x-y)f(y)dy. \tag{5.83}$$

Once we differentiate both sides, we get that

$$\frac{\sigma^2}{2\lambda}\big(f'(x)(x+r)^2+2(x+r)f(x)\big)-\int_{\underline{\nu}}^{x}f(y)dy=0. \tag{5.84}$$

Upon realizing that $F(\cdot)=\int_{\underline{\nu}}^{\cdot}f(y)dy$, if the equation above holds, then $F(\cdot)$ solves the ODE below

$$\begin{cases}F''(x)(x+r)^2+2F'(x)(x+r)-(2\lambda)/\sigma^2F(x)=0 \quad \text{for} \quad x\in(\underline{\nu},0)\\ F(\underline{\nu})=0 \quad \text{and} \quad F(0)=1-\phi.\end{cases} \tag{5.85}$$

The boundary conditions formalize the mass preservation on losses. Cauchy-Euler solution suggests $F(x)=A(x+r)^{\vartheta_1}+B(x+r)^{\vartheta_2}$ with ϑ_1 and ϑ_2 are the roots of $\vartheta^2+\vartheta-2\lambda/\sigma^2=0$, and $A,B\in\mathbb{R}$ such that

$$\begin{cases}F(\underline{\nu})=A(\underline{\nu}+r)^{\vartheta_1}+B(\underline{\nu}+r)^{\vartheta_2}=0\\ F(0)=A(r)^{\vartheta_1}+B(r)^{\vartheta_2}=1-\phi\end{cases} \qquad \vartheta_{1,2}=\frac{1}{2}\left[-1\pm\sqrt{1+\frac{8\lambda}{\sigma^2}}\right]. \tag{5.86}$$

Let us assume that $\underline{\nu}=-r$, then the first boundary condition is "freely" satisfied. For $B=0$, we need A such that.

$$F(x)=A(x+r)^{\vartheta_1}=(1-\phi)\left(\frac{x+r}{r}\right)^{\vartheta} \qquad \text{where} \qquad \vartheta=\vartheta_1=\frac{1}{2}\left[-1+\sqrt{1+\frac{8\lambda}{\sigma^2}}\right]. \tag{5.87}$$

By differentiating $F(\cdot)$ defined as in equation (5.87), we get a candidate solution for the integral equation in (5.82) we started with. We then proceed with verification

$$\begin{aligned}f(x)&=\frac{2\lambda}{\sigma^2}\int_{\underline{\nu}}^{x}\frac{x-y}{(x+r)^2}f(y)dy \qquad (5.88)\\ \frac{(1-\phi)\vartheta}{r^{\vartheta}}(x+r)^{\vartheta-1}&=\frac{2\lambda}{\sigma^2}\int_{\underline{\nu}}^{x}\frac{x-y}{(x+r)^2}\frac{(1-\phi)\vartheta}{r^{\vartheta}}(y+r)^{\vartheta-1}dy\\ \vartheta(\vartheta+1)&=2\lambda/\sigma^2\end{aligned}$$

This is true since ϑ is the positive root for $\vartheta^2+\vartheta-2\lambda/\sigma^2=0$. So this proves that $f(\cdot)$ obtained differentiating $F(\cdot)$ in equation (5.87) solves the integral equation in (5.82). $f(\cdot)$ corresponds to the density of $\hat{\nu}$ obtained by averaging the distributions solving the optimization problem in [10] with respect to an arbitrary density $\varphi(\cdot)$ defined over $\mathcal{K}\subseteq(0,\infty)$. The aim now is to look for the proper specification of $\varphi(\cdot)$ and $\mathcal{K}\subseteq(k_{\min},\infty)$.

$$F(x)=\hat{\nu}((-\infty,x])=\int_{\mathcal{K}}\nu_k((-\infty,x])\varphi(k)dk=\int_{\mathcal{K}}F_k(x)\varphi(k)dk. \tag{5.89}$$

By combining these results, one has that, for any $x\in(\underline{\nu},0)=(-r,0)$

$$(1-\phi)\left(\frac{x+r}{r}\right)^{\vartheta}=\int_{\mathcal{K}}(1-\phi)\mathbb{1}_{(c_1(k)\leq x)}\varphi(k)dk. \tag{5.90}$$

On the LHS, we have the distribution function on losses that solves the ODE in (5.85) while on the RHS, we used the fact that, for the single agent problem, the only mass on losses is $1-\phi$, which corresponds to the size of the jump in that region. While the jump size is *k-independent*, the jump location in optimality can be derived by using the scaling property in k, since we work under the assumption $k>k_{\min}$. Indeed, if $\bar{c}_1<0$ is the jump location on losses for the case $k=1$, then $c_1(k)=\bar{c}_1g(k)$ is the analogous quantity for $k\in\mathcal{K}$, where $g(\cdot)$ is the usual scaling function. The function $c_1(\cdot)$ is invertible and, under the assumption of $\alpha_->\alpha_+$, it is increasing. Therefore, we have that

$$\left(\frac{x+r}{r}\right)^{\vartheta}=\int^{c_1^{-1}(x)}\varphi(k)dk=\int^{g^{-1}(x/\bar{c}_1)}\varphi(k)dk. \tag{5.91}$$

By differentiating both terms with respect to x we get

$$\frac{\vartheta}{r}\left(\frac{x+r}{r}\right)^{\vartheta-1} = \varphi\left(c_1^{-1}(x)\right)\frac{d}{dx}\left(g^{-1}\left(\frac{x}{\bar{c}_1}\right)\right) = \frac{\alpha_- - \alpha_+}{|\bar{c}_1|}\varphi\left(c_1^{-1}(x)\right)\left(\frac{x}{\bar{c}_1}\right)^{-\alpha_- + \alpha_+ - 1}. \quad (5.92)$$

We can now change variable by setting $x = c_1(k)$, then, by rearranging terms, one has that

$$\begin{aligned}\varphi(k) &= \frac{\vartheta}{r}\left(\frac{c_1(k)+r}{r}\right)^{\vartheta-1}\frac{|\bar{c}_1|}{\alpha_- - \alpha_+}(g(k))^{\alpha_- - \alpha_+ + 1} \\ &= \frac{\vartheta}{r^{\vartheta}}\left(c_1(k)+r\right)^{\vartheta-1}\frac{|\bar{c}_1|}{\alpha_- - \alpha_+}\frac{g(k)}{k}.\end{aligned} \quad (5.93)$$

This function is well defined for values of k such that $c_1(k) = \bar{c}_1 g(k) > -r$, which is equivalent to saying that $k > (r/|\bar{c}_1|)^{\alpha_+ - \alpha_-} = k_{\min}$. This is satisfied since it coincides with the condition we imposed to use the scaling property in Proposition 3.2. So, if we use $\varphi(\cdot)$ defined as in equation (5.93) on $\mathcal{K} = (k_{\min}, \infty) \subset (0, \infty)$ in the averaging procedure to define $\hat{\nu}$, then the (interior of the) support of $\hat{\nu}$ will be $(-r, \infty)$ (it will cover the whole state space of the process X we use in the embedding) and the related market selling propensity will be $\mathtt{D}(x) = \lambda > 0$ for any $x \in (-r, 0)$. □

Lemma 5.7. For $\lambda > 0$ arbitrary and $x \in (\underline{\nu}, 0)$, let $F(\cdot)$ be a solution to the ODE in (5.85) on $(\underline{\nu}, 0)$. Then $f(\cdot) = F'(\cdot)$ does not solve the integral equation in (5.82) if $\underline{\nu} \neq -r$.

Proof of Lemma 5.7. As described in the proof of Proposition 3.10, Cauchy-Euler solution suggests $F(x) = A(x+r)^{\vartheta_1} + B(x+r)^{\vartheta_2}$ with ϑ_1 and ϑ_2 as below and $A, B \in \mathbb{R}$ such that

$$\begin{cases} F(\underline{\nu}) = A(\underline{\nu}+r)^{\vartheta_1} + B(\underline{\nu}+r)^{\vartheta_2} = 0 \\ F(0) = A(r)^{\vartheta_1} + B(r)^{\vartheta_2} = 1 - \phi \end{cases} \qquad \vartheta_{1,2} = \frac{1}{2}\left[-1 \pm \sqrt{1 + \frac{8\lambda}{\sigma^2}}\right] \quad (5.94)$$

$$A = -B(\underline{\nu}+r)^{\vartheta_2 - \vartheta_1} \qquad \text{and} \qquad B = \frac{1-\phi}{r^{\vartheta_2}\left[1 - \left(\frac{r}{\underline{\nu}+r}\right)^{\vartheta_1 - \vartheta_2}\right]}. \quad (5.95)$$

We can differentiate $F(\cdot)$ and check if $f(\cdot) = F'(\cdot)$ solves the integral equation in (5.82), which is equivalent to saying that $\mathtt{D}(x) = \lambda$ for $x \in (\underline{\nu}, 0)$. Let us plug $f(\cdot)$ into the integral equation (5.82). Once we integrate by parts the RHS, since $F(\underline{\nu}) = 0$, we get $\mathtt{D}(x) = \lambda$ for $x \in (\underline{\nu}, 0)$ if and only if

$$\begin{aligned}A(x+r)^{\vartheta_1 - 1}\left[\vartheta_1 - \frac{2\lambda}{\sigma^2(\vartheta_1+1)}\right] + B(x+r)^{\vartheta_2 - 1}\left[\vartheta_2 - \frac{2\lambda}{\sigma^2(\vartheta_2+1)}\right] = \\ - \frac{2\lambda}{(x+r)^2\sigma^2}\left[\frac{A}{\vartheta_1+1}(\underline{\nu}+r)^{\vartheta_1+1} + \frac{B}{\vartheta_1+1}(\underline{\nu}+r)^{\vartheta_1+1}\right].\end{aligned} \quad (5.96)$$

The LHS cancels out since ϑ_1 and ϑ_2 are roots of the polynomial $\vartheta^2 + \vartheta - 2\lambda/\sigma^2$. So, the solution to the ODE $F(\cdot)$ is such that $f(\cdot) = F'(\cdot)$ solves the integral equation in (5.82) if and only if

$$\begin{aligned}\frac{A}{\vartheta_1+1}(\underline{\nu}+r)^{\vartheta_1+1} + \frac{B}{\vartheta_2+1}(\underline{\nu}+r)^{\vartheta_2+1} = 0 \\ \frac{-B(\underline{\nu}+r)^{\vartheta_2-\vartheta_1}}{\vartheta_1+1}(\underline{\nu}+r)^{\vartheta_1+1} + \frac{B}{\vartheta_2+1}(\underline{\nu}+r)^{\vartheta_2+1} = 0 \\ -B(\underline{\nu}+r)^{\vartheta_2-1}\left[\frac{1}{1+\vartheta_1} + \frac{1}{1+\vartheta_2}\right] = 0.\end{aligned} \quad (5.97)$$

Since $B \neq 0$, and $\underline{\nu} > -r$ this equation holds if and only if we have a zero multiplicative factor. However, the difference between the ratios involving ϑ_1 and ϑ_2 is positive as long as $\lambda > 0$. Therefore, if $\underline{\nu} > -r$, the solution to the ODE in (5.85) can not used to derive a solution to the integral equation (5.82). Hence, we can not have $\mathtt{D}(x) = \lambda$ for any $x \in (\underline{\nu}, 0)$. □

Proof of Proposition 3.12. Let us assume $\underline{\nu} = -r$. Proposition 3.10 shows that, under this assumption, we can construct $\hat{\nu}$ by using mixing density defined in equation 3.17, such that $\mathtt{D}(x) = 0$ for any $x \in (\underline{\nu}, 0)$, and preserving $1 - \phi$ mass on losses.

Let $\lambda > 0$, let $\hat{\nu}$ be the measure obtained through the usual mixing procedure ensuring constant selling propensity, and probability mass preservation on losses. Let us argue by contradiction assuming that $\underline{\nu} \in (-r, 0)$[30], and let $x \mapsto F(x) = \hat{\nu}((-\infty, x])$. Then saying that the related selling propensity is such that $\mathtt{D}(x) = \lambda$ for any $x \in (-r, 0)$ is equivalent to saying that $f(\cdot) = F'(\cdot)$ solves the integral equation in (5.82). Since one can derive the ODE in (5.85) from the integral equation (5.82), we can conclude that $F(x) = A(x+r)^{\vartheta_1} + B(x+r)^{\vartheta_2}$ for any $x \in (-r, 0)$ solves the ODE in $(-r, 0)$ with A and B satisfying the usual boundary conditions. But this contradicts Lemma 5.7, therefore we can conclude that $\underline{\nu} = -r$. □

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