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Valuing the option to invest in an incomplete market

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Abstract This paper considers the impact of entrepreneurial risk aversion and incompleteness on investment timing and the value of the option to invest. A risk averse entrepreneur faces the irreversible decision of when to pay a cost in order to receive a one-off investment payoff. The uncertainty associated with the investment payoff can be partly offset by hedging, but the remaining unhedgeable risk is idiosyncratic. Nested within our incomplete set-up is the complete model of McDonald and Siegel (Q J Econ 101:707–727, 1986) which assumes investment payoffs are perfectly spanned by traded assets. We find risk aversion and idiosyncratic risk erode option value and lower the investment threshold. Our main finding is that there is a parameter region within which the complete and incomplete models give differing investment signals. In this region, the option is never exercised (and investment never occurs) in the complete model, whereas the entrepreneur exercises the option in the incomplete setting. Strikingly, this parameter region corresponds to a negative implicit dividend yield on the payoff, and so this exercise behavior contrasts with conventional wisdom of Merton (Bell J Econ Manage 4:141–183, 1973) for complete markets. Finally, in this parameter region, increased volatility speeds-up investment and option values are not strictly convex in project value, in sharp contrast to the conclusion of standard real options models.

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The real options theory of corporate investment, dating back to Myers [16], recognizes that investment opportunities are options on real assets. Under the real options approach, the investment timing decision is made to maximize option value. This decision depends crucially upon the volatility of the project value, and a higher volatility leads to a manager waiting longer to invest, since the option to wait is more valuable. In deriving such conclusions, the literature assumes either the real asset is traded, or other assets perfectly span the risk of the real asset. These assumptions result in a complete market model. The canonical models of Brennan and Schwartz [3] and McDonald and Siegel [11] (see also Dixit and Pindyck [6]) amongst others, fit this description.

In reality, the assets underlying real options are not traded in capital markets, and other assets may (at best) partially span risk. In this paper we investigate the impact of risk aversion and incompleteness on investment timing and option value. As we shall show, this will significantly alter the conclusions of the traditional complete real options models.

We consider a risk averse entrepreneur with exponential utility who can choose at any time to undertake an irreversible investment project for a cost. We assume that the reward from undertaking the project is a one-off payoff at the time of investment. This one-off investment payoff is random and is not a traded asset, so there is unhedgeable or idiosyncratic risk associated with waiting.

The entrepreneur does not make the decision to invest in isolation. He can also trade in a risk-less bond and a risky asset which is correlated with the investment payoff. This provides him with a hedging opportunity since he can offset some of the risk associated with his unknown investment payoff. Despite hedging the market risk, the entrepreneur still faces some remaining idiosyncratic risk. It is his aversion to such risk which will alter his investment behavior in an incomplete market. The entrepreneur selects an investment time, and a hedge position to maximize his expected utility of wealth, where his wealth consists of the option payoff in addition to the value of his hedge portfolio. The formulation reflects the entrepreneur's behavior over an infinite horizon, as his portfolio choice after he exercises the option is considered. It is a natural generalization of the benchmark complete market real options framework, and leads to a certainty equivalent valuation of the real option and investment threshold in closed form.

Comparative statics show the higher the entrepreneur's risk aversion, or the lower the correlation between the project value and hedging asset, the lower will be the investment threshold and option value. A lower correlation means more idiosyncratic risk remains as the project value fluctuates. This causes the risk averse entrepreneur to exercise at a lower threshold to reduce uncertainty and lock-in a value for the investment payoff. Similarly, if he is more risk averse, he prefers to act earlier since waiting involves facing idiosyncratic risk. The rationale is that the entrepreneur resolves uncertainty when he exercises the option. The explicit nature of our solution enables us to compare easily to the benchmark complete model of McDonald and Siegel [11], and a model where the idiosyncratic risk is not priced. Both are limiting cases of our model for a risk averse entrepreneur, either as correlation between the investment payoff and risky asset approaches one, or as risk aversion tends to zero.

The key contribution of our paper is to show the presence of risk aversion and idiosyncratic risk gives rise to an additional parameter region within which both benchmark models recommend never exercising the option (to take advantage of potential rises in the investment payoff) but our incomplete model gives a finite threshold at which the option is exercised. The intuition is that when the entrepreneur is risk averse, there is an additional incentive to exercise to avoid exposure to idiosyncratic risk, and in this new region, this effect dominates over the benefit from waiting. In fact, in this region, the benefit from waiting arises from a negative implicit dividend yield on the investment payoff. Our results can be compared with the conventional finding of Merton [14] that American call options will only be exercised early when there are positive dividends. This is true in the complete market, but in the incomplete setting the additional incentive to exercise to avoid risk means it is possible to exercise early even with negative dividends. Since we find this additional parameter region does not disappear in the limit as either correlation approaches one or risk aversion approaches zero, we conclude that approximating with the benchmark real options models when the entrepreneur is risk averse could lead to an incorrect decision. Further, we show that in the common parameter regime where exercise or investment occurs under both the benchmark and incomplete models, exercising sub-optimally according to either of the benchmark models can lead to an economically significant loss in value.

Uncertainty increases the value of waiting and delays investment in complete market real options models. In contrast, we find in the incomplete setting, the investment–uncertainty relationship differs in the two parameter regimes. In the additional regime (where implicit dividends are negative) we find that idiosyncratic risk can cause the investment threshold and option value to fall with volatility. In this region, increased volatility speeds-up rather than delays investment. This occurs because the value-decreasing impact of idiosyncratic risk outweighs the value-increasing effect arising from a convex payoff. Additionally, in this parameter region, the option value is not strictly convex in the project value.

We now briefly review the related literature. Miao and Wang [15] consider the impact of incomplete markets on investment timing in a model with consumption and portfolio allocations. In contrast to our setting, their entrepreneur maximizes exponential utility from consumption and investment payoffs follow arithmetic Brownian motion.[1] Whilst they are able to study the effect of incomplete markets on consumption, our set-up remains much closer to the canonical complete model of McDonald and Siegel [11] and Dixit and Pindyck [6] by maximizing utility of wealth and using geometric Brownian motion.

In the case where investment results in a one-off payoff, Miao and Wang [15] show via asymptotic expansions (complementing their numerical results) that incompleteness results in earlier investment. However, we show that not only does idiosyncratic risk and risk aversion speed-up investment, but that there is an additional range of parameters under which the entrepreneur exercises the option. In the complete setting for the same parameter values, the option would never be exercised. We also find that in this parameter region, the entrepreneur exercises the option despite implicit dividends on the investment payoff being negative. In addition, our set-up is tractable and leads to closed-form expressions which enable comparative statics to be performed. Miao and Wang [15] also show investment may be delayed due to incomplete markets when investment payoffs are delivered over time in flows rather than in a lump-sum. We also mention the model of Hugonnier and Morellec [9] where the focus is on agency issues and the effect of control challenges on manager behavior in an incomplete setting. Their manager chooses the investment time but does not directly benefit from exercise of the option, although he may be replaced if his exercise strategy deviates from

[1] Miao and Wang [15] remark on how geometric Brownian motion would alter their results at least in the simpler situation where there is no correlated asset with which to hedge.

the optimal shareholder policy. Finally, we mention modeling analogies between investment options in incomplete markets and the exercising of executive stock options when managers cannot trade the stock.[2]

1 The investment problem and modeling assumptions

Consider an infinitely-lived, risk averse entrepreneur[3] with exponential utility who can choose to undertake an irreversible investment project with payoff V_τ at time τ for cost K.[4] The project value is not spanned by traded assets so markets are incomplete. Our entrepreneur can also invest in risk-free bonds and the traded risky asset P correlated with the project value V. The values V and P follow geometric Brownian motion processes

$$\frac{\mathrm{d}V_t}{V_t} = \nu \mathrm{d}t + \eta \mathrm{d}W_t \tag{1}$$

and

$$\frac{\mathrm{d}P_t}{P_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t \tag{2}$$

where expected returns and volatilities ν, μ, σ, η are constants. Denote by $\xi = \frac{\nu}{\eta}$ and $\lambda = \frac{\mu}{\sigma}$ the instantaneous Sharpe ratios of V and P, respectively.

The driving Brownian motions B and W are correlated with $\rho \in [-1, 1]$ and we can write $\mathrm{d}W = \rho \mathrm{d}B + \sqrt{1-\rho^2}\mathrm{d}Z$ for some Brownian motion Z independent of B. The role of trading in the risky asset P is that it enables the entrepreneur to hedge the market risk, represented by Brownian motion B. The remaining risk generated by Z is unhedgeable and represents idiosyncratic or private risk. When correlation is one, the asset P spans all risks, so the model consisting of the project value V together with the risky asset P, is complete. The entrepreneur faces idiosyncratic risk and incomplete markets provided $|\rho| < 1$.

If the entrepreneur exercises the option to invest at time τ, he pays the cost K and receives the one-off investment payoff V_τ, generated by the project. The random value V could also be interpreted as the value received upon selling-on a finished project. To illustrate the applicability of our framework, we give two examples. Consider first a biotechnology firm specializing in research and development devoted to drug discovery. These technologies may be of value to larger pharmaceutical companies. The biotech is a small company run by an entrepreneur. The biotech could sell-on its drug technologies to larger companies, however, it incurs research and development expenditures first. If the biotech sells its technologies, the pharmaceutical pays the biotech the amount V, and subsequently develops and commercializes the technology. The value received by the biotech for drug technologies is not spanned

[2] Ingersoll [10] considers a risk averse executive but concentrates on the marginal value of an option. In contrast, we do not want to restrict ourselves to small quantities. The method relies on numerical approximations in the case of American options. Earlier, Detemple and Sundaresan [5] examine a binomial model and focus on the impact of a short sales restriction on valuation.

[3] We consider a single owner-manager or entrepreneur to abstract from agency issues between the manager and shareholders. However the model is easily adaptable to consider such agency issues. Shareholders are well-diversified so their preferred investment timing is modeled via the benchmark model where idiosyncratic risk is not priced (see Sect. 2). The manager is risk averse and his timing choice is reflected in the model of this section. The analysis in the paper concerning the differences under the benchmark and utility-based model can be interpreted to give conclusions on agency costs of incompleteness.

[4] We immediately express all amounts in discounted units or equivalently take the risk-free bond as numeraire. Hence the rates of return μ and ν are excess growth rates. Note this means we have a investment cost/strike of $K\mathrm{e}^{rt}$ rather than the more common formulation of a constant strike.

by traded assets and so the entrepreneur faces incomplete markets.[5] A second example would be a real estate developer who pays to construct buildings on land he owns, and then decides when to sell-on the finished site. He receives a lump-sum payoff when he sells the property. The developer cannot perfectly hedge risk from this fluctuating property value, and so faces an incomplete market. Amongst many others, Titman [20], Quigg [18] and Grenadier [7] treat applications of real options to real estate.

Let X^θ denote the entrepreneur's (discounted) wealth from his position in the risky asset P and risk-free bond. This position arises for both investment and hedging purposes. Wealth X^θ has dynamics

$$\mathrm{d}X^\theta = \theta_t \frac{\mathrm{d}P}{P} \tag{3}$$

where θ_t denotes the holdings in the asset P. Prior to exercise of the option at τ, the entrepreneur's holdings in the risky asset P are both for investment purposes and to provide a partial hedge against the uncertain investment payoff. After τ, the entrepreneur solves a Merton [13] style portfolio choice problem to invest his total wealth.

We can now formulate the objective of the entrepreneur. His problem is to find

$$I_* = \sup_{(\tau,\theta)\in\mathcal{A}} \mathbb{E}\left[\mathrm{e}^{-\zeta\tau} U\left(X_\tau^\theta + (V_\tau - K)^+\right) | X_t^\theta = x, V_t = v\right] \tag{4}$$

where $U(x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}$ is exponential utility, the discount factor ζ is taken to be $\zeta = -\frac{1}{2}\lambda^2$, and $\mathcal{A}$ is a suitable class of admissible[6] pairs of stopping times and strategies.

We now explain where this objective comes from. It is a non-standard situation since we need to evaluate wealth at an intermediate time τ prior to the (infinite) horizon. That is, our problem is to evaluate utility from wealth in a consistent fashion across different dates. If the horizon were finite, we would simply map back to τ from the utility function $U(x)$ at T via the value function. Such an argument would give us an objective (4) where there is a particular discount factor which depends on market parameters. This choice of discount factor exactly accounts for the fact that after τ, the entrepreneur invests wealth optimally in the traded asset. We describe this in more detail in the Appendix. However, since we have an infinite horizon, we need another criteria, and argue that we want to formulate the problem such that there are no biases arising from the portfolio choice problem influencing the manager's choice of exercise/investment time. A different choice of ζ would create artificial incentives to exercise early, or may even lead to a degenerate situation where the investment option should never be exercised.[7] We also elaborate on this interpretation in the Appendix. Further mathematical development of such horizon-unbiased utilities is given in Henderson and Hobson [8].

In Sect. 3 we will solve the risk averse entrepreneur's problem as given in (4). However first we will give a discussion of some benchmark models.

5 See Berk et al. [2] and Nicholson et al. [17] for studies of real options applied to biotechs and pharmaceuticals.

6 We show in the Appendix that a suitable class is $\mathcal{A} = \bigcup_{T,J,L} \mathcal{A}_{T,J,L}$ where $\mathcal{A}_{T,J,L} = \{\tau : \tau \le T \wedge H_J^V, |\theta_t| \le L\}$ and $H_J^V = \inf\{u : V_u \ge J\}$.

7 Note however that the specification $\zeta = -\frac{1}{2}\lambda^2$ is a modeling choice, and is not essential to solve the model in closed-form. We could solve the model in (4) for a general discount factor ζ, although the resulting investment times would be biased towards early or late exercise.

2 Benchmark models

This section briefly reviews the benchmark models against which we will later compare the risk averse entrepreneur's investment timing. We give two benchmark models - the well known complete markets real options model of McDonald and Siegel [11] (see also Dixit and Pindyck [6]) and the model where the entrepreneur does not price idiosyncratic risk. The latter model where idiosyncratic risk is not compensated is appropriate for a well-diversified manager. Of course, typically managers are not well diversified and thus our aim in this paper is to treat the case where the entrepreneur requires compensation for idiosyncratic risk. We will return to this case in the next section.

If idiosyncratic risks are not priced, the option value, denoted $p^{(\rho)}$ solves[8]

$$0 = \frac{1}{2}\eta^2 v^2 \frac{\partial^2 p^{(\rho)}}{\partial v^2} - \eta(\lambda\rho - \xi) v \frac{\partial p^{(\rho)}}{\partial v} \tag{7}$$

subject to boundary, value-matching and smooth pasting conditions:

$$p^{(\rho)}(0) = 0; \quad p^{(\rho)}(\tilde{V}^{(\rho)}) = \tilde{V}^{(\rho)} - K; \quad \left.\frac{\partial p^{(\rho)}}{\partial v}\right|_{\tilde{V}^{(\rho)}} = I_{\{\tilde{V}^{(\rho)} > K\}}$$

This gives the usual first passage time criteria where the manager invests the first time the (discounted) investment payoff V_t is greater than or equal to a constant threshold level $\tilde{V}^{(\rho)}$. We solve for this threshold and associated value of the option to invest in the standard way to give the following result.[9]

Proposition 2.1 *Denote by* $\beta^{(\rho)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$ *the non-zero root of the quadratic*

$$\phi(\phi - 1)\eta^2/2 - \eta\phi(\lambda\rho - \xi) = 0.$$

There are two possibilities depending on the parameter $\beta^{(\rho)}$*:*
(i) *If* $\beta^{(\rho)} > 1$*, investment/exercise takes place at the first passage time* $\tau = \inf\{V_t \geq \tilde{V}^{(\rho)}\}$ *where*

$$\tilde{V}^{(\rho)} = \frac{\beta^{(\rho)}}{\beta^{(\rho)} - 1} K \tag{8}$$

[8] The manager solves the following for the optimal investment time τ:

$$p^{(\rho)}(v) = \sup_{\tau} \mathbb{E}\left[D^0_\tau (V_\tau - K)^+ | V_0 = v\right] \tag{5}$$

where D^0_t denotes the state price density which assigns zero market price of risk to the independent Brownian motion Z. Define the family of state price densities D^ϑ_t by $D^\vartheta_t = e^{-\lambda B_t - \frac{1}{2}\lambda^2 t} e^{-\vartheta Z_t - \frac{1}{2}\vartheta^2 t}$ where λ represents the market price of risk on the traded Brownian motion B and ϑ represents the market price of risk on the non-traded Brownian motion Z. Taking $\vartheta = 0$ gives the state price density D^0_t under which Z-risk is not compensated. We have $B^0_t = B_t + \lambda t$ and $Z^0_t = Z_t$ are independent Brownian motions, giving P and V follow

$$\frac{dP}{P} = \sigma dB^0, \quad \frac{dV}{V} = \eta\left[\rho dB^0 + \sqrt{1-\rho^2} dZ - (\lambda\rho - \xi) dt\right]. \tag{6}$$

[9] We remark here that in both of the standard real options models described in this Section, it is equivalent to consider an investment paying a stream of cash flows over time or an investment paying the present value of those cash flows at the time of investment (a one-off or lump-sum case). Since $\mathbb{E}\int_0^\infty D^0_s V_s \, ds = V_0/\eta(\lambda\rho - \xi)$, it is equivalent to consider a payoff based on cash flows of $(V_s I_{(s>\tau)} - K)^+$ and a one-off payoff of $(R(V_\tau) - K)^+$ where $R(v) = v/\eta(\lambda\rho - \xi)$. This is no longer true in the incomplete setting and thus we consider the case of a one-off payoff.

The value of the investment option is

$$p^{(\rho)}(v) = \begin{cases} (\tilde{V}^{(\rho)} - K)(\frac{v}{\tilde{V}^{(\rho)}})^{\beta^{(\rho)}}; & v < \tilde{V}^{(\rho)} \\ v - K; & v \geq \tilde{V}^{(\rho)} \end{cases} \tag{9}$$

(ii) *If $\beta^{(\rho)} \leq 1$, the option value is maximized by $\tilde{V}^{(\rho)} = \infty$, and investment never occurs and is postponed indefinitely. If $\beta^{(\rho)} < 1$, the option value is infinite. If $\beta^{(\rho)} = 1$, the option value is v.*

Typically in real options theory, it is assumed that either the project value V is itself a traded asset, or it is spanned by traded assets, resulting in a complete market model. An example where these assumptions might be reasonable is if the investment cashflow arose from the sale of a commodity on which futures contracts are liquidly traded. The standard real options model of McDonald and Siegel [11] (see also Dixit and Pindyck [6]) can be recovered from the above framework which accounts for non-priced idiosyncratic risk. Under the assumption that $\rho = 1$, the risky asset P is a spanning asset for V. The corresponding investment threshold and option value are obtained from Proposition 2.1 with the substitution $\rho = 1$. This gives

Corollary 2.2 *In the complete market where $\rho = 1$, we have $\beta^{(1)} = 1 - \frac{2(\xi - \lambda)}{\eta}$ and again there are two possibilities depending on the parameter $\beta^{(\rho)}$:*
(i) *If $\beta^{(1)} > 1$, the investment threshold is given by*

$$\tilde{V}^{(1)} = \frac{\beta^{(1)}}{\beta^{(1)} - 1} K \tag{10}$$

and the value of the investment option is[10]

$$p^{(1)}(v) = \begin{cases} (\tilde{V}^{(1)} - K)(\frac{v}{\tilde{V}^{(1)}})^{\beta^{(1)}}; & v < \tilde{V}^{(1)} \\ v - K; & v \geq \tilde{V}^{(1)} \end{cases}$$

(ii) *If $\beta^{(1)} \leq 1$, the option value is maximized by $\tilde{V}^{(1)} = \infty$, and investment never occurs and is postponed indefinitely. If $\beta^{(1)} < 1$, the option value is infinite. If $\beta^{(1)} = 1$, the option value is v.*

As is well known, since $\tilde{V}^{(\rho)} > K$, taking the option to invest into account gives a rule which leads to waiting beyond the standard net present value criteria, specifically waiting for investment payoff V_t to reach the higher level, $\tilde{V}^{(\rho)}$. Model parameters determine how large the threshold is relative to K. In some instances, parameters are such that the threshold is in fact infinite, and waiting has infinite option value. In this case, investment never takes place. Notice that in Proposition 2.1 (and Corollary 2.2), there were two scenarios depending on the value of the parameter $\beta^{(\rho)}$ (or $\beta^{(1)}$). If this parameter exceeded the critical value of one, investment should take place at the given (finite) threshold. However, if this parameter was one or lower, investment does not occur. In this latter case, the investment threshold is infinite, and so investment never takes place and the option is retained.

The condition for investment to occur is often stated either in terms of model parameters or via an implicit dividend yield. In terms of the underlying model parameters, the condition

[10] This is a standard perpetual American option problem and was solved by McKean [12] in an appendix to Samuelson [19], see also Merton [14].

becomes $\xi < \lambda\rho$ for investment to occur at a finite threshold. In the case of a complete market, the condition for investment simplifies to $\xi < \lambda$.[11]

In both models, the intuition is that the option should remain unexercised when it offers better opportunities (than the risky asset P) to obtain a greater payoff in the future. This occurs when the Sharpe ratio of V is high relative to that of the risky asset P. In the non-priced idiosyncratic risk model the correlation also appears in the comparison. In the extreme case of zero correlation, all risks are idiosyncratic and provided $\xi > 0$, it is always beneficial to wait, despite the idiosyncratic risk, since this risk is not priced by the entrepreneur. However, McDonald and Siegel [11] note that $\xi < 0$ may arise for firms in competitive industries exhibiting temporary rents, such as high technology industries. In this situation, there will be a finite investment threshold.

We define ξ^*_{np} to be the critical value of the Sharpe ratio in the non-priced idiosyncratic risk model which distinguishes between the two possibilities of investing at a finite threshold and not investing. From the previous condition, we see $\xi^*_{np} = \lambda\rho$. We define similarly ξ^*_c to be the critical Sharpe ratio in the complete model, and see that $\xi^*_c = \lambda$.

The second usual way to state the condition for investment to take place is via the interpretation of $\delta^{(\rho)} = \lambda\rho - \xi$ (or $\delta^{(1)} = \lambda - \xi$ in the complete market) as an implicit proportional dividend yield (see Dixit and Pindyck [6]). With this interpretation, the condition for investment to occur becomes $\delta^{(\rho)} > 0$. From (6) and (7) we see the expected return on the investment payoff V is decreasing when $\delta^{(\rho)} > 0$ and thus there is a reason to exercise the option since on average, waiting results in a lower amount upon exercise. The dividend yield $\delta^{(\rho)}$ represents the opportunity cost of waiting. However, if $\delta^{(\rho)} \le 0$, there is no reason to exercise the option early and it is optimal to keep waiting. This is exactly the conclusion that waiting is optimal for an American call option with no dividends made by Merton [14]. We will see later that this intuition is no longer sufficient in the incomplete setting.

3 The risk averse entrepreneur's threshold and option value

The remainder of the paper concentrates on the risk averse entrepreneur's timing and portfolio choice problem described in Sect. 1 and given in (4). To solve the problem, we need a number of steps which are given in detail in the proof of the following proposition.[12] Define

$$G(x, v) = \sup_{(\tau,\theta)\in\mathcal{A}} \mathbb{E}\left[-\frac{1}{\gamma}e^{\frac{1}{2}\lambda^2(\tau-t)}e^{-\gamma(X^\theta_\tau+(V_\tau-K)^+)}|X^\theta_t = x, V_t = v\right]$$

Finding $G(x, v)$ at $t = 0$ is equivalent to solving (4) (with $\zeta = -\frac{1}{2}\lambda^2$). By time-homogeneity, we deduce the manager invests at the first passage time of V to a constant threshold $\tilde{V}^{(\rho,\gamma)}$,

$$\tau = \inf\{t : V_t \ge \tilde{V}^{(\rho,\gamma)}\} \tag{11}$$

[11] Often in the real options literature (see [6]) the condition for investment to occur is expressed in terms of expected returns rather than Sharpe ratios, because it is assumed that the volatilities of V and P are equal. Making this assumption gives the condition $\nu < \mu$ for investment to occur in the complete model. Once we do not have perfect spanning, there is no reason to assume this a priori and we retain distinct volatilities, $\eta \neq \sigma$ throughout. A further clarification with the standard models concerns the equilibrium approach of McDonald and Siegel [11]. The CAPM specifies the required return on the asset P via $\lambda = \lambda^M\rho^{PM}$ where λ^M is the Sharpe ratio of the market portfolio, and ρ^{PM} is the correlation between the returns on the asset P and the market portfolio. This results in the condition for investment occurring being $\xi < \lambda^M\rho^{PM}$. Although we do not insist on the CAPM choice for λ, at any point in the paper, the reader may specialize to this choice.

[12] The proofs of propositions in this section are given in the Appendix.

The following proposition characterizes G.[13]

Proposition 3.1 *In the continuation region, G solves the following non-linear HJB equation*

$$0 = \frac{1}{2}\lambda^2 G + \xi\eta v G_v + \frac{1}{2}\eta^2 v^2 G_{vv} - \frac{1}{2}\frac{(\lambda G_x + \rho\eta v G_{xv})^2}{G_{xx}} \tag{12}$$

with boundary, value matching and smooth pasting conditions

$$\begin{aligned} G(x,0) &= -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}, \\ G(x,\tilde{V}^{(\rho,\gamma)}) &= -\frac{1}{\gamma}\mathrm{e}^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}, \\ G_v(x,\tilde{V}^{(\rho,\gamma)}) &= I_{\{\tilde{V}^{(\rho,\gamma)}>K\}}\mathrm{e}^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}. \end{aligned}$$

We can now solve the above HJB equation and characterize the threshold $\tilde{V}^{(\rho,\gamma)}$ as the solution of a non-linear equation. We summarize the results in a Proposition.

Proposition 3.2 *Recall* $\beta^{(\rho)} = 1 - \frac{2(\xi-\lambda\rho)}{\eta}$. *If* $\beta^{(\rho)} > 0$ (*correspondingly* $\xi < \lambda\rho + \frac{\eta}{2}$), *the entrepreneur will invest at time* τ *given in* (11). *The optimal investment threshold,* $\tilde{V}^{(\rho,\gamma)}$, *is the unique solution to*

$$\tilde{V}^{(\rho,\gamma)} - K = \frac{1}{\gamma(1-\rho^2)}\ln\left[1 + \frac{\gamma(1-\rho^2)\tilde{V}^{(\rho,\gamma)}}{\beta^{(\rho)}}\right] \tag{13}$$

where the solution is such that $\tilde{V}^{(\rho,\gamma)} > K$.
If $\beta^{(\rho)} \le 0$ (*or equivalently* $\xi \ge \lambda\rho + \frac{\eta}{2}$) *then smooth pasting fails and there is no solution. In this case, the entrepreneur postpones exercise indefinitely, and* $\tilde{V}^{(\rho,\gamma)} = \infty$.

The value function $G(x,v)$ (*solving* (12) *and associated conditions*) *is given by*

$$G(x,v) = \begin{cases} -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\left[1 - (1 - \mathrm{e}^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)(1-\rho^2)})\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta^{(\rho)}}\right]^{\frac{1}{1-\rho^2}} & v \in [0,\tilde{V}^{(\rho,\gamma)}) \\ -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\mathrm{e}^{-\gamma(v-K)} & v \in [\tilde{V}^{(\rho,\gamma)},\infty) \end{cases}$$

Given we have the value function, the value of the option to the entrepreneur can be derived via a standard certainty equivalence (or utility indifference) argument. We evaluate the certainty equivalent value by finding the amount of incremental wealth which can be invested optimally, which gives the same utility as having the option. That is, the certainty equivalent value $p^{(\rho,\gamma)}(v)$ solves $G(x,v) = G(x + p^{(\rho,\gamma)}(v), 0)$.

Proposition 3.3 *If* $\beta^{(\rho)} > 0$ (*or* $\xi < \lambda\rho + \frac{\eta}{2}$), *the entrepreneur's certainty equivalent valuation of the option is given by*

$$p^{(\rho,\gamma)}(v) = -\frac{1}{\gamma(1-\rho^2)}\ln\left(1 - (1 - \mathrm{e}^{-\gamma(1-\rho^2)(\tilde{V}^{(\rho,\gamma)}-K)})\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta^{(\rho)}}\right) \tag{14}$$

where $\tilde{V}^{(\rho,\gamma)}$ *and* $\beta^{(\rho)}$ *are given in Proposition* 3.2.

[13] The solution to (12) is a candidate solution of the problem. In the Appendix we sketch a verification argument to show that the candidate solution is indeed the value function I_*.

The option value $p^{(\rho,\gamma)}(v)$ also has the representation

$$p^{(\rho,\gamma)}(v) = \sup_{\tau \text{ bounded}} -\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}[D_\tau^0 e^{-\gamma(1-\rho^2)(V_\tau - K)^+} | V_0 = v] \tag{15}$$

where D_τ^0 is the state price density under which idiosyncratic risks are not priced, used in (5).

If $\beta^{(\rho)} \le 0$ (or $\xi \ge \lambda\rho + \frac{\eta}{2}$), waiting for the project value to rise is preferable, hence the entrepreneur postpones indefinitely and the value of the option is infinite.

These propositions give us both the constant threshold $\tilde{V}^{(\rho,\gamma)}$ and the value of the option to invest in closed-form. The risk averse entrepreneur exercises the option when the investment payoff V reaches the threshold given in (13), and the certainty equivalent value of the option to the entrepreneur is given in (14). Similarly to the benchmark models, the threshold is always greater than K and so the option to wait is always valuable. In contrast to the complete market model (of Corollary 2.2), both the threshold and option value depend on the entrepreneur's risk aversion level as well as the correlation between the investment payoff V and the risky asset P. The correlation reflects the degree to which the entrepreneur can hedge the uncertainty associated with receiving the random payoff V_τ at the exercise time. Higher correlation corresponds to a larger portion of the risk being market (or hedgeable) risk versus idiosyncratic (or unhedgeable) risk. When $|\rho| = 1$, the asset P provides a perfect hedge and the market is complete. In fact, we show that the complete models of Dixit and Pindyck [6] and McDonald and Siegel [11] and non-priced idiosyncratic risk model are nested within the incomplete utility-based model as special cases when $|\rho| \to 1$ and $\gamma \to 0$ respectively.

Proposition 3.4 *Two special cases of the incomplete utility-based model are*:
(A) *Complete model*: *As $\rho \to 1$,*

(i) $\beta^{(\rho)} \to \beta^{(1)}$; (ii) $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(1)}$; (iii) $p^{(\rho,\gamma)}(v) \to p^{(1)}(v)$
where $\beta^{(1)}$, $\tilde{V}^{(1)}$, $p^{(1)}(v)$ were given in Corollary 2.2.

(B) *Non-priced idiosyncratic risk model*: *As $\gamma \to 0$,*

(i) $\beta^{(\rho)} = \beta^{(\rho)}$; (ii) $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(\rho)}$; (iii) $p^{(\rho,\gamma)}(v) \to p^{(\rho)}(v)$
where $\beta^{(\rho)}$, $\tilde{V}^{(\rho)}$, $p^{(\rho)}(v)$ were given in Proposition 2.1.

The above proposition holds for all values of $\beta^{(\rho,\gamma)}$. Note for instance that when $0 < \beta^{(\rho,\gamma)} < 1$, the threshold $\tilde{V}^{(\rho,\gamma)}$ and option value $p^{(\rho,\gamma)}$ are both finite whilst the thresholds and option values in the benchmark models are infinite in this case (see Proposition 2.1 and Corollary 2.2). The proposition says that in the limit, the finite thresholds and option values tend to infinity.

We now investigate the impact of risk aversion and correlation on the entrepreneur's threshold and option value. Since the threshold and option value were given in closed form in Propositions 3.2 and 3.3, comparative statics can be obtained easily via differentiation.[14]

Proposition 3.5 *The investment threshold $\tilde{V}^{(\rho,\gamma)}$ and option value $p^{(\rho,\gamma)}$ are increasing in $|\rho|$ and decreasing in γ.*

The intuition behind this result is as follows. By paying the investment cost (or equivalently exercising the option), the entrepreneur is locking-in the payoff he receives. Waiting

[14] We regard $\delta^{(\rho)}$ as a fixed parameter. Dixit and Pindyck [6] "regard δ as a basic parameter independent of η" and McDonald and Siegel [11] note that not doing so leads to "ambiguity in the comparative statics results".

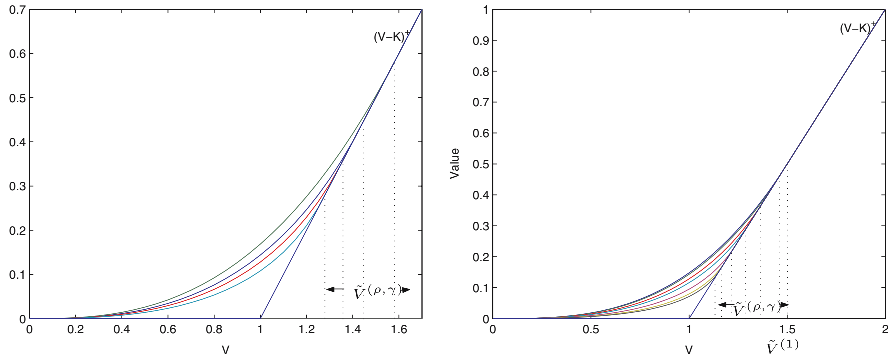

Fig. 1 The value of the option to invest, where investment costs are $K = 1$ and in the case where $\beta^{(\rho)} > 1$ (or $\xi < \xi^*_{np}$). The left panel gives option values and investment thresholds as γ varies, fixing $\rho = 0.9$. The right panel fixes $\gamma = 10$ and plots option values and thresholds for varying ρ. The *lines* in the left panel take $\gamma = 0, 5, 10, 20$ (from highest to lowest) with $\beta^{(\rho)} = 2.7$. The *lines* in the right panel take $\rho = 1$, $\rho = 0.99, 0.95, 0.9, 0.75, 0.5, 0.0$ (from highest to lowest) and $\beta^{(\rho)} = 3 = \beta^{(1)}$

involves facing random fluctuations in V which can only be partially hedged away by trading. Remaining risk is idiosyncratic and by waiting, the entrepreneur is exposed to this risk. If the manager is more risk averse, he dislikes uncertainty concerning V and so prefers to exercise earlier to lock-in value, and reduce this exposure to idiosyncratic risk. In the extreme situation where $\rho = 0$, the option value and threshold will be lowest. In this case, all risk is idiosyncratic. In comparison to the $\rho = 0$ situation, the ability to hedge increases option value and the investment threshold.

We notice from Proposition 3.5, that either higher risk aversion or a lower correlation will impact on the threshold and value in the same direction. In fact, risk aversion and correlation appear together as $\gamma(1-\rho^2)$ in (13) and (14). The presence of the correlated risky asset with which to hedge means that the entrepreneur has a scaled down "effective risk aversion" of $\gamma(1-\rho^2)$. The higher the correlation, the less idiosyncratic risk he is exposed to, and the lower his effective risk aversion.[15]

We now illustrate these observations in Figs. 1 and 2. The leftmost panels of Figs. 1 and 2 give the impact of the entrepreneur's risk aversion on the value of the option. Both show that the investment threshold and the option value are decreasing in risk aversion, γ. The rightmost panels of Figs. 1 and 2 show the effect of the correlation between the project value and the hedging asset. Both panels show the threshold and option value fall as correlation is lowered.

Figures 1 and 2 correspond to different parameter regimes. The panels of Fig. 1 take $\beta^{(\rho)} > 1$ whilst those of Fig. 2 take $0 < \beta^{(\rho)} < 1$. We will explain in the next section how the existence of these two distinct parameter regimes has important implications for investment, and later we will show the threshold and option value behave very differently in the two regimes.

Finally, we quantify the impact of following an incorrect model in a world where the risk averse entrepreneur faces incompleteness. Figure 3 plots option values given

[15] Another fruitful way to interpret this observation is as follows. Two entrepreneurs face the same investment opportunity—one (with risk aversion γ) has access to a hedge asset with correlation $\rho \neq 0$, and the other does not have any hedge asset available to him, but has lower risk aversion coefficient, $\gamma(1-\rho^2)$. Two such entrepreneurs will act identically with respect to investment timing and will place the same value on the option to invest.

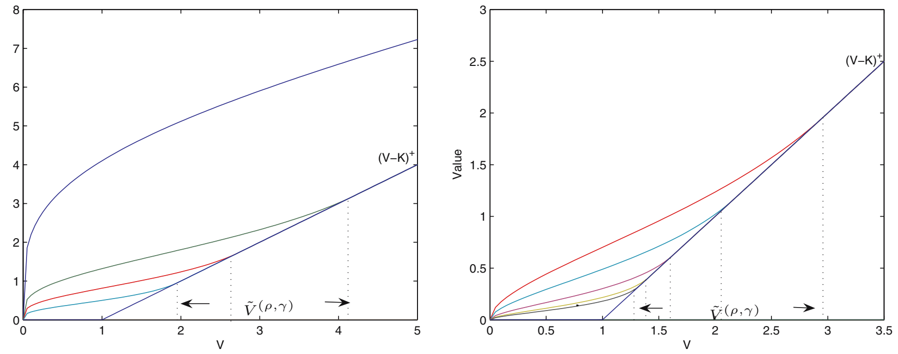

Fig. 2 The value of the option to invest, where investment costs are $K = 1$ and in the case where $0 < \beta^{(\rho)} < 1$ (or $\xi^*_{np} \leq \xi < \xi^*_i$). The left panel gives option values and investment thresholds as γ varies, fixing $\rho = 0.9$. The right panel fixes $\gamma = 10$ and plots option values and thresholds for varying ρ. The *lines* in the left panel correspond to values $\gamma = 1, 5, 10, 20$ (from highest to lowest), and $\beta^{(\rho)} = 0.2$. The *lines* in the right panel take $\rho = 0.95, 0.9, 0.75, 0.5, 0.0$ (from highest to lowest), with $\beta^{(\rho)} = 0.5 = \beta^{(1)}$

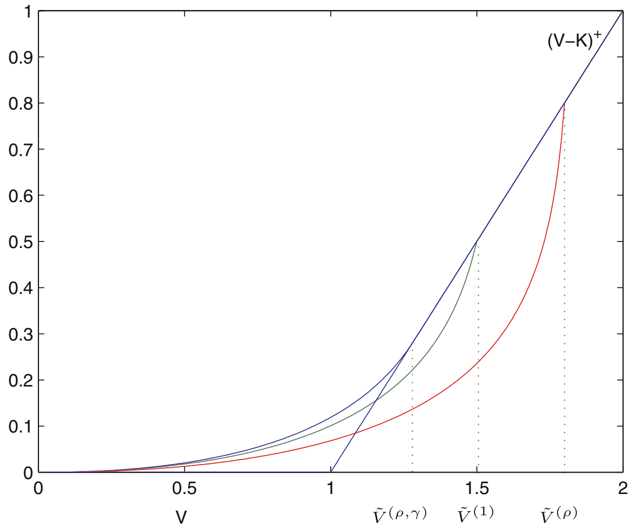

Fig. 3 Value of the option to invest under optimal versus suboptimal exercise with $\beta^{(\rho)} > 1$ and $\rho = 0.75$. The *top line* is the incomplete utility-based model value from Proposition 3.3 where $\tilde{V}^{(\rho,\gamma)} = 1.29$. The *middle line* represents suboptimal exercise where the entrepreneur exercises at $\tilde{V}^{(1)} = 1.5$, the complete model threshold. The *lowest line* is suboptimal exercise when $\tilde{V}^{(\rho)} = 1.8$, the threshold from the non-priced idiosyncratic risk model. Parameters are $K = 1, \gamma = 10$. For $\rho = 0.75$, $\beta^{(\rho)} = 2.25$. For the complete model, $\rho = 1$ and $\beta^{(1)} = 3$

investment decisions are made under the three possible models. We take $\rho = 0.75$. The highest line corresponds to the entrepreneur using the incomplete model and investing at threshold $\tilde{V}^{(\rho,\gamma)} = 1.29$. The middle and lower lines represent suboptimal exercise decisions. The middle line is the option value when the entrepreneur waits to invest at the complete market threshold of $\tilde{V}^{(1)} = 1.5$. The lowest line is the value when he waits to invest at the threshold arising from the non-priced idiosyncratic risk model, $\tilde{V}^{(\rho)} = 1.8$. Given our

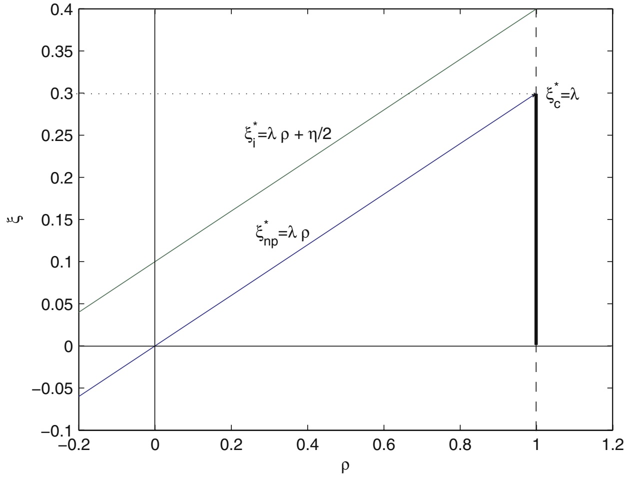

Fig. 4 Critical values of the project Sharpe ratio ξ, as a function of correlation, ρ. Parameter values are $\lambda = 0.3$, $\eta = 0.2$. The *higher line*, ξ_i^*, with slope λ, is the largest Sharpe ratio in the incomplete utility-based model for which there is a finite investment threshold. Below this is the line ξ_{np}^*, also with slope λ. The line ξ_{np}^* goes through the point $\xi_c^* = \lambda$ when $\rho = 1$

parameter choice, Fig. 3 shows that the loss in value resulting from an incorrect model is economically significant. For example, at the incomplete threshold of 1.29, the option value is 0.29. However if the entrepreneur waits until threshold 1.5, he loses about 20% of the option value. If he waits until threshold 1.8, about 50% of the option value is lost.

4 An additional parameter region where investment occurs

Recall that in the benchmark models presented in Sect. 2, we highlighted that for some parameter values, the option would never be exercised and investment would not occur. In this section we undertake an analogous investigation for the incomplete model, and find that the parameter region where exercise occurs is extended, and that in this additional region, the option is exercised even though the implicit dividend yield is negative. This has implications for investment timing since there are situations where the option is exercised in the incomplete model but never exercised in the corresponding complete model.

Proposition 3.2 tells us that the entrepreneur exercises at the constant threshold given in (13) provided $\beta^{(\rho)} > 0$, or equivalently, $\xi < \lambda\rho + \frac{\eta}{2}$. In the case where $\xi \geq \lambda\rho + \frac{\eta}{2}$, the entrepreneur postpones exercise and waits indefinitely. Define $\xi_i^* = \lambda\rho + \frac{\eta}{2}$ to be the critical value of the Sharpe ratio distinguishing these two situations. The key observation is that this critical value ξ_i^* has changed compared with those obtained for the benchmark models of Sect. 2. Recall for the complete model, we found $\xi_c^* = \lambda$. The non-priced idiosyncratic risk model gave $\xi_{np}^* = \lambda\rho$.

This is illustrated in Fig. 4. Each of ξ_i^*, ξ_c^* and ξ_{np}^* is represented on the graph as a function of ρ. ξ_c^* is just the single value λ when ρ is one. The higher line with slope of $\lambda = 0.3$ represents $\xi_i^* = \lambda\rho + \eta/2$, the critical Sharpe ratio for the incomplete model. The corresponding line $\xi_{np}^* = \lambda\rho$ for the non-priced idiosyncratic risk model is the lower one, also with slope of λ. The first observation from Fig. 4 is that there is a gap between the lines ξ_i^* and

ξ^*_{np} corresponding to a parameter region where the non-priced idiosyncratic risk model signals investment should be postponed indefinitely, whilst the incomplete utility-based model says investment should be undertaken at a finite threshold, described in Proposition 3.2. We call the parameter region $\lambda\rho < \xi < \lambda\rho + \eta/2$ the additional region, since it is an extra region in which the incomplete model recommends investment is undertaken. When parameters are such that $\xi < \lambda\rho$, both the incomplete and benchmark models give a finite threshold at which investment occurs. This is the common parameter region where the investment option is exercised at some finite threshold in each of the models.

Just as we noted earlier for the benchmark models, the criteria for investment to occur can also be stated in terms of an implicit proportional dividend yield. Recall we denoted $\delta^{(\rho)} = \lambda\rho - \xi$ and the aforementioned common parameter region corresponds to $\delta^{(\rho)} > 0$. Re-writing the condition for investment in the incomplete model of $\xi < \lambda\rho + \frac{\eta}{2}$ gives $\delta^{(\rho)} > -\eta/2$. This implies that even if dividends are *negative*, investment may still occur at some finite threshold. This appears to contradict conventional wisdom as the standard Merton [14] result says an American call will never be exercised early in the absence of positive dividends. The additional parameter region is described in dividend yield terms by $-\eta/2 < \delta^{(\rho)} < 0$, so implicit dividends are always negative in this region.

The explanation for the existence of this additional exercise region is as follows. In order for the entrepreneur to leave the option unexercised, he requires a higher Sharpe ratio than in the non-priced idiosyncratic risk benchmark model, since the benefit from waiting must outweigh the incentive to exercise to reduce exposure to idiosyncratic risk. The higher the idiosyncratic risk, the higher is η and the higher is the critical threshold the Sharpe ratio must reach for him to leave the option unexercised. For correlations $\rho > 1 - \frac{\eta}{2\lambda}$, the entrepreneur requires a higher Sharpe ratio (to leave the option unexercised) than in the complete benchmark model. Again, as η increases, the effect of idiosyncratic risk is larger, and he requires a higher Sharpe ratio for a wider range of correlation values. This has significant impact on the option to invest since the option is exercised when V is high (relative to investment costs). That is, the entrepreneur faces uncertainty regarding a large amount and therefore the fact he is risk averse has a significant impact on his decision.

Now we return to the interpretation via an implicit dividend yield. The conventional wisdom concerning the necessity of positive dividends for exercise of the American call is no longer the full story. Again, as in the benchmark models, the expected return on V is decreasing with $\delta^{(\rho)}$ (see (6) and (15)), which means a positive $\delta^{(\rho)}$ encourages the exercise of the option. However, we already noted that the entrepreneur has an additional incentive to exercise to reduce idiosyncratic risk. He actually requires a negative dividend to encourage waiting in order to counterbalance the impact of idiosyncratic risk. At $\delta^{(\rho)} = -\eta/2$, the benefit from the "negative dividends" on the expected return of V balances the cost of waiting in terms of exposure to risk. When $-\eta/2 < \delta^{(\rho)} < 0$ (or $\lambda\rho < \xi < \lambda\rho + \eta/2$), the impact of idiosyncratic risk dominates the incentive to wait because of negative dividends, and the option is exercised. In this same region, the model where idiosyncratic risk is not priced does not experience this tradeoff since a manager in such a model is not concerned with idiosyncratic risk and so waits indefinitely if there are negative dividends. Finally, when $\delta^{(\rho)} > 0$ (or $\xi < \lambda\rho$), both idiosyncratic risk and positive dividends impact in the same direction to encourage early exercise. As we saw in Sect. 3, the additional impact of idiosyncratic risk caused the entrepreneur to exercise at a lower threshold than in the complete or non-priced idiosyncratic risk models.

A final result that we draw attention to is the behavior of the additional parameter region in limiting cases. Since we showed in Proposition 3.4 that the benchmark models are recovered in the limit as correlation tends to one or risk aversion tends to zero, we also might expect that the additional parameter region disappears in the limit. If this were the case, then if correlation were high, or risk aversion believed to be low, then the benchmark models would provide a good approximation to the incomplete model. However, this is not the case, and this additional region does not shrink as we take limits $\rho \to 1$ or $\gamma \to 0$. This can be expressed in the following result which is now immediate.

Theorem 4.1 (i) ξ_i^* *does not tend to* ξ_c^* *as* $\rho \to 1$*;* (ii) ξ_i^* *does not tend to* ξ_{np}^* *as* $\gamma \to 0$

This result (and Fig. 4) shows the gap between the various model critical values remains in the limit. This has important implications for whether the benchmark models provide a reasonable approximation. In fact, since the parameter region does not shrink in the limit, the benchmark models do not provide a good approximation to the incomplete model even when correlation is high or risk aversion is small. This is because if the complete market or non-priced idiosyncratic risk model give the recommendation that exercise should be postponed indefinitely, it is not possible to conclude that it is also optimal for a risk averse entrepreneur to postpone investment even if correlation is very close to one or risk aversion is low. Using a complete model when markets are incomplete can lead to an incorrect conclusion with regard to investment timing. Similarly, using a model where idiosyncratic risks are not compensated when the entrepreneur is risk averse can lead to an incorrect decision. These differences in investment timing are more significant than the numerical differences we demonstrated earlier in Fig. 3 when parameters were such that $\xi < \lambda\rho$ or $\delta^{(\rho)} > 0$. For these parameters, recall, all models gave a finite investment threshold, although, of course, these thresholds all differ in value.

We comment now on the practical relevance of the parameter region $\lambda\rho < \xi < \lambda\rho + \frac{\eta}{2}$ in which the models lead to different conclusions, via an illustrative example. Consider the biotech-pharmaceutical example given earlier in the paper. The entrepreneur has the option to receive a one-off payoff for drug technologies from a pharmaceutical company. As a first approximation suppose the dynamics of the payoff received by the biotech can be related to the NASDAQ Biotech Index, which has annual return of about 13% and standard deviation of 45%.[16] This gives an approximate Sharpe ratio of $\xi = 0.29$. Assume the entrepreneur uses the AMEX Pharmaceutical Index as a partial hedge against the risk arising from his option. This index has annual returns of around 12% with standard deviation 27%, giving an approximate Sharpe ratio of $\lambda = 0.44$. The two indices have correlation around 0.40. If the entrepreneur had not required compensation for idiosyncratic risk when valuing his option to invest, he would retain the option and wait indefinitely since $\xi > \lambda\rho = 0.17$. However, in fact $0.17 < \xi < \lambda\rho + \eta/2 \approx 0.4$ so the incomplete model would recommend the entrepreneur acts at some finite threshold, given by $\tilde{V}^{(\rho,\gamma)}$. Risk aversion towards the unhedgeable portion of η changes the behavior of the entrepreneur.

5 The investment–uncertainty relationship

In this final section we illustrate the impact of uncertainty on the value of the option and the timing of investment. We show the investment–uncertainty relationship arising from the incomplete model differs in the two distinct parameter regions.

16 These numbers were taken from the April 15, 2005 "Statistical Analysis of the Historical Performance of the Biotechnology and Pharmaceutical Sectors", BioPharma Fund. The time period used was 1994–2005. We ignore interest rates for simplicity in this illustrative example.

It is well known that in complete real options models, uncertainty in the project value increases the option value and delays investment.[17] In such models, volatility increases option value purely via the convexity of the option payoff.[18] The intuition in the complete market model is that volatility gives a greater spread of outcomes for the investment payoff, and since the option is never exercised for $V < K$, this increases option value. In a complete market, the option value itself is also convex in the project value or investment payoff. We show in this section that both of these properties are no longer universally true in an incomplete market.

Entrepreneurial risk aversion together with incompleteness introduces another effect of increased volatility which acts in the opposite direction to the usual convexity effect. When some of the volatility η is idiosyncratic, higher idiosyncratic volatility is bad news for the risk averse entrepreneur. As discussed earlier, idiosyncratic volatility encourages exercise at a lower threshold in order to lock-in a value for the project.

The combination of these two effects in the incomplete utility-based model means that volatility can either raise or lower the option value, and simultaneously raise or lower the investment threshold, which is in sharp contrast to traditional thinking on the effect of uncertainty on real options. We investigate precisely how these effects interact in the following result.

Proposition 5.1 (i) *Let* $\beta^{(\rho)} > 1$ (*or equivalently* $\xi < \xi^*_{np}$ *or* $\delta^{(\rho)} > 0$)
Incomplete utility-based model and the Benchmark models:
I. The value of the option is convex in v *under all models.*
II. The threshold and value of the option are increasing in volatility under all models

(ii) *Let* $0 < \beta^{(\rho)} < 1$ (*or equivalently* $\xi^*_{np} < \xi < \xi^*_i$ *or* $-\frac{\eta}{2} < \delta^{(\rho)} < 0$)
Incomplete utility-based model:
I. The value of the option $p^{(\rho,\gamma)}(v)$ *is not strictly convex in* v*. For low* v*, the option value is concave, and for larger* v *(near* $\approx \tilde{V}^{(\rho,\gamma)}$*), the option value is convex.*
II. The threshold and value of the option are decreasing in volatility.
Benchmark models:
In both the non-priced idiosyncratic risk model and complete model, the thresholds and option values are infinite.

(iii) *Let* $\beta^{(\rho)} = 1$ (*or equivalently* $\xi = \xi^*_{np}$ *or* $\delta^{(\rho)} = 0$)
Incomplete utility-based model:
I. The value of the option is convex in v*.*
II. The threshold and option value do not depend on η*.*
Benchmark models:
In both the non-priced idiosyncratic risk model and complete model, the thresholds are infinite and the option values are equal to v*.*

We find the behavior of the threshold and option value differs in the two parameter regions. When the Sharpe ratio on the project is low enough ($\xi < \xi^*_{np}$), or equivalently, dividends are positive $\delta^{(\rho)} > 0$, the behavior of the option value and threshold with respect to volatility is the same for all three models. Figure 5 displays option value as a function of volatility η for the two regions. The left panel (corresponding to $\xi < \xi^*_{np}$) gives the non-priced idiosyncratic

[17] See Dixit and Pindyck [6] and McDonald and Siegel [11] amongst many others. This hinges on the assumption that $\delta^{(\rho)}$ is a fixed parameter and Dixit and Pindyck [6] "regard δ as a basic parameter independent of η". We make the same assumption here.

[18] Convexity properties have been well studied in the context of financial options, see Bergman et al. [1].

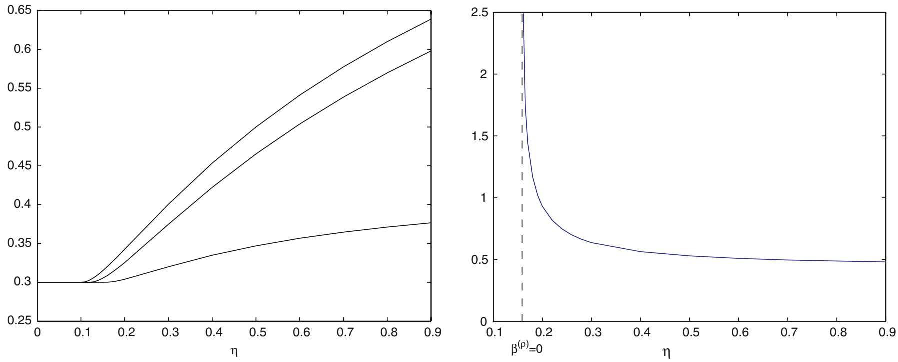

Fig. 5 Value of the option to invest for a range of project volatilities η. In the left panel, $\beta^{(\rho)} > 1$, whilst in the right panel, $0 < \beta^{(\rho)} < 1$. In the left panel, the *highest line* is the value under the non-priced idiosyncratic risk model ($\gamma = 0$, $\rho = 0.9$) and the *middle line* is the complete model value ($\rho = 1$). The *lower line* is the incomplete (utility model) value with $\rho = 0.9$. In the left panel, $\delta^{(\rho)} = 0.17$ when $\rho = 0.9$ and $\delta^{(1)} = 0.2$. In the right panel, $\delta^{(\rho)} = -0.08$ and the line represents the incomplete value. The value is infinite in this case under both benchmark models. Parameters common to both panels are $K = 1$, $\gamma = 10$

risk model (highest), complete (middle) and incomplete utility-based model (lowest) option values, which are all increasing in η.

However, if the Sharpe ratio is in the additional region ($\xi^*_{np} < \xi < \xi^*_i$), or equivalently implicit dividends satisfy $-\eta/2 < \delta^{(\rho)} < 0$, then in the incomplete model, the dislike of idiosyncratic risk dominates, resulting in the option value and threshold both decreasing with volatility. In this case, a higher volatility and a resulting lower threshold will cause volatility to speed-up investment. The right panel of Fig. 5 shows how the option value in the incomplete utility-based model varies with volatility, in the situation $\xi^*_{np} < \xi < \xi^*_i$. Recall, in this parameter region, both the complete and non-priced idiosyncratic risk models give infinite thresholds and option values. The graph shows that, (as in Part II. of (ii) of the proposition), the option value decreases with η.

The proposition also states results concerning the shape of the option value itself. In the common parameter region where $\xi < \xi^*_{np}$, the option value remains convex everywhere in all three models. However, in the additional parameter region, the option value under the incomplete model is not strictly convex anymore, and we can show it is concave for low values of V (see Fig. 2). Again, this result is in contrast to the usual belief that option values are convex, and is another demonstration of the striking behavior that can occur once managerial risk aversion is accounted for in real options.

Although Miao and Wang [15] also find that volatility and uncertainty can speed-up investment (in the situation of a lump-sum investment payoff), this only occurs in their model when risk aversion is sufficiently large. In contrast, we find that for all (non-zero) values of risk aversion, the threshold will decrease with volatility (and hence speed-up investment) provided parameters are such that $\lambda\rho < \xi < \lambda\rho + \eta/2$. Since this depends on observable quantities, and not on an unobservable risk aversion parameter, it should be more amenable to empirical testing than the conclusion of Miao and Wang [15]. In addition, we also show that the option value need not be convex everywhere in the incomplete model, and again, when parameters are such that $\lambda\rho < \xi < \lambda\rho + \eta/2$, the option value is concave for low values of v.

6 Conclusions

This paper studies the impact of incompleteness and risk aversion on the investment timing decision and value of the option to invest. In our framework, an entrepreneur decides when to pay an investment cost to receive a random one-off investment payoff, the risk of which cannot be fully hedged against.

We find that incompleteness and risk aversion reduce the investment threshold and option value. More importantly, we find there is an additional parameter region (in which dislike of idiosyncratic risk outweighs the benefit of waiting for a higher investment payoff) wherein the incomplete and benchmark real options models give conflicting investment signals. The incomplete model where the entrepreneur is risk averse, recommends exercising at some finite threshold, whilst both the complete model and the model where idiosyncratic risk is not priced suggest retaining the option to wait indefinitely for a higher investment payoff. Such conflicting signals imply that the benchmark real options models are not necessarily a good approximation to an incomplete setting even if correlation is high or risk aversion is small.

Merton [14] showed American call options are never exercised unless dividends are positive. However, we find that the additional region (where the option is exercised under the incomplete model) corresponds to parameters which imply a negative implicit dividend yield. This apparent contradiction is resolved because in the incomplete model there is an additional incentive to exercise to avoid idiosyncratic risk. In this parameter region, some conventional properties of thresholds and option values are no longer true. We saw that the value of the option is not strictly convex in the investment value itself, and the option value and investment threshold decrease with volatility. Strikingly, uncertainty speeds-up investment in this parameter region, which is in contrast to traditional thinking that uncertainty increases the option value of waiting, and leads to a higher threshold.[19] The sign of the investment–uncertainty relationship is also of empirical interest, so we offer new insights into this relationship for risk averse entrepreneurs facing one-off investment payoffs. In fact, our conclusion that volatility speeds-up investment holds for any level of risk aversion, provided parameters place the investment in the additional region, and so is amenable to testing.

To remain as close to the canonical model of McDonald and Siegel [11] as possible, we have chosen to develop an incomplete markets framework where the entrepreneur maximizes his expected utility of wealth. The advantage of our set-up (and unlike in Miao and Wang [15]) is that standard real options models are recovered as a special case when the correlation between the investment payoff and hedging asset is one. We also made a number of assumptions. The use of exponential utility allowed us to eliminate wealth-dependence and reduce to a one-dimensional free boundary problem which we could solve. Other utilities could be studied at the cost of an increase in dimension. Similarly, studying the perpetual problem lead to stationary solutions or a constant threshold (see Dixit and Pindyck [6] or McDonald and Siegel [11]). There is no reason to expect the main conclusions of the paper would change if these were altered, however, the solutions would be much less tractable.

[19] It is interesting to note that Brock et al. [4] show that when V has a lower absorbing barrier sufficiently close to the current value, an increase in volatility can lower the value of the option. See also Sect. 5, Chapt. 5 of Dixit and Pindyck [6] where a similar effect is observed for a mean-reverting model. However, both of these models are under the assumption of complete markets, and the behavior is being driven by the special features of the chosen process for V.

Appendix

Remarks on the Manager's Optimization Problem in (4)

In this section we provide further explanation and justification for the choice of discount factor $\zeta = -\frac{1}{2}\lambda^2$ in the manager's optimization problem in (4). We will do this in two parts. First, in (i), we will show that this choice reflects the fact that the manager has an infinite horizon and is investing optimally in the risky asset and risk-free bond after the exercise time τ of the option. The factor $\mathrm{e}^{-\frac{1}{2}\lambda^2\tau}$ is accounting for this optimal portfolio choice after τ. In (ii), we provide a second interpretation that the choice $\zeta = -\frac{1}{2}\lambda^2$ ensures there are no biases in the manager's choice of investment/exercise time resulting from the underlying portfolio choice problem.

(i) For sake of argument, first consider the manager's problem under the restriction that $\tau < T$ for some horizon $T < \infty$. That is, assume the option to invest has a finite horizon T. The manager's preferences are described by a utility function $U(x)$ for discounted wealth.

Temporarily ignoring the option to invest, consider solving the optimal portfolio choice problem with terminal horizon T

$$J(t,x) = \sup_{(\theta_s)_{t\leq s\leq T}} \mathbb{E}[U(X_T^\theta)|X_t^\theta = x] \tag{16}$$

where X^θ represents the entrepreneur's wealth from holding the risky asset P, see (3) and $J(t,x)$ denotes the value function or indirect utility at $t < T$. For technical reasons we assume the class of admissible strategies θ_s are bounded, see the verification argument for details.

We now want to choose a time $\tau \leq T$ at which to exercise the investment option to receive $(V_\tau - K)^+$. Conditioning shows that

$$\sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq T}} \mathbb{E}U(X_T^\theta + (V_\tau - K)^+) = \sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq \tau}} \mathbb{E}J(\tau, X_\tau^\theta + (V_\tau - K)^+)$$

It is thus *equivalent* to either solve the exercise problem over the finite horizon T with utility $U(x)$, or to solve the exercise problem up to the (exercise) time τ but using $J(t,x)$ defined in (16) to evaluate wealth at τ. In this sense, the problem we are interested in (with horizon T and the manager has preferences described by utility function $U(x)$) is translated into one in which it is *as if* the optimization ends at τ but $J(t,x)$ is used to evaluate wealth at τ.

In the case of exponential utility, $U(x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}$, standard calculations show that $J(t,x) = -(\mathrm{e}^{-\lambda^2 T/2})\frac{1}{\gamma}\mathrm{e}^{\lambda^2 t/2-\gamma x}$ and then from the above,

$$\sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq T}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{-\gamma(X_T^\theta+(V_\tau-K)^+)}\right] = \mathrm{e}^{-\lambda^2 T/2}\sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq \tau}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{\lambda^2\tau/2-\gamma(X_\tau^\theta+(V_\tau-K)^+)}\right]$$

Clearly, it is equivalent to solve the problem on the right-hand-side without the pre-factor $\mathrm{e}^{-\lambda^2 T/2}$, (so as if $J(t,x)$ becomes $-\frac{1}{\gamma}\mathrm{e}^{\lambda^2 t/2-\gamma x}$) and the relevant problem to solve becomes

$$\sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq \tau}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{\lambda^2\tau/2-\gamma(X_\tau^\theta+(V_\tau-K)^+)}\right] = \sup_{\tau\leq T}\sup_{(\theta_s)_{s\leq \tau}} \mathbb{E}\left[\mathrm{e}^{\lambda^2\tau/2}U(X_\tau^\theta + (V_\tau - K)^+)\right]. \tag{17}$$

Now we want to consider the infinite horizon problem. In this case the natural generalization is to remove the restriction $\tau \leq T$ in (17) and to consider

$$\sup_{(\tau,\theta)\in\mathcal{A}} \mathbb{E}[\mathrm{e}^{\lambda^2\tau/2}U(X_\tau^\theta + (V_\tau - K)^+)].$$

This is exactly the formulation in (4) with the choice $\zeta = -\frac{1}{2}\lambda^2$.

(ii) Our second explanation of the choice $\zeta = -\frac{1}{2}\lambda^2$ is based on the idea that we do not want the manager to have an in-built preference for early or late exercise/investment based on the set-up of the underlying portfolio choice/hedging problem. That is, we do not want the manager to have a preference for particular horizon times τ in absence of the option to invest.

Temporarily ignoring the option to invest, we just consider the underlying portfolio choice problem of the manager. For the moment, τ denotes the terminal horizon of the portfolio choice problem, that is, the date at which the manager evaluates his expected utility of wealth. We will show that the choice $\zeta = -\frac{1}{2}\lambda^2$ is the one for which the solution of the portfolio choice problem

$$\sup_{(\theta_u)_{t\le u\le\tau}} \mathbb{E}[\mathrm{e}^{-\zeta\tau}U(X_\tau^\theta)|X_t = x] \tag{18}$$

does not depend on the horizon τ. If the manager was not indifferent as to the choice of τ in (18), then when we include the option, the entrepreneur would already have a preference for some exercise time which would not arise purely from the option and there would be a bias in the model set-up.

To give a less technical presentation, we first consider the simpler case where the horizon of the portfolio choice problem is non-random, denoted by T. This enables us to employ a simple proof based on HJB arguments. We return to the case of a random horizon τ later.

For a non-random horizon T, and taking $U(x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}$, the problem becomes

$$M(t,x) = \sup_{(\theta_u),t\le u\le T} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{-\gamma X_T^\theta}\mathrm{e}^{-\zeta T}\,\middle|\, X_t^\theta = x\right]. \tag{19}$$

Apart from the presence of the discount factor ζ, this is a very similar portfolio choice problem to the standard Merton [13] problem with exponential utility where utility of terminal wealth is maximized, and the investor can only invest in risk-free bonds or a single risky asset. The solution to that standard problem depends on the time remaining until the terminal horizon. We show for a particular choice of ζ, the dependence of the solution on the terminal horizon is removed.

Using (3), the HJB equation is given as

$$\sup_\theta \left\{\dot{M} + M_x\theta\mu + \frac{1}{2}M_{xx}\theta^2\sigma^2\right\} = 0 \tag{20}$$

with boundary condition $M(T,x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\mathrm{e}^{-\zeta T}$. Performing the maximization over θ gives

$$\theta_t^* = -\frac{M_x\lambda}{M_{xx}\sigma}$$

and substitution into (20) results in

$$\left\{\dot{M} - \frac{1}{2}\frac{\lambda^2(M_x)^2}{M_{xx}}\right\} = 0.$$

The solution with the given boundary condition can be verified (by substitution) to be

$$M(t,x) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\mathrm{e}^{-\frac{1}{2}\lambda^2(T-t)}\mathrm{e}^{-\zeta T}.$$

We see that the solution will depend on the terminal horizon T in general, unless we take $\zeta = -\frac{1}{2}\lambda^2$. If $\zeta > -\frac{1}{2}\lambda^2$, then, if the manager can choose T, he would choose $T = \infty$. That is, larger horizons would be better. On the other hand, if $\zeta < -\frac{1}{2}\lambda^2$, then smaller horizons are preferable, and $T = 0$ would be chosen.

We now provide an argument for the case where the horizon is a random time, denoted by τ. We again find that the choice $\zeta = -\frac{1}{2}\lambda^2$ is the one which gives indifference over the horizon. Recall the problem (18) with $\zeta = -\frac{1}{2}\lambda^2$ can be written as

$$\sup_{(\theta_u)_{t\le u\le\tau}} \mathbb{E}[J(\tau, X_\tau^\theta)|X_t = x] \tag{21}$$

If we can show that $J(t, X_t^\theta)$ is a super-martingale in general, and a martingale for the optimal θ ($J \le 0$), then

$$J(t,x) = \sup_{(\theta_u)_{t\le u\le\tau}} \mathbb{E}[J(\tau, X_\tau^\theta)|X_t^\theta = x]$$

and we can write

$$J(t,x) = \sup_{\tau} \sup_{(\theta_u)_{t\le u\le\tau}} \mathbb{E}[J(\tau, X_\tau^\theta)|X_t^\theta = x]$$

since $J(t,x)$ does not depend on the horizon τ.

We now show these properties. Applying Itô's formula to $J(t, X_t^\theta)$ and integrating gives

$$J(\tau, X_\tau^\theta) = J(t, X_t^\theta) + \int_t^\tau \frac{J(s, X_s^\theta)}{2}\left[\lambda - \gamma\theta_s\sigma\right]^2 \mathrm{d}s - \int_t^\tau \gamma\theta_s\sigma J(s, X_s^\theta)\mathrm{d}B_s$$

Provided the last term has zero expectation (e.g. if τ and θ are bounded) it follows that $\mathbb{E}J(\tau, X_\tau^\theta) \le J(t, X_t^\theta)$ for any θ, and using the optimal strategy solving the problem (21), $\theta_s^* = \frac{\lambda}{\gamma\sigma}$, we have

$$\sup_{(\theta_u)_{t\le u\le\tau}} \mathbb{E}[J(\tau, X_\tau^\theta)] = J(t, X_t^\theta).$$

Hence $J(t, X_t^\theta)$ is a super-martingale in general and a martingale for the optimal θ.

Proof of Proposition 3.1 We develop the HJB equation and associated conditions given in the Proposition. In the continuation region (where $G(x,v) > -\frac{1}{\gamma}\mathrm{e}^{-\gamma(x+(v-K)^+)}$), $\mathrm{e}^{\frac{1}{2}\lambda^2 t}G(x,v)$ is a martingale under the optimal strategy and a supermartingale otherwise. The HJB equation is derived using Ito's formula, giving

$$0 = \frac{1}{2}\lambda^2 G + \xi\eta v G_v + \frac{1}{2}\eta^2 v^2 G_{vv} + \sup_\theta\left\{\theta\lambda\sigma G_x + \frac{1}{2}\theta^2\sigma^2 G_{xx} + \theta\sigma\rho\eta v G_{xv}\right\}$$

Optimizing over θ gives

$$\theta_t^* = \frac{-\lambda G_x - G_{xv}\rho\eta v}{G_{xx}\sigma} \tag{22}$$

and so

$$0 = \frac{1}{2}\lambda^2 G + \xi\eta v G_v + \frac{1}{2}\eta^2 v^2 G_{vv} - \frac{1}{2}\frac{(\lambda G_x + \rho\eta v G_{xv})^2}{G_{xx}}$$

We solve subject to the associated boundary condition, as well as at the constant exercise threshold $\tilde{V}^{(\rho,\gamma)}$, we must have value-matching and smooth-pasting conditions:

$$G(x,0) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x} \tag{23}$$

$$G(x,\tilde{V}^{(\rho,\gamma)}) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\mathrm{e}^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)^+} \tag{24}$$

$$G_v(x,\tilde{V}^{(\rho,\gamma)}) = \mathrm{e}^{-\gamma x}\mathrm{e}^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)^+} I_{\{\tilde{V}^{(\rho,\gamma)}>K\}} \tag{25}$$

Now we have a non-linear pde for the candidate solution for the optimization problem. In order to complete the proof, we need to solve the pde and verify the resulting candidate solution does indeed correspond to the value function. This is done in Proposition 3.2 and the verification argument directly following.

Proof of Proposition 3.2 We want to solve the non-linear pde (12) subject to the associated boundary, value matching and smooth pasting conditions. Proposing a solution of the form $G(x,v) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}\Gamma(v)^g$ gives

$$0 = \left[v\Gamma_v\eta\,(\xi-\lambda\rho) + \frac{1}{2}\eta^2v^2\Gamma_{vv} + \frac{1}{2}\frac{\Gamma_v^2}{\Gamma}\eta^2v^2(g(1-\rho^2)-1)\right]. \tag{26}$$

Choosing $g = \frac{1}{1-\rho^2}$ eliminates the non-linear term, leaving

$$0 = \left[v\Gamma_v\eta\,(\xi-\lambda\rho) + \frac{1}{2}\eta^2v^2\Gamma_{vv}\right] \tag{27}$$

with corresponding conditions on $\Gamma(v)$ (translated from the conditions in Proposition 3.1)

$$\Gamma(0) = 1 \tag{28}$$

$$\Gamma(\tilde{V}^{(\rho,\gamma)}) = \mathrm{e}^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)^+(1-\rho^2)} \tag{29}$$

$$\frac{\Gamma_v(\tilde{V}^{(\rho,\gamma)})}{\Gamma(\tilde{V}^{(\rho,\gamma)})} = -\gamma I_{\{\tilde{V}^{(\rho,\gamma)}>K\}}(1-\rho^2) \tag{30}$$

We can now re-express the optimal strategy θ^* given in (22) as

$$\theta_t^* = \frac{\lambda}{\sigma\gamma} + \frac{\rho\eta v\Gamma_v(v)}{\sigma\gamma(1-\rho^2)\Gamma(v)} \tag{31}$$

Notice the first term in θ^* is the optimal position in the risky asset P in the absence of the option. The second term is a hedging component to reflect the optimal hedge for the option risk with the risky asset P.

We propose a solution of the form $\Gamma(v) = Cv^{\psi}$, for some constant C which results in the fundamental quadratic in ψ,

$$\psi(\psi-1)\frac{\eta^2}{2} + \psi\eta(\xi-\lambda\rho) = 0. \tag{32}$$

The two roots of the quadratic are

$$\psi = \beta^{(\rho)} = 1 - \frac{2(\xi-\lambda\rho)}{\eta}, \quad \psi = 0. \tag{33}$$

That is, there are one non-zero and one zero root. It can be seen that the general form of the solution must be $\Gamma(v) = Cv^{\beta^{(\rho,\gamma)}} + B$, and (28) gives $B = 1$. We now have to decide when we can build a solution that value-matches and smooth-pastes. There are two possibilities. If $\beta^{(\rho)} \le 0$ (or equivalently $\xi \ge \lambda\rho + \frac{\eta}{2}$) then $C = 0$, smooth pasting fails and there is no solution. In this case, the entrepreneur postpones indefinitely. If $\beta^{(\rho)} > 0$ (correspondingly $\xi < \lambda\rho + \frac{\eta}{2}$), the entrepreneur will exercise at time τ.

In the case $\beta^{(\rho)} > 0$, (29) gives the constant $C < 0$ and via (30) we solve for the optimal investment threshold, $\tilde{V}^{(\rho,\gamma)}$, as the solution to (13). We have finally (substituting for C),

$$\Gamma(v) = 1 - (1 - \mathrm{e}^{-\gamma(1-\rho^2)(\tilde{V}^{(\rho,\gamma)}-K)}) \left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta^{(\rho)}}. \tag{34}$$

and hence the solution is of the form given.

Denote $c = \gamma(1-\rho^2)$ for convenience. We now show that (13) has a unique solution $\tilde{V}^{(\rho,\gamma)} > K$. It is sufficient to show that

$$f(v) = \frac{1}{c} \ln\left[1 + \frac{cv}{\beta^{(\rho)}}\right]$$

satisfies $f(0) = 0$, $f(v) > 0$ and $f''(v) < 0$. Since $\beta^{(\rho)} > 0$ we have $f(v) > 0$. Differentiation shows $f(v)$ is indeed concave in v, and thus (13) has a unique solution satisfying $\tilde{V}^{(\rho,\gamma)} > K$.

Verification argument

The problem in (4) is to find

$$I_* = \sup_{(\tau,\theta)\in\mathcal{A}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{\frac{\lambda^2}{2}\tau}\mathrm{e}^{-\gamma(X_\tau^\theta+(V_\tau-K)^+)}|X_0^\theta = x, V_0 = v\right]$$

where $\mathcal{A}$ is a set of admissible pairs of stopping times τ and strategies $(\theta_t)_{0\le t\le\tau}$. In particular, $\mathcal{A} = \bigcup_{T,J,L}\mathcal{A}_{T,J,L}$ where

$$\mathcal{A}_{T,J,L} = \{\tau : \tau \le T \wedge H_J^V, |\theta_t| \le L\}$$

and $H_J^V = \inf\{u : V_u \ge J\}$.

We first remark that even in the standard Merton investment problem (16) (with a fixed horizon), some condition on admissible strategies is required for the problem to be non-degenerate. Further, once there is also a stopping problem, conditions are also required on the stopping times τ. In the latter case, $Q_t = -\frac{1}{\gamma}\mathrm{e}^{\frac{\lambda^2}{2}t-\gamma X_t^\theta}$ is a negative local supermartingale. For $\theta_t = \frac{\lambda}{\sigma\gamma}$, it is a negative martingale which converges to zero almost surely. Hence

$$\sup_{\tau\,\text{finite},\theta} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{\frac{\lambda^2}{2}\tau-\gamma X_\tau^\theta}\right] = 0$$

However if we insist τ is bounded by T, say, then the martingale property gives

$$\sup_{\substack{\tau\,\text{bounded}\\ \theta\,\text{bounded}}} \mathbb{E}\left[-\frac{1}{\gamma}\mathrm{e}^{\frac{\lambda^2}{2}\tau-\gamma X_\tau^\theta}\right] = -\frac{1}{\gamma}\mathrm{e}^{-\gamma x}$$

Now we return to the verification argument for our problem. Recall from (29) and (34) that

$$\Gamma(v) = \begin{cases} \mathrm{e}^{-\gamma(1-\rho^2)(v-K)^+}; & v \ge \tilde{V}^{(\rho,\gamma)} \\ (1 - Cv^{\beta^{(\rho)}}); & v < \tilde{V}^{(\rho,\gamma)} \end{cases}$$

where C, $\tilde{V}^{(\rho,\gamma)}$ are chosen to satisfy value-matching and smooth-pasting. Let $Z_t = -\frac{1}{\gamma} e^{\frac{\lambda^2}{2}t-\gamma X_t^\theta}\Gamma(V_t)^{\frac{1}{1-\rho^2}}$. Then, given $(\tau,\theta) \in \mathcal{A}$ it is straightforward to show that Z_t is a true supermartingale. Hence for $(\tau,\theta) \in \mathcal{A}$,

$$\mathbb{E}\left[-\frac{1}{\gamma}e^{\frac{\lambda^2}{2}\tau}e^{-\gamma X_\tau^\theta-\gamma(V_\tau-K)^+}\right] \le Z_0 = -\frac{1}{\gamma}e^{-\gamma x}\Gamma(v)^{\frac{1}{1-\rho^2}}$$

and it follows that

$$I_* \le -\frac{1}{\gamma}e^{-\gamma x}\Gamma(v)^{\frac{1}{1-\rho^2}}$$

To deduce the reverse inequality it is sufficient to show that for some admissible (τ_n,θ_n) we obtain

$$\mathbb{E}\left[-\frac{1}{\gamma}e^{\frac{\lambda^2}{2}\tau_n}e^{-\gamma X_{\tau_n}^{\theta_n}-\gamma(V_{\tau_n}-K)^+}\right] \ge I_* - \epsilon$$

This can be achieved by using $\tau_n = \inf\{u : V_u \ge \tilde{V}^{(\rho,\gamma)}\} \wedge n$ and $\theta_n = \frac{\lambda}{\sigma\gamma} + \frac{\rho\eta V\Gamma_v(V)}{\sigma\gamma(1-\rho^2)\Gamma(V)}$ for $t \le \tau_n$. (Note θ_n is bounded for $V \le \tilde{V}^{(\rho,\gamma)}$.) The proof of this part of the verification relies on the fact that even though $\mathbb{P}(\tau > n)$ does not tend to zero, on $(\tau > n)$ we have $V_t \to 0$ and $e^{-\gamma(V_{\tau_n}-K)^+} = 1 = \lim_{V\downarrow 0}\Gamma(V)^{\frac{1}{1-\rho^2}}$.

Proof of Proposition 3.3 Let $\beta^{(\rho)} > 0$. We compare the value function in Proposition 3.2 with the value function achieved having no option, but with an adjusted initial wealth of $x + p^{(\rho,\gamma)}$. That is, $p^{(\rho,\gamma)}(v)$ solves $G(x,v) = G(x + p^{(\rho,\gamma)}(v), 0)$.

To obtain the second result of the proposition, we want to express $p^{(\rho,\gamma)}(v)$ as an optimal stopping problem. Rewrite the option value as $p^{(\rho,\gamma)}(v) = -\frac{1}{\gamma(1-\rho^2)}\ln\Gamma(v)$ where $\Gamma(v)$ was given in (34). We guess an optimal stopping representation for $\Gamma(v)$ as

$$\hat{\Gamma}(v) = \inf_{\tau \text{ bounded}} \mathbb{E}[D_\tau^0 e^{-\gamma(1-\rho^2)(V_\tau-K)^+}|V_0 = v]$$

It can be verified that $\hat{\Gamma}(v)$ satisfies the pde (27) and associated conditions, hence $\hat{\Gamma}(v) = \Gamma(v)$ and the representation in the proposition holds.

Proof of Proposition 3.4 Part (i) in both (A) and (B) are straightforward from the definitions. We distinguish between the different cases for $\beta^{(\rho,\gamma)}$. Consider first the case where $\beta^{(\rho,\gamma)} > 1$. Write $c = \gamma(1-\rho^2)$. To show (ii) in both (A) and (B), observe from (13)

$$\tilde{V}^{(\rho,\gamma)} - K = \frac{1}{c}\ln\left(1 + \frac{c\tilde{V}^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}}\right) \le \frac{1}{c}\frac{c\tilde{V}^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}} = \frac{\tilde{V}^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}}$$

So if $\beta^{(\rho,\gamma)} > 1$, we have $\tilde{V}^{(\rho,\gamma)} \le \frac{\beta^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}-1}K$. Observe also that $x - \frac{x^2}{2} \le \ln(1+x) \le x$ gives $\tilde{V}^{(\rho,\gamma)} \ge \frac{\beta^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}-1}K - \frac{c(\tilde{V}^{(\rho,\gamma)})^2}{2(\beta^{(\rho,\gamma)})^2(\beta^{(\rho,\gamma)}-1)}$. Letting $c \downarrow 0$ gives $\tilde{V}^{(\rho,\gamma)} \ge \frac{\beta^{(\rho,\gamma)}}{\beta^{(\rho,\gamma)}-1}K$. Putting these observations together gives $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(1)}$ as $\rho \to 1$ and $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(\rho)}$ as $\gamma \to 0$. Part (iii) can be shown similarly via the valuation formula in Proposition 3.3.

Now we turn to the case where $0 < \beta^{(\rho,\gamma)} < 1$. There exists $y^* = y^*(\beta^{(\rho,\gamma)})$ such that for $y \in (0, y^*)$, $\ln(1+y) \ge \beta^{(\rho,\gamma)}y$. Observe also that if $\ln(1 + \frac{cx}{\beta^{(\rho,\gamma)}}) \ge c(x-K)$ then $\tilde{V}^{(\rho,\gamma)} \ge x$. So for $\frac{xc}{\beta^{(\rho,\gamma)}} \le y^*(\beta^{(\rho,\gamma)})$,

$$\ln\left(1 + \frac{xc}{\beta^{(\rho,\gamma)}}\right) \ge \beta^{(\rho,\gamma)}\frac{xc}{\beta^{(\rho,\gamma)}} = xc > c(x-K)$$

and so $\tilde{V}^{(\rho,\gamma)} \geq x$. Since this is true for each $x \leq \frac{\beta^{(\rho,\gamma)} y^*(\beta^{(\rho,\gamma)})}{c}$, it is true for $x = \frac{\beta^{(\rho,\gamma)} y^*(\beta^{(\rho,\gamma)})}{c}$. Hence $\tilde{V}^{(\rho,\gamma)} \geq \frac{\beta^{(\rho,\gamma)} y^*(\beta^{(\rho,\gamma)})}{c}$. Let $c \downarrow 0$. Then $\tilde{V}^{(\rho,\gamma)} \uparrow \infty$. So for $0 < \beta^{(\rho,\gamma)} < 1$, $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(1)} = \infty$ as $\rho \to 1$ and $\tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(\rho)} = \infty$ as $\gamma \to 0$.

The case where $\beta^{(\rho,\gamma)} = 1$ can be treated similarly. The final possibility where $\beta^{(\rho,\gamma)} < 0$ is special because all quantities are infinite in this case.

Proof of Proposition 5.1 We first show the convexity results in Part I of each (i)–(iii). Straightforward differentiation of the option value in the non-priced idiosyncratic risk model and complete model gives these are convex in v. The incomplete utility-based model is more complicated. Differentiating (14) twice in v, we obtain

$$\frac{\partial^2}{\partial v^2}\left\{p^{(\rho,\gamma)}(v)\right\} = -\frac{\beta^{(\rho,\gamma)}}{cv^2}(1 - \mathrm{e}^{cp^{(\rho,\gamma)}})\left[\beta^{(\rho,\gamma)}\mathrm{e}^{cp^{(\rho,\gamma)}} - 1\right]$$

where $c = \gamma(1-\rho^2)$. The term outside the square brackets is positive for any $\beta^{(\rho,\gamma)} > 0$. Now observe that if $\beta^{(\rho,\gamma)} \geq 1$, the term inside the square brackets is positive also, and overall the second derivative is greater than zero. We have now shown all results in Part I of (i) as well as Part I of (iii).

We show the result of Part I of (ii) by noting that if $\beta^{(\rho,\gamma)} < 1$, the sign of the square bracket term is indeterminate and the second derivative may be of either sign. Using the expression for $p^{(\rho,\gamma)}$ and rearranging gives convexity will hold if $(1 - \mathrm{e}^{-c(\tilde{V}^{(\rho,\gamma)}-K)})(v/\tilde{V}^{(\rho,\gamma)})^{\beta} > 1-\beta$. For small values of v, this inequality will never hold, and hence $p^{(\rho,\gamma)}(v)$ is concave. For large values of $v \approx \tilde{V}^{(\rho,\gamma)}$, $(v/\tilde{V}^{(\rho,\gamma)})^{\beta} \approx 1$ and it can be shown that $1-\mathrm{e}^{-c(\tilde{V}^{(\rho,\gamma)}-K)} > 1-\beta$. For v close to the optimal threshold, $p^{(\rho,\gamma)}$ is convex. Note that these conclusions are independent of the size of c and therefore do not depend on the size of γ.

Now we prove the relationships between option value and volatility given in II of (i)–(iii). Recall $\delta^{(\rho)}$ is fixed. We treat the benchmark models first. It is straightforward to show if $\delta^{(\rho)} > 0$ or equivalently $\beta^{(\rho)} > 1$ then $\frac{\partial}{\partial\eta}\beta^{(\rho)} < 0$ and $\frac{\partial}{\partial\eta}\tilde{V}^{(\rho)} > 0$. Both derivatives are zero if $\beta^{(\rho)} = 1$. Expressing the derivative of the option value with respect to volatility as

$$\frac{\partial p^{(\rho)}(v)}{\partial\eta} = \frac{\partial p^{(\rho)}(v)}{\partial\beta^{(\rho)}}\frac{\partial\beta^{(\rho)}}{\partial\eta} + \frac{\partial p^{(\rho)}(v)}{\partial\tilde{V}^{(\rho)}}\frac{\partial\tilde{V}^{(\rho)}}{\partial\eta} \tag{35}$$

it is easy to show that $\frac{\partial p^{(\rho)}}{\partial\tilde{V}^{(\rho)}} = 0$ and $\frac{\partial p^{(\rho)}}{\partial\beta^{(\rho)}} < 0$. Putting these together gives $\frac{\partial p^{(\rho)}}{\partial\eta} > 0$ in the non-priced idiosyncratic risk model. All of the above goes through for the complete model with the superscripts changed from ρ to 1. We have therefore shown Part II of (i) for the benchmark models.

Now we turn to the incomplete utility-based model. If $\beta^{(\rho)} > 1$ then $\frac{\partial}{\partial\eta}\beta^{(\rho)} < 0$ and $\frac{\partial}{\partial\eta}\tilde{V}^{(\rho,\gamma)} > 0$. These signs are reversed if $\beta^{(\rho)} < 1$. Both derivatives are zero if $\beta^{(\rho)} = 1$. Now consider the option value $p^{(\rho,\gamma)}(v)$. Again, straightforward differentiation gives $\frac{\partial}{\partial\beta^{(\rho)}}p^{(\rho,\gamma)}(v) < 0$ and $\frac{\partial}{\partial\tilde{V}^{(\rho,\gamma)}}p^{(\rho,\gamma)}(v) = 0$. Combining these results via the equivalent expression to (35), we see that in the case $\beta^{(\rho)} > 1$, $\frac{\partial p^{(\rho,\gamma)}}{\partial\eta} > 0$ and in the case $\beta^{(\rho)} < 1$, we have $\frac{\partial p^{(\rho,\gamma)}}{\partial\eta} < 0$. This shows the remaining part of II (i), and the results of II of (ii) and (iii).

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