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VALUATION OF CLAIMS ON NONTRADED ASSETS USING UTILITY MAXIMIZATION

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A topical problem is how to price and hedge claims on nontraded assets. A natural approach is to use for hedging purposes another similar asset or index which *is* traded. To model this situation, we introduce a second nontraded log Brownian asset into the well-known Merton investment model with power law and exponential utilities. The investor has an option on units of the nontraded asset and the question is how to price and hedge this random payoff. The presence of the second Brownian motion means that we are in the situation of incomplete markets. Employing utility maximization and duality methods we obtain a series approximation to the optimal hedge and reservation price using the power utility. The problem is simpler for the exponential utility, and in this case we derive an explicit representation for the price. Price and hedging strategy are computed for some example options and the results for the utilities are compared.

Key Words: nontraded assets, option pricing, incomplete markets, unhedgeable risks, constant relative risk aversion, basis risk, exponential utility

1. INTRODUCTION

Valuing claims on nontraded assets presents new challenges in option pricing theory. An agent expects to receive or pay out a claim on a nontraded asset, and must decide how to best manage this risk. One method is to choose another similar asset or index which is traded and use this for hedging purposes. Clearly the higher the correlation between the traded and nontraded assets, the better we expect the hedge to perform. However, there is a need to quantify such statements and to give a framework under which we evaluate the optimal hedge and reservation price using a close asset. This is the objective of the paper.

To model these ideas mathematically, we introduce a second asset into the Merton investment model (Merton 1969) on which no trading is allowed. In the Merton model, the agent seeks to maximize expected utility of terminal wealth using either the exponential utility or the power law utility. We consider each of these utilities in this paper.

When the asset price follows exponential Brownian motion, the optimal behavior for an agent in the two models is well known: for the power utility a constant proportion of

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wealth is invested in the risky asset; for the exponential case, the cash amount invested is constant. Now suppose the investor has an option on the second nontraded asset, payable at time T. The problem is how to price and hedge this random payoff when trading in the second asset is not permissible. This is an incomplete markets problem and this type of risk is often called *basis risk*. In the finance literature, these problems are described as involving background risks, a typical example being labor income risk (see Franke, Stapleton, and Subrahmanyam 2001).

These problems occur often in practice. In many situations, the underlying assets can be traded (e.g., stocks in a basket); however, transactions costs may make it preferable to hedge with an index. Examples of this include a portfolio of illiquid shares hedged with index futures, a basket option hedged with an index, or a five-year futures contract hedged with a one-year futures contract. Illiquidity is also common in problems of commodity hedging. Another area where claims involving nontraded assets occur frequently is that of real options (see Dunbar 2000 and Dixit and Pindyck 1994).

A related problem involving stochastic income has been examined in the literature, beginning with He and Pagès (1993). El Karoui and Jeanblanc-Pique (1998) and Cuoco (1997) both assumed the income is spanned by assets but imposed a liquidity constraint. Duffie et al. (1997) and Koo (1998) considered infinite-horizon optimal consumption and investment with stochastic income imperfectly correlated with the risky asset. Numerical solutions were given in Munk (2000) using a Markov chain approximation. Duffie and Jackson (1990) and Svensson and Werner (1993) each considered a number of simple examples, and Duffie and Richardson (1991) found explicit solutions under a quadratic utility.

Detemple and Sundaresan (1999) studied a nontraded asset model as a special case of a portfolio constraint. Values were obtained numerically in a trinomial model of asset prices. Zariphopoulou (2001) studied a related general problem of utility maximization under constant relative risk aversion (CRRA) and employed a transformation to reduce the PDE to a linear one. The coefficients of the diffusion price process for a traded asset depend on a "stochastic factor" correlated with the asset price, creating unhedgeable risks. An example in Zariphopoulou looks at nontraded assets by obtaining price bounds for claims on the traded asset, where the price process is affected by the nontraded asset. The present paper and an earlier paper by Henderson and Hobson (2000) differ from Zariphopoulou by directly pricing a claim on a nontraded asset by including it in the utility from wealth.

Davis (2000) applied the dual approach to nontraded assets with the exponential utility function. Under exponential Brownian motion, he obtained an expression for the optimal hedge involving the solution to a nonlinear PDE. Hobson (1994) took the primal approach to the same problem and also obtained the hedge as a solution to a nonlinear PDE. Rouge and El Karoui (2000) used exponential utility in a portfolio constraints model. They related the price equation, a backward SDE, with minimal entropy. In a general semimartingale model, Becherer (2001) examined utility methods for nonreplicable claims under exponential utility. He specialized to a semicomplete product model and a Markov-type model driven by an Itô process for the traded asset prices and a multivariate point process representing some untradable factors of risk.

Our model considers agents with either constant relative risk aversion (power law utility) or constant absolute risk aversion (exponential utility). As is often the case, it appears that for the power utility there is no closed-form solution for the utility maximization problem in our model, as the PDE resulting from the stochastic control problem is highly nonlinear. We assume that the money value in the nontraded asset is small compared with wealth and that the payoff is bounded below. Under these assumptions, we "guess" the hedge and prove optimality using a dual approach. From this we obtain a series expansion for the value function and reservation price. The use of an expansion enables us to avoid solving the PDE numerically and allows for easier interpretation.

However, in the case of exponential utility, the problem is simpler. By separating wealth out of the problem, we reduce the PDE to a linear one and solve it directly to obtain an explicit price. Although the optimization problem for the exponential utility is more straightforward, the resulting price is wealth independent. As highlighted by Rouge and El Karoui (2000), this is not always desirable since it is unrealistic to assume that agents with different endowments have the same attitude toward risk. This may be particularly important in some applications. We compare the results from the two utilities with some interesting results.

Two examples, a call option and a power payoff, are used throughout the paper and prices and hedges are calculated from the general results. See Tompkins (1999) for a discussion of the uses of power style options in practice.

The remainder of the paper is organized as follows. Section 2 sets up our model with an additional nontraded asset and defines the value function for the problem. The complete markets case when the nontraded asset can be perfectly replicated is treated in Section 3. Section 4 considers the incomplete case for the power utility and we give an expansion for the value function of the agent as well as for her reservation price and the optimal strategy. The price and strategy are computed for the two example options. In Section 5 we give explicit results for the exponential utility and specialize to the examples. Section 6 concludes.

2. THE MERTON PROBLEM WITH AN ADDITIONAL NONTRADED ASSET

We consider the problem of an agent faced with receiving (or paying) a claim on a risky asset on which trading is not possible, or not allowed. The agent must decide how best to price and hedge this claim. Note that we refer to the asset as nontraded; however, this can be interpreted in a number of ways. The asset may not be traded at all or it may be traded but the agent would prefer not to trade for efficiency reasons. One case is when it is illiquid and too expensive to trade, and another may be when the agent is not permitted to trade in the asset, as with many executive stock options.

Begin by assuming the nontraded asset S follows an exponential Brownian motion

$$\frac{dS}{S} = \nu\,dt + \eta\,dZ, \tag{2.1}$$

where Z is a Brownian motion and ν and η are constants. The price S can be observed in the market at all times. We will take $r = 0$ throughout the paper for simplicity, although this is equivalent to using discounted variables. The agent is to receive (or pay) an option with payoff $h(S_T)$ at a future time $T < \infty$. A natural idea to approaching this problem is to look for a close or similar asset that is traded in the market, and use this asset to hedge the position. Introduce a traded asset P,

$$\frac{dP}{P} = \mu\,dt + \sigma\,dB, \tag{2.2}$$

where B is correlated to the Brownian motion Z, with correlation ρ. The idea is to choose P such that ρ is high, so we are mainly concerned with high, positive ρ.

In practice, the asset P may be a related index or another stock from the same industry group. This setup was used by Hubalek and Schachermayer (2001) to show that no-arbitrage is not sufficient to price the claim uniquely, and it was also used by Davis (2000).

It is convenient to think of Z as a linear combination of two independent Brownian motions B and W. Thus

$$Z_t = \rho B_t + \sqrt{1-\rho^2}W_t.$$

For $|\rho| < 1$ we are in an incomplete market situation since we cannot trade on S and therefore W.

Our agent's aim is to maximize expected utility of wealth, where, in addition to funds generated by trading, the agent is to receive (or pay out) λ units of the claim $h(S_T)$. The value function of the agent is given by

$$V(t, X_t, S_t; \lambda) = \sup_{(\theta_u)_{u \geq t}} \mathbb{E}_t[U(X_T + \lambda h(S_T))]. \tag{2.3}$$

We will consider two forms of utility function. First, those utilities with constant relative risk aversion of the form $U(x) = \frac{x^{1-R}}{1-R}$ for $R > 0, R \neq 1$, and second, those with constant absolute risk aversion $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, $\gamma > 0$. The results for the power law utility can be adapted for logarithmic utility by setting $R = 1$. We leave this for the interested reader.

For the first choice of family of utility functions, utility is only defined for positive wealth. Wealth is given by $X_T = X_t + \int_t^T \theta_u(dP_u/P_u)$ for some adapted θ which is constrained to ensure that $X_T + \lambda h(S_T) > 0$ almost surely; see Karatzas and Shreve (1987, Chap. 5.8). Note that θ_t is the cash amount invested in the traded asset P at time t. For this wealth restriction to hold, we need the following assumption on the payoff.

Assumption 2.1. *Either*
(i) $0 \leq h \leq b$ (e.g., put option) and λ can be positive or negative; or
(ii) $h \geq 0$ but not bounded above (e.g., call option) and λ can only be positive.
If h is not nonnegative but is bounded below by $-c$, then it will turn out that by considering $\tilde{h} = h + c$ our results still hold.

This assumption allows for three of the four simple option positions. When this assumption does not hold (say for a short call, where h is not bounded above but $\lambda < 0$) we have that V is identically minus infinity for the power law utility (a problem common to many utility functions) because the potential obligation is unbounded, and no hedging strategy can completely remove the risk. To see this, recall that S_T is the product of a term measurable with respect to the filtration of B and a random part $e^{\eta\sqrt{1-\rho^2}W_T}$ that is independent of B and unbounded above. In particular, for any X_T that can be generated from a finite initial fortune x, and investments in the traded asset P, we have

$$\mathbb{P}(X_T + \lambda h(S_T) < 0) > 0.$$

Since for the power utility, $U \equiv -\infty$ on the negative real line, we have

$$V(t, X_t, S_t, \lambda) = -\infty, \qquad |\rho| < 1, t < T.$$

This problem with unbounded payoffs also prevents the exponential utility from coping with the short call since $\mathbb{E}U(-(S_T - K)^+) = -\infty$. However in this case Assumption 2.1 is a stronger assumption on the payoff than is necessary. (This is discussed further in Section 5.)

The value function in (2.3) is a modification of the traditional Merton (1969) problem to include the additional payoff. We can thus think of this problem as the Merton wealth problem adjusted to include the nontraded asset. In the simple Merton problem with power utility we have

$$V(t,x) = \sup_{(\theta_u)_{u\geq t}} \mathbb{E}_t[U(X_T)|X_t = x] = \frac{x^{1-R}}{1-R}\exp\left\{\frac{1}{2}\frac{\mu^2}{\sigma^2}\frac{(1-R)}{R}(T-t)\right\}, \tag{2.4}$$

and if $\pi_t = \frac{\theta_t}{X_t}$ is the proportion of wealth invested in the risky asset, then

$$\pi_t^* = \frac{\mu}{\sigma^2 R}, \tag{2.5}$$

which is constant, the so-called *Merton proportion*. Similarly for the Merton problem with exponential utility, we have

$$V(t,x) = \sup_{(\theta_u)_{u\geq t}} \mathbb{E}_t[U(X_T)|X_t = x] = -\frac{1}{\gamma}e^{-\gamma x}e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)} \tag{2.6}$$

and

$$\theta_t^* = \frac{\mu}{\gamma\sigma^2}. \tag{2.7}$$

Now return to the problem with random endowment, h. We first show that V exists in $(-\infty,\infty)$ for each of the utility functions. Under Assumption 2.1, if (ii) holds so $h \geq 0$ and $\lambda > 0$ then $V(t,x,s;\lambda) \geq V(t,x,s;0)$ where the "no claim" position is given in (2.4) or (2.6) above. When (i) holds, $\lambda h \geq -|\lambda| b$ so $X_T + \lambda h(S_T) \geq X_T - |\lambda| b$ and $V(t,x,s;\lambda) \geq V(t,x-|\lambda|b,s;0)$.

Now we can find a simple upper bound for V by considering the dual problem. The reader is referred to Karatzas et al. (1991) for a description of the dual approach. The problem is to maximize $\mathbb{E}(U(X_T + \lambda h(S_T)))$ over feasible values of the terminal wealth X_T. For a positive random variable Λ, consider

$$\begin{aligned}
&\mathbb{E}\left\{U(X_T+\lambda h(S_T)) - \Lambda\left(X_T - \left(x + \int_t^T \theta_t \frac{dP}{P}\right)\right)\right\} \\
&\quad = \mathbb{E}\{U(X_T+\lambda h(S_T)) - \Lambda(X_T+\lambda h(S_T))\} + \mathbb{E}\{\Lambda(x+\lambda h(S_T))\} + \mathbb{E}\left\{\Lambda\int_t^T \theta_t\frac{dP}{P}\right\}.
\end{aligned}$$

Suppose Λ is of the form $\Lambda = \alpha d\mathbb{Q}/d\mathbb{P}$ for some change of measure $\mathbb{Q}$ such that P is a $\mathbb{Q}$ martingale and positive constant α. Then, with $\tilde{U}(y) = \sup_x(U(x) - xy)$,

$$\begin{aligned}
\sup_{X_T}\mathbb{E}\{U(X_T+\lambda h(S_T))\} &\leq \inf_{\Lambda}\mathbb{E}\{\tilde{U}(\Lambda) + \Lambda(x+\lambda h(S_T))\} \\
&= \inf_{\alpha}\inf_{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) + \alpha x + \lambda\alpha\mathbb{E}^{\mathbb{Q}}h(S_T)\right\}.
\end{aligned}$$

The problem is now to choose Λ in an optimal fashion. Set

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T\right) = \frac{d\mathbb{Q}^0}{d\mathbb{P}},$$

where $\mathbb{Q}^0$ is the minimal martingale measure. This makes the price process P into a martingale without affecting the Brownian motion W. Under the minimal martingale measure of Föllmer and Schweizer (1990), processes contained within the span of the traded assets (such as $\sigma B_t + \mu t$) become martingales, and martingales that are orthogonal to this space are unchanged in law. Thus, under the minimal martingale measure $\mathbb{Q}^0$,

$$\begin{aligned}\frac{dS}{S} &= \nu\,dt + \eta\rho\,dB + \eta\sqrt{1-\rho^2}\,dW \\ &= \left(\nu - \frac{\eta\rho\mu}{\sigma}\right)dt + \frac{\eta\rho}{\sigma}(\sigma\,dB + \mu\,dt) + \eta\sqrt{1-\rho^2}\,dW,\end{aligned}$$

where the final two terms in the last expression are both martingales. Thus S has drift $\delta = \nu - \frac{\mu\rho\eta}{\sigma}$ under $\mathbb{Q}^0$.

We now use the form of the two utilities. For the power law utility $U(x) = x^{1-R}/(1-R)$ we have $\tilde{U}(y) = (R/(1-R))y^{(R-1)/R}$. The exponential utility $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$ gives $\tilde{U}(y) = \frac{y}{\gamma}[\ln y - 1]$.

Then

$$\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \frac{R}{1-R}\alpha^{(R-1)/R}A$$

for the power utility, where

$$A = e^{\{m(1-R)/R\}T}$$

and $m = \frac{1}{2}\frac{\mu^2}{\sigma^2 R}$. Similarly,

$$\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \frac{\alpha}{\gamma}(\ln\alpha - 1) + \frac{\alpha}{\gamma}\frac{\mu^2}{2\sigma^2}T$$

for the exponential utility.

The minimization over α involves finding the minimum of

$$\frac{R}{1-R}\alpha^{(R-1)/R}A + \alpha(x + \lambda\mathbb{E}^0 h(S_T))$$

for the power law, and the exponential utility gives

$$\alpha\left(x + \lambda\mathbb{E}^0 h(S_T) + \frac{\mu^2}{2\gamma\sigma^2}T - \frac{1}{\gamma}\right) + \frac{\alpha}{\gamma}\ln\alpha.$$

The minimum and the upper bound for $U(x) = \frac{x^{1-R}}{1-R}$ is now easily seen to be

$$\frac{1}{1-R}A^R(x + \lambda\mathbb{E}_t^0 h(S_T))^{1-R} = V(t,x,s;0)\left(1 + \frac{\lambda}{x}\mathbb{E}_t^0 h(S_T)\right)^{1-R},$$

where $V(t,x,s;0)$ is given in (2.4), while $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$ gives

$$-\frac{1}{\gamma}e^{-\gamma x - \gamma\lambda\mathbb{E}^0 h(S_T) - \frac{\mu^2}{2\sigma^2}T} = V(t,x,s;0)e^{-\gamma\lambda\mathbb{E}^0 h(S_T)}$$

with $V(t,x,s;0)$ from (2.6).

The value function can be used to find the price that the agent is prepared to pay for the claim $\lambda h(S_T)$. The common procedure for pricing in a utility maximization framework is to compare the expected utility for an agent who does not receive any units of the claim to the expected utility of the agent who receives $\lambda h(S_T)$. The adjustment to the initial wealth which makes these values equal gives the so-called reservation price of the option. Equivalently, the investor is indifferent between the investment problem with zero endowment and the problem with the additional opportunity to buy the claim. Mathematically, given an initial (time 0) wealth of x_0, the reservation price is the solution to the equation $V(0, x_0 - p, s_0; \lambda) = V(0, x_0, s_0; 0)$; see Hodges and Neuberger (1989).

3. THE COMPLETE MARKETS CASE

If the correlation between B and Z is one, there is only one source of risk. We may compute the price and hedge directly, as there will be a unique martingale measure. With $\rho = 1$, $dS/S = \eta\, dB + v\, dt$, giving the relationship

$$\frac{dS}{S} = \frac{\eta}{\sigma}\frac{dP}{P} + \left(v - \frac{\mu\eta}{\sigma}\right)dt.$$

The measure under which P is a martingale must also make S into a martingale. Call this measure $\bar{\mathbb{Q}}$ and we must have $v = \eta\mu/\sigma$, otherwise the model would allow for arbitrage.

We wish to solve the utility maximization problem in (2.3). By considering the new wealth variable $Y_t = X_t + \lambda \bar{C}_t$, where $\bar{C}_t = \mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T)$, we can solve the problem explicitly in this case. Define $\bar{C}_t^S = \frac{\partial}{\partial S}\mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T)$, $\bar{C}_t^{SS} = \frac{\partial}{\partial S}\bar{C}_t^S$. By using the PDE for $\bar{C}$,

$$\dot{\bar{C}} + \tfrac{1}{2}\bar{C}^{SS} S^2 \eta^2 = 0,$$

Y solves

$$dY = \left(\theta + \frac{\lambda \bar{C}_t^S S_t \eta}{\sigma}\right)\sigma\, dB + \left(\theta + \frac{\lambda \bar{C}_t^S S_t \eta}{\sigma}\right)\mu\, dt = \tilde{\theta}_t(\sigma\, dB + \mu\, dt),$$

where $\tilde{\theta} = (\theta + \lambda \bar{C}_t^S S_t \eta/\sigma)$, and the agent seeks to maximize $\mathbb{E}U(Y_T)$. This corresponds to the Merton (1969) problem, with a modified strategy.

We treat each utility in turn. From the results of Section 2, the optimal $\tilde{\theta}$ for the power utility is $\frac{\mu}{\sigma^2 R} Y_t$ and θ^*, the optimal amount of cash invested in P, is

$$\theta^*(t, x, s; \lambda) = \frac{\mu}{\sigma^2 R} y - \frac{\eta}{\sigma}\lambda \bar{C}_t^S s = \frac{\mu}{\sigma^2 R}(x + \lambda \bar{C}_t) - \frac{\eta}{\sigma}\lambda \bar{C}_t^S s. \tag{3.1}$$

Using wealth Y, the value function is given by

$$\begin{aligned} V(t, x, s; \lambda) = \mathbb{E}_t[U(Y_T)] &= \frac{y^{1-R}}{1-R} e^{(1-R)m(T-t)} \\ &= \frac{x^{1-R}}{1-R} e^{(1-R)m(T-t)} \left(1 + \frac{\lambda \bar{C}_t}{x}\right)^{(1-R)}. \end{aligned}$$

Using the exponential utility and the results of Section 2, the optimal $\tilde{\theta}$ is $\frac{\mu}{\gamma\sigma^2}$ and

$$\theta^*(t, x, s; \lambda) = \frac{\mu}{\gamma\sigma^2} - \frac{\eta}{\sigma}\lambda \bar{C}_t^S s.$$

The value function is given by

$$V(t, x, s; \lambda) = -\frac{1}{\gamma} e^{-\gamma y} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)} = -\frac{1}{\gamma} e^{-\gamma(x + \lambda \bar{C}_t)} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)}.$$

For both utilities, we can follow the arguments at the end of the previous section to obtain the price the agent will pay for $\lambda h(S_T)$:

$$p = \lambda \bar{C}_t = \lambda \mathbb{E}_t^{\bar{\mathbb{Q}}} h(S_T). \tag{3.2}$$

This is the expected value of the claim under the risk neutral measure $\bar{\mathbb{Q}}$, as must be the case in a complete market. This complete market analysis does not require the restrictions on the payoff given in Assumption 2.1, and it is valid for short call options.

We treat the two utility functions separately in Sections 4 and 5. Comparisons are drawn between the two sets of results in Section 5.

4. THE INCOMPLETE CASE: POWER LAW UTILITY

In this section we concentrate on the power utility $U(x) = \frac{x^{1-R}}{1-R}$. As described earlier, if $|\rho| < 1$, the market is incomplete as the position in S cannot be replicated with P. We assume that the value of the claim under the minimal martingale measure $\lambda \mathbb{E}_t^0 h(S_T)$ is small relative to current wealth x. Henderson and Hobson (2000) concentrate on the case where the claim is proportional to the share price S_T; i.e., $h(S_T) = S_T$. This allowed scalings within the problem to be exploited to reduce the dimensionality by one. The resulting nonlinear PDE was approached using a series expansion. If we follow a similar approach here and derive the PDE associated with the value function, we have an extra variable. Using the value function in (2.3), write

$$V(t, X_t, S_t; \lambda) = \sup_{(\theta_u)_{u \geq t}} \frac{X_t^{1-R}}{1-R} \mathbb{E}\left(\frac{X_T}{X_t} + \frac{\lambda h(S_T)}{X_t}\right)^{1-R} = \frac{X_t^{1-R}}{1-R} g(T-t, S_t, X_t), \tag{4.1}$$

where $g(0, s, x) = (1 + \frac{\lambda h(s)}{x})^{1-R}$.

Using Itô on V and the fact that V is a supermartingale under any θ and a martingale under the optimal strategy gives

$$\left(-\dot{g} + g_s S v + \frac{1}{2} g_{ss} S^2 \eta^2\right) + \frac{[\mu(g_x + \frac{g(1-R)}{X}) + \sigma\rho\eta(g_{xs} S + g_s(1-R)\frac{S}{X})]^2}{4\sigma^2(\frac{1}{2} g_{xx} - \frac{1}{2} g R(1-R)/X^2 + g_x(1-R)/X)} = 0. \tag{4.2}$$

This is a nonlinear PDE in three variables, and the method used in the linear case does not seem straightforward. In this paper we take a different approach to treat the general claim $h(S_T)$. Since scaling can no longer be used, we conjecture the form of the optimal strategy. We verify that this strategy gives an upper and lower bound on the value function and these bounds agree to order λ^2. For this approach, we need to assume $\lambda \mathbb{E}_t^0 h(S_T)/x$ is small.

In the main theorem below, we prove that our conjecture for the optimal strategy is indeed optimal, and we derive an expansion for the value function V_2 up to order λ^2. Recall that $\mathbb{Q}^0$ is the minimal martingale measure for P.

THEOREM 4.1.

(1) Define $C_t = \mathbb{E}_t^0 h(S_T)$, $C_t^S = \frac{\partial}{\partial S} \mathbb{E}_t^0 h(S_T)$. *For* h *and* λ *satisfying Assumption 2.1 and* $U(x) = \frac{x^{1-R}}{1-R}$, *the optimal strategy* θ^* *is given by*

$$\theta^*(t, x, s; \lambda) = \theta_1(t, x, s; \lambda) + o(\lambda) \tag{4.3}$$

with

$$\theta_1(t, x, s; \lambda) = \frac{\mu}{\sigma^2 R}(x + \lambda C_t) - \frac{\eta\rho}{\sigma} \lambda s C_t^S. \tag{4.4}$$

(2) Using θ_1 *we define*

$$V_2(t,x,s;\lambda) = \frac{x^{1-R}}{1-R} e^{(1-R)m\tau} \left(1 + \lambda \frac{C_t}{x} - . \frac{\lambda^2}{2} R\eta^2 (1-\rho^2) \hat{\mathbb{E}} \int_t^T \frac{S_u^2 C_u^{S^2}}{X_u^{0^2}} \, du \right)^{1-R}, \tag{4.5}$$

where $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \exp(\frac{\mu(1-R)}{\sigma R} B_T - \frac{1}{2}(\frac{\mu(1-R)}{\sigma R})^2 T)$. *Then for* λ *and* h *satisfying Assumption 2.1 and* $U(x) = \frac{x^{1-R}}{1-R}$, *the value function* $V(t,x,s;\lambda)$ *is given by*

$$V(t,x,s;\lambda) = V_2(t,x,s;\lambda) + o(\lambda^2). \tag{4.6}$$

We first consider some examples before returning to prove the theorem.

EXAMPLE 4.1. Taking $h(S_T) = (S_T - \bar{K})^+$ gives the important example of a call option. We must have $\lambda > 0$ (since h is not bounded above), thus the agent is long a call option. We can evaluate the price under the minimal martingale measure $\mathbb{Q}^0$. Recall that under this measure S has drift $\delta = \nu - \frac{\mu\rho\eta}{\sigma}$; hence

$$\mathbb{E}_t^0 (S_T - \bar{K})^+ = C_t = e^{\delta(T-t)} sN(d_+) - \bar{K}N(d_-), \tag{4.7}$$

where $d_\pm = \frac{\ln\frac{s}{\bar{K}} + (\delta \pm \frac{1}{2}\eta^2)(T-t)}{\eta(T-t)}$ and

$$\frac{\partial}{\partial S} \mathbb{E}_t^0 (S_T - \bar{K})^+ = C_t^S = e^{\delta(T-t)} N(d_+),$$

giving an optimal hedging strategy of

$$\theta_t^* = \frac{\mu x}{\sigma^2 R} + \lambda e^{\delta(T-t)} sN(d_+) \left[\frac{\mu}{\sigma^2 R} - \frac{\eta\rho}{\sigma} \right] - \lambda \frac{\mu \bar{K}}{\sigma^2 R} N(d_-) + o(\lambda). \tag{4.8}$$

If $\lambda = 0$ we regain the Merton "constant proportion of wealth" hedge of (2.5). Taking $\delta = 0$ and $\rho = 1$ we recover the complete case of Section 3.

We can examine the effect of changing ρ on hedge. To get a comparison, we fix μ and δ, the drift of S under $\mathbb{Q}^0$. This means that ν, the real world drift, varies with ρ. Figure 4.1

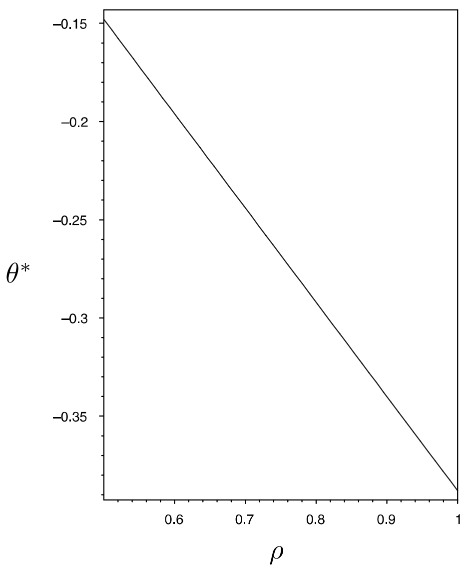

FIGURE 4.1. The optimal hedge θ_1 for the claim $(S_T - \bar{K})^+$ for $0.5 \le \rho \le 1$. Utility is power law. Note that the hedge is net of the Merton hedge (2.5), which is 326.53 in this example. Parameter values are $\lambda = 0.01$, $s = 100$, $T = 1$, $\bar{K} = 100$, $x = 500$, $R = 0.5$, $\mu = 0.04$, $\eta = 0.30$, $\sigma = 0.35$, $\nu = \frac{\mu\eta\rho}{\sigma}$, and $\delta = 0$.

shows hedge θ_1 net of the Merton hedge in (2.5). Thus zero represents the Merton strategy. For this choice of parameters, the agent holds less of the asset than the Merton hedge, and this decreases with correlation. When $\rho = 0$, the agent follows a strategy close to the no claim Merton strategy (in (2.5)) as the traded asset is of no use in reducing risk. This strategy deviates from Merton as correlation increases. In Figure 4.2, the effect of changing S on the strategy is displayed. If we fix S then we recover the behavior displayed in Figure 4.1. If S is low, and the option is out-of-the-money, then it is optimal to use the no claim Merton hedge given as zero on the graph. If S is large, and the option is far in-the-money, the hedge differs most from the Merton hedge.

EXAMPLE 4.2. Taking $h(S_T) = S_T^2$, we have a "power" payoff. Again we need $\lambda > 0$ for Assumption 2.1 to hold. This example is used because the price can be calculated explicitly.

$$
\begin{aligned}
C_t &= \mathbb{E}^0 S_T^2 = s^2 e^{(2\delta+\eta^2)(T-t)} \\
C_t^S &= 2se^{(2\delta+\eta^2)(T-t)} \\
\theta_t^* &= \frac{\mu x}{\sigma^2 R} + \lambda s^2 e^{(2\delta+\eta^2)(T-t)}\left[\frac{\mu}{\sigma^2 R} - \frac{2\eta\rho}{\sigma}\right] + o(\lambda).
\end{aligned} \tag{4.9}
$$

Interestingly, the sign of the λ term in the hedge depends on $(\frac{\mu}{\sigma^2 R} - \frac{2\eta\rho}{\sigma})$ where the power 2 appears in the second term. For the call example in (4.8) the same factor decides the sign but the power is 1.

Proof of Theorem 4.1. We demonstrate that the strategy θ_1 is optimal and derive an expansion for the value function by exhibiting upper and lower bounds for the supremum of expected utility which agree to order λ^2.

The exposition for the lower bound requires the fact that $\lambda h(S_T) \geq 0$ almost surely. Under Assumption 2.1(ii), this is satisfied. Under (i), we have two cases. If $\lambda > 0$ then

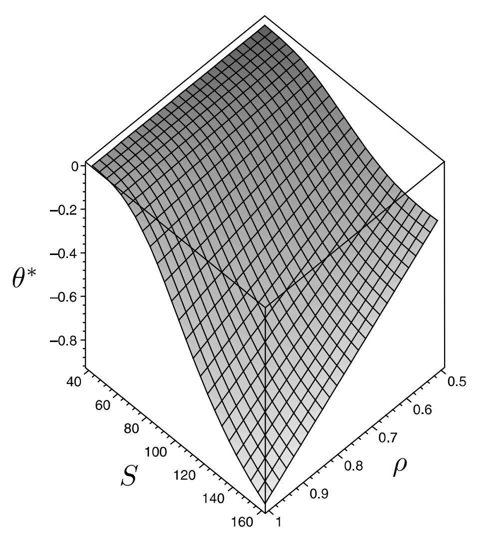

FIGURE 4.2. The optimal hedge θ_1 for the claim $(S_T - \bar{K})^+$ for $0.5 \leq \rho \leq 1$ and $40 \leq s \leq 160$. Utility is power law. Note that the hedge is net of the Merton hedge (2.5). Parameter values are $\lambda = 0.01$, $T = 1$, $\bar{K} = 100$, $x = 500$, $R = 0.5$, $\mu = 0.04$, $\eta = 0.30$, $\sigma = 0.35$, $v = \frac{\mu\eta\rho}{\sigma}$, and $\delta = 0$.

again $\lambda h(S_T) \geq 0$. If $\lambda < 0$ we write $-|\lambda|h = -|\lambda|b + |\lambda|(b-h)$. Thus the payoff has a positive component minus a constant. In our model, prices for claims are nonlinear due to the appearance of $(C^S)^2$. Thus, taking two claims h_1 and h_2, we could not simply compute the second-order term for the sum $h_* = h_1 + h_2$ by adding components. However, if h_1 is a constant, then $(C_1^S)^2 + (C_2^S)^2 = (C_*^S)^2$ and the sum of the second-order terms (only one of which is nonzero) is equal to the second-order term for the sum. Hence we can split our claim h into a constant part $(-b)$ and a positive component $(b-h)$.

The Lower Bound. Consider first the zero endowment problem where (X_t^0, θ_t^0) is the optimal wealth, strategy pair. Then $dX_t^0 = \theta_t^0 dP_t/P_t$ with $\theta_t^0 = (\mu X_t^0)/(\sigma^2 R)$ and

$$X_t^0 = x_0 \exp\left(\frac{\mu}{\sigma R} B_t + \frac{\mu^2}{\sigma^2 R} t - \frac{\mu^2}{2\sigma^2 R} t\right).$$

Now consider the problem with a random endowment of $\lambda h(S_T)$ at time T. We would like to consider the strategy in Theorem 4.1. However, with this strategy we cannot guarantee that wealth remains positive, so we use a localized version.

Fix K and let

$$H_K = \inf\left\{u : \int_0^u \frac{1}{X_t^0}\left(-\frac{\mu}{\sigma^2 R} C_t + \frac{\eta\rho}{\sigma} S_t C_t^S\right)\left(\frac{dP}{P} - \frac{\mu\, dt}{R}\right) = K\right\}.$$

Suppose $\lambda < \frac{1}{2}K^{-1}$. Consider the wealth process $X^{1,K}$ generated from an initial fortune x_0 using the strategy

$$\theta_t^{1,K} = \frac{\mu}{\sigma^2 R}(X_t^{1,K} + \lambda C_t I_{(t<H_K)}) - \frac{\eta\rho}{\sigma} \lambda S_t C_t^S I_{(t<H_K)}.$$

Then $X_t^{1,K}$ is given by

$$X_t^{1,K} = X_t^0 \left\{1 + \lambda \int_0^{t\wedge H_K} \frac{1}{X_u^0}\left(\frac{\mu}{\sigma^2 R} C_u - \frac{\rho\eta}{\sigma} S_u C_u^S\right)\left(\frac{dP_u}{P_u} - \frac{\mu}{R} du\right)\right\}.$$

Note that on $H_K < T$ we have $X_T^{1,K} = X_T^0(1-\lambda K)$ and indeed more generally $X_T^{1,K} \geq X_T^0(1-\lambda K)$. In particular, the localization times H_K allow us to bound the wealth process from below.

Now consider the sum of the wealth process and the random endowment. It is convenient to consider $Z_t^{\lambda,K} = X_t^{1,K} + \lambda C_t$. On $t \leq H_K$, using the PDE satisfied by the option price C we have

$$dZ_t^{\lambda,K} = \frac{\mu}{\sigma^2 R} Z_t^{\lambda,K} \frac{dP}{P} + \lambda S_t C_t^S \eta \sqrt{(1-\rho^2)}\, dW$$

so that, still with $t \leq H_K$,

$$Z_t^{\lambda,K} = X_t^0 \left\{1 + \lambda\left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^t \frac{S_u C_u^S}{X_u^0} \eta \sqrt{(1-\rho^2)}\, dW\right)\right\}.$$

Also, $Z_T^{\lambda,K} = X_T^{1,K} + \lambda h(S_T) \geq X_T^0(1-\lambda K) + \lambda h(S_T) \geq X_T^0(1-\lambda K)$.

From Taylor's expansion we have $U(y+h) = U(y) + hU'(y) + \frac{1}{2}h^2 U''(y+\xi h)$ with $\xi = \xi(\lambda, K, \omega) \in [0,1]$. We will take $y = X_T^0$ and $h = Z_T^{\lambda,K} - X_T^0$, and consider the expected value of this expansion term by term. The first term yields $\mathbb{E}(U(X_T^0)) = V(0, x_0, s_0; 0)$. For the second term, note that

$$U'(X_T^0) = x_0^{-R} \exp\left(\frac{\mu^2}{2\sigma^2}\frac{(1-R)}{R} T\right)\frac{d\mathbb{Q}^0}{d\mathbb{P}},$$

where $\mathbb{Q}^0$ is the minimal martingale measure. Then since both X^0 and $X^{1,K}$ are martingales under $\mathbb{Q}^0$, we have

$$\mathbb{E}[(Z_T^{\lambda,K} - X_T^0)U'(X_T^0)] = x_0^{-R} e^{(1-R)mT} \mathbb{E}^0(\lambda h(S_T)).$$

For the final term in the Taylor expansion we have that for $\xi = \xi(\lambda, K, \omega) \in [0,1]$,

$$X_T^0 + \xi(Z_T^{\lambda,K} - X_T^0) \geq X_T^0(1 - \lambda K).$$

Then, since U'' is increasing,

$$\begin{aligned}
&\frac{1}{\lambda^2}(Z_T^{\lambda,K} - X_T^0)^2 U''(X_T^0 + \xi(Z_T^{\lambda,K} - X_T^0)) \\
&\quad \geq (X_T^0)^2 \left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)}\, dW \right)^2 U''(X_T^0(1-\lambda K)) I_{(H_K \geq T)} \\
&\quad + (h(S_T) - X_T^0 K)^2 U''(X_T^0(1-\lambda K)) I_{(H_K < T)}.
\end{aligned}$$

By the dominated convergence theorem, on taking expectations and letting $\lambda \downarrow 0$, we find for each K that $\lambda^{-2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda \mathbb{E}[h(S_T)U'(X_T^0)])$ is bounded below by

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[(X_T^0)^2 U''(X_T^0) \left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)}\, dW \right)^2 I_{(H_K \geq T)} \right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[(h(S_T) - X_T^0 K)^2 U''(X_T^0) I_{(H_K < T)} \right].
\end{aligned}$$

If we let $K \uparrow \infty$ this expression becomes

$$\frac{1}{2}\mathbb{E}\left[U''(X_T^0)(X_T^0)^2 \left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)}\, dW \right)^2 \right]. \tag{4.10}$$

We can interpret $U''(X_T^0)(X_T^0)^2$ as a constant multiplied by a change of measure that affects the drift of dP/P. With this interpretation it is straightforward to show that (4.10) becomes

$$\begin{aligned}
&-\frac{R}{2}\mathbb{E}\left[(X_T^0)^{1-R} \left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)}\, dW \right)^2 \right] \\
&= -\frac{R}{2} x_0^{1-R} e^{m(1-R)T} \hat{\mathbb{E}}\left[\left(\frac{\mathbb{E}_0^0 h(S_T)}{x_0} + \int_0^T \frac{S_t C_t^S}{X_t^0} \eta \sqrt{(1-\rho^2)}\, dW \right)^2 \right] \\
&= -\frac{R}{2} x_0^{1-R} e^{m(1-R)T} \left[\frac{\mathbb{E}_0^0 h(S_T)^2}{x_0^2} + \eta^2(1-\rho^2)\hat{\mathbb{E}}\left(\int_0^T \frac{S_t^2 (C^S)_t^2}{X^{0^2}}\, dt \right) \right],
\end{aligned} \tag{4.11}$$

where $\hat{\mathbb{P}}$ is the measure under which both $\hat{B}_t \equiv B_t - (\mu(1-R)/\sigma R)t$ and $\hat{W}_t \equiv W_t$ are Brownian motions.

We have shown that

$$\limsup_{K \uparrow \infty} \lim_{\lambda \downarrow 0} \frac{1}{\lambda^2}(\mathbb{E}U(Z_T^\lambda) - \mathbb{E}U(X_T^0) - \lambda \mathbb{E}[U'(X_T^0)h(S_T)])$$

is greater than the expression (4.11) and

$$\sup_{X_T} \mathbb{E}U(X_T + \lambda h(S_T)) \geq \mathbb{E}U(Z_T^\lambda) \geq V_2(0, x_0, s_0; \lambda) + o(\lambda^2),$$

where V_2 is given in Theorem 4.1 (4.5). Hence V_2 is a lower bound to order λ^2.

The Upper Bound. An upper bound on the value function will be found by considering the dual problem as in Section 2. We refine our choice of measure $\mathbb{Q}$ to obtain a higher order bound. For each $\epsilon > 0$ we show that $V_2 + \epsilon\lambda^2$ is an upper bound.

Let $M_u = \eta\sqrt{1-\rho^2}\int_0^u (S_t C_t^S / X_t^0) dW_t$ and for any $K > 0$ define

$$T_K = \inf\{u : |M_u| + [M]_u = K\}.$$

Now choose K large enough so that

$$\hat{\mathbb{E}}[[M]_T - [M]_{T_K}] < \epsilon.$$

Let $\mathbb{Q}_K$ be given by

$$\frac{d\mathbb{Q}_K}{d\mathbb{P}} = \exp\left(-\frac{\mu}{\sigma}B_T - \frac{\mu^2}{2\sigma^2}T\right)\exp\left(-R\lambda M_{T_K} - \frac{1}{2}R^2\lambda^2[M]_{T_K}\right).$$

Then

$$\mathbb{E}^{\mathbb{P}}\left(\tilde{U}\left(\alpha\frac{d\mathbb{Q}_K}{d\mathbb{P}}\right)\right) = \frac{R}{1-R}\alpha^{(R-1)/R}A_K,$$

where

$$\begin{aligned} A_K &= \mathbb{E}\left[\exp\left(\frac{\mu(1-R)}{\sigma R}B_T + \frac{\mu^2(1-R)}{2\sigma^2 R}T\right)\exp\left((1-R)\lambda M_{T_K} + \frac{1}{2}R(1-R)\lambda^2[M]_{T_K}\right)\right] \\ &= e^{\{m(1-R)/R\}T}\hat{\mathbb{E}}\left[\exp\left(\frac{1}{2}(1-R)\lambda^2[M]_{T_K}\right)\right]. \end{aligned}$$

Note that the measure $\hat{\mathbb{P}}$ is the measure that arose in the calculation of the lower bound. Here $[M]_{T_K}$ is bounded so A_K can be written as an expansion in λ:

$$\begin{aligned} A_K &= e^{\{m(1-R)/R\}T}\left[1 + \frac{1}{2}(1-R)\lambda^2\hat{\mathbb{E}}[M]_{T_K} + O(\lambda^4)\right]. \\ &\le e^{\{m(1-R)/R\}T}\left[1 + \frac{1}{2}(1-R)\lambda^2(\hat{\mathbb{E}}[M]_T - \epsilon I_{(R>1)}) + O(\lambda^4)\right]. \end{aligned} \tag{4.12}$$

Now

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_K}h(S_T) &= \mathbb{E}\left(h(S_T)\frac{d\mathbb{Q}^0}{d\mathbb{P}}\left[e^{-\lambda RM_{T_K} - \frac{1}{2}\lambda^2R^2[M]_{T_K}}\right]\right) \\ &= \mathbb{E}^0(h(S_T)[1 - \lambda RM_{T_K} + o(\lambda)]) \\ &= \mathbb{E}^0 h(S_T) - \lambda R\mathbb{E}^0(M_{T_K}h(S_T)) + o(\lambda) \end{aligned}$$

If we now show that

$$\mathbb{E}^0 M_{T_K}h(S_T) = x_0\hat{\mathbb{E}}M_{T_K}^2 = x_0\hat{\mathbb{E}}[M]_{T_K} \tag{4.13}$$

then

$$\mathbb{E}^{\mathbb{Q}_K}h(S_T) = \mathbb{E}^0 h(S_T) - \lambda Rx_0\hat{\mathbb{E}}[M]_{T_K} + o(\lambda) \le \mathbb{E}^0 h(S_T) - \lambda Rx_0(\hat{\mathbb{E}}[M]_T - \epsilon) + o(\lambda). \tag{4.14}$$

Using Itô on M_tC_t and since M_t, C_t are $\mathbb{Q}^0$-martingales, we have

$$\mathbb{E}^0 M_{T_K}h(S_T) = \mathbb{E}^0\int_0^T d(M_{t\wedge T_K}C_t) = \eta^2(1-\rho^2)\mathbb{E}^0\int_0^{T_K}\frac{S_t^2(C_t^S)^2}{X_t^0}dt. \tag{4.15}$$

Now using $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{\frac{\mu(1-R)}{\sigma R}B_T - \frac{1}{2}(\frac{\mu(1-R)}{\sigma R})^2 T}$ and $\frac{d\mathbb{Q}^0}{d\mathbb{P}}$, we derive the relationships

$$\frac{(X_T^0)^{1-R}}{x_0^{1-R}} = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} e^{m(1-R)T}$$

and

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}} = \frac{(X_T^0)^{-R}}{x_0^{-R}} e^{-m(1-R)T}.$$

Thus

$$\hat{\mathbb{E}} M_{T_K}^2 = \mathbb{E}\left[\frac{M_{T_K}^2}{e^{m(1-R)T}} \frac{(X_T^0)^{1-R}}{x_0^{1-R}}\right] = \frac{1}{x_0} \mathbb{E}^0 M_{T_K}^2 X_T^0$$
$$= \frac{1}{x_0} \mathbb{E}^0 \int_0^{T_K} X^0 (dM)^2 = \frac{\eta^2(1-\rho^2)}{x_0} \mathbb{E}^0 \int_0^{T_K} \frac{S_t^2 (C^S)_t^2}{X_t^0} dt \tag{4.16}$$

using Itô on $M_t^2 X_t^0$. Thus (4.13) holds using (4.15) and (4.16).

Now

$$\sup_{X_T} \mathbb{E}(U(X_T + \lambda S_T)) \le \inf_\alpha \left\{ \frac{R}{1-R} \alpha^{\frac{1-R}{R}} A_K + \alpha x_0 + \lambda \alpha \mathbb{E}^{\mathbb{Q}_K} h(S_T) \right\}$$
$$= \frac{1}{1-R} (A_K)^R (x_0 + \lambda \mathbb{E}^{\mathbb{Q}_K} h(S_T))^{1-R}, \tag{4.17}$$

and using (4.12) and (4.14) we see for some constant c_0 that

$$\inf_\alpha \mathbb{E}\left(V\left(\alpha \frac{d\mathbb{Q}_K}{d\mathbb{P}}\right) + \alpha(x_0 + \mathbb{E}^{\mathbb{Q}_K} h(S_T))\right) \le V_2(0, x_0, s_0; \lambda) + c_0 \epsilon \lambda^2. \qquad \square$$

We wish to calculate the expansion for the reservation price, p, that the agent would be willing to pay for λ units of the claim $h(S_T)$. As discussed at the end of Section 2, this involves solving $V(t, x, S_t; 0) = V(t, x - p, S_t; \lambda)$, which can be written as

$$x^{1-R} = (x-p)^{1-R} \left[1 + \frac{\lambda \mathbb{E}^0 h(S_T)}{x-p} - \frac{\lambda^2}{2} R \eta^2 (1-\rho^2) \hat{\mathbb{E}} \int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^{0\,2}} du\right]^{1-R}, \tag{4.18}$$

where $X_t^0 = x - p$. To first order, we find $p = \lambda \mathbb{E}^0 h(S_T)$. The second-order term is calculated by finding c_2 in $p = \lambda \mathbb{E}^0 h(S_T) + \lambda^2 c_2$. Combining finally the definitions of measures $\mathbb{Q}^0, \hat{\mathbb{P}}$, we have proved the following result.

THEOREM 4.2. *For h and λ satisfying Assumption 2.1, and $U(x) = \frac{x^{1-R}}{1-R}$, the time t price p for λ units of $h(S_T)$ delivered at time T, given a current wealth x, is*

$$p(t, x, s; \lambda) = p = \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2} x \hat{\mathbb{E}}[M]_T + o(\lambda^2)$$
$$= \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2} \frac{\eta^2}{x} (1-\rho^2) \hat{\mathbb{E}} \int_t^T \frac{S_u^2 (C^S)_u^2}{(X_u^0/x)^2} du + o(\lambda^2)$$
$$= \lambda \mathbb{E}_t^0 h(S_T) - \lambda^2 \frac{R}{2} \eta^2 (1-\rho^2) \mathbb{E}^0 \int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} du + o(\lambda^2). \tag{4.19}$$

Note that when $\rho = 1$, $\delta = 0$, we recover the price in the complete market case (3.2).

EXAMPLE 4.1 CONTINUED. We wish to calculate the price of the call with $h(S_T) = (S_T - \bar{K})^+$. From (4.19)

$$p = \lambda \mathbb{E}_t^0 (S_T - \bar{K})^+ - \lambda^2 \frac{R}{2} \eta^2 (1-\rho^2) \mathbb{E}^0 \int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} \, du + o(\lambda^2). \tag{4.20}$$

The first term is simple, we have a closed-form expression in (4.7). The second term is more involved and we simulate this. The simulation results gave standard errors of less than 0.2%.

Note that if we fix δ the second-order term is increasing in ρ. The minimal martingale measure term is unchanged with ρ, so the price in (4.20) is increasing in ρ. This is consistent with the idea that as the correlation approaches 1, the traded asset gives a better hedge and the position is less risky. The agent is thus willing to pay more for the claim.

A plot of the second-order term in (4.20) is given in Figure 4.3. Parameters used are: $\lambda = 0.01, s_0 = 100, T = 1, \rho = 0.8, R = 0.5, \mu = 0.04, \eta = 0.30, \sigma = 0.35, \quad \delta = 0$, $x_0 = 500$, and $0 \le \bar{K} \le 200$. In both Figures 4.3 and 4.4 we plot the second-order term divided by λ^2:

$$\frac{R}{2} \eta^2 (1-\rho^2) \mathbb{E}^0 \int_t^T \frac{S_u^2 (C^S)_u^2}{X_u^0} \, du, \tag{4.21}$$

or equivalently the second-order term with $\lambda = 1$. In Figure 4.3, this is about 0.07 for $\bar{K} = 100$. For comparison, the first-order term (with $\lambda = 1$ and for $s_0 = 100$, $\bar{K} = 100$, say) is 11.924. Thus the second term is about 0.6% of the first. Of course, if we use a larger value for R, we would get a larger ratio here. To obtain the comparison for different λ, we can simply multiply this percentage by λ.

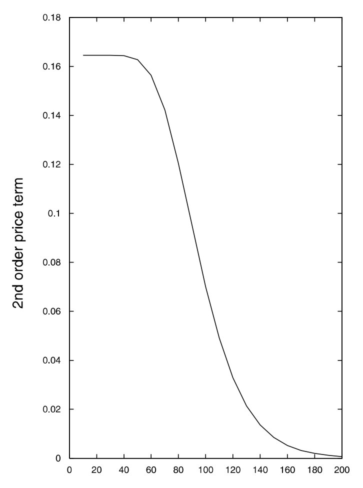

FIGURE 4.3. The second-order term of the reservation price of the claim $(S_T - \bar{K})^+$ for $0 \le \bar{K} \le 200$, as given in (4.21). Utility is power law. Parameter values are $\lambda = 0.01$, $s_0 = 100$, $T = 1$, $R = 0.5$, $\mu = 0.04$, $\rho = 0.8$, $\eta = 0.30$, $\sigma = 0.35$, $\delta = 0$, and $x_0 = 500$.

Returning to Figure 4.3, as $\bar{K} \to 0$ the payoff approaches S_T and we recover the linear case of Henderson and Hobson (2000). In this case we have a simple second-order term and it has greatest effect on the price. As $\bar{K} \to 200$ the option will not pay out and hence the second-order term tends to zero.

For a strike of $\bar{K} = 100$ the second-order term in (4.20) is graphed in Figure 4.4 for varying values of the asset price S. Note that this appears to look like the price of a call option. If S is large, the option is in-the-money and the price correction term is large (as it is likely the claim will pay out). Alternatively, if S is low, the option is unlikely to pay out and therefore the problem is close to the Merton problem with no option. With the parameter $\delta = 0$ note that the first term is simply the complete (risk neutral) price.

EXAMPLE 2 CONTINUED. For $h(S_T) = S_T^2$ the price can be calculated explicitly.

$$p = \lambda s^2 e^{(2\delta+\eta^2)(T-t)} - 2\lambda^2 R\eta^2(1-\rho^2)\frac{s^4}{x}\left(\frac{e^{(3\eta^2+6\delta+\frac{\mu^2}{\sigma^2R^2})(T-t)} - 1}{3\eta^2 + 6\delta + \frac{\mu^2}{\sigma^2R^2}}\right) + o(\lambda^2).$$

Returning to some more general remarks on Theorem 4.2, if we consider the reservation price for the random payment of $\lambda h(S_T)$, and convert it into a unit price, we find

$$\frac{p}{\lambda} = \mathbb{E}_t^0 h(S_T) - \lambda\frac{R}{2}\frac{\eta^2}{x}(1-\rho^2)\hat{\mathbb{E}}\int_t^T \frac{S_u^2 (C^S)_u^2}{(X_u^0/x)^2}\,du + o(\lambda). \tag{4.22}$$

The *marginal price of a derivative* is the price at which diverting a little money into the derivative at time zero has a neutral effect on the achievable utility. This is given by

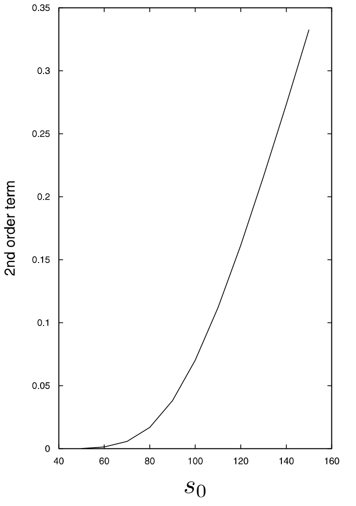

FIGURE 4.4. The second-order term of the reservation price of the claim $(S_T - \bar{K})^+$ for $50 \le s_0 \le 150$, as given in (4.21). Utility is power law. Parameter values are $\lambda = 0.01, T = 1, \bar{K} = 100, R = 0.5, \mu = 0.04, \rho = 0.8, \eta = 0.30, \sigma = 0.35, \quad \delta = 0,$ and $x_0 = 500$.

$$\lim_{\lambda \downarrow 0} \frac{p}{\lambda} = \mathbb{E}_t^0 h(S_T). \tag{4.23}$$

Of note is that the marginal price is independent of the risk-aversion parameter R. This is an example of a general result which states that the marginal price is independent of the utility function; see Davis (1999) or Hobson (1994, Thm. 1). Further, the marginal price is the expected payoff under the minimal martingale measure $\mathbb{Q}^0$. Importantly, and unlike in the complete market scenario of Section 3, the marginal price the agent is prepared to pay for $h(S_T)$ depends on the drift μ of the traded asset.

As we remarked above, conclusions about the marginal price the agent is prepared to pay for the asset are independent of the agent's utility. However the reservation price for a nonnegligible quantity of nontraded asset does depend on the utility as expressed in the λ^2 term in the expansion (4.19). Note that the correction term to order λ^2 is negative since $[M]_T \geq 0$, using (4.19). This is because utilities are concave, so the agent is prepared to pay a lower (unit) price for larger quantities.

Pricing in this model is nonlinear in that the reservation price the agent is prepared to pay for $2x$ units of the claim is less than twice the price for x units. We now examine the unit price in (4.22). Putting $\lambda > 0$ gives a buy price of

$$\mathbb{E}_t^0 h(S_T) - \phi$$

and likewise $\lambda < 0$ gives a sell price of

$$\mathbb{E}_t^0 h(S_T) + \phi,$$

where $\phi = |\lambda| \frac{R}{2} x \hat{\mathbb{E}}[M]_T > 0$ using the equivalent formulation in (4.19). Perhaps this difference 2ϕ can be seen as a proxy for the bid–ask spread and on the option and might be useful for comparing two claims on nontraded assets.

We can see also from (4.22) that if initial wealth increases, with fixed S and λ, then the holding in derivatives is diluted and the price is larger. For the power law utility, the absolute risk aversion $-\frac{U''(x)}{U'(x)} = \frac{R}{x}$ is a decreasing function of wealth and thus the higher the wealth, the higher the price the agent is willing to pay. In contrast, for the exponential utility considered in the next section, we will see that the price is independent of wealth.

5. THE INCOMPLETE CASE: EXPONENTIAL UTILITY

The second utility function we focus on is the exponential utility $U(x) = -\frac{1}{\gamma} e^{-\gamma x}$ (see, e.g., Hodges and Neuberger 1989; Svensson and Werner 1993; Duffie and Jackson 1990; Davis 2000; Cvitanic et al. 2001; Delbaen et al. 2000; and Rouge and El Karoui 2000). This utility has constant absolute risk aversion, and its popularity is derived in part from its separability properties.

For this utility, we are able to obtain an explicit representation for the price. This is an advantage compared to the power utility where we used expansions. However, to compare the results to the power utility we will also calculate an expansion for the price.

Again, let $V(t, X_t, S_t; \lambda)$ be the value function for the agent who at time t has wealth X_t and who will receive $\lambda h(S_T)$ at time T. Here we take $U(x) = -\frac{1}{\gamma} e^{-\gamma x}$. Then

$$V(t, X_t, S_t; \lambda) = \sup_\theta \mathbb{E}_t U(X_T + \lambda h(S_T)) = -\frac{1}{\gamma} e^{-\gamma X_t} \inf_\theta \mathbb{E}_t (e^{-\gamma \int_t^T \theta_u (dP_u/P_u) - \gamma\lambda h(S_T)})$$
$$= -\frac{1}{\gamma} e^{-\gamma X_t} g(T - t, \log S_t),$$

where $g(0, z) = e^{-\lambda\gamma h(e^z)}$. Using the fact that V is a supermartingale for any strategy, and a martingale for the optimal strategy, we find that g solves the PDE

$$\dot{g} - v g_z + \frac{1}{2}\eta^2 g_z - \frac{1}{2}\eta^2 g_{zz} + \frac{1}{2}\frac{(\sigma\eta\rho g_z + \mu g)^2}{\sigma^2 g} = 0.$$

An expression for the optimal hedge θ^* can be given as

$$\theta^* = \frac{(\mu g + \sigma g_z \eta\rho)}{g\gamma\sigma^2}.$$

When there is no option, this collapses to the Merton strategy given in (2.7). We follow Hobson (1994) and the example in Henderson and Hobson (2000) to solve this equation. This trick is also used in many of Zariphopolou's papers (e.g., see Zariphopoulou 2001) and converts a nonlinear PDE into a linear one. If we set $g(\tau, y) = e^{\alpha\tau} G(\tau, y + \beta\tau)^b$ then we find that G solves

$$b\dot{G} - \frac{1}{2}\eta^2 b G_{yy} - \frac{1}{2}\eta^2 (b(b-1) - \rho^2 b^2)\frac{G_y^2}{G}$$
$$+ \left[b(\beta + \frac{1}{2}\eta^2 - v + \frac{\eta\rho\mu}{\sigma}) \right] G_y + \left[\alpha + \frac{\mu^2}{2\sigma^2} \right] G = 0.$$

Choosing

$$b = \frac{1}{(1-\rho^2)}, \qquad \alpha = -\frac{\mu^2}{2\sigma^2}, \qquad \beta = v - \frac{\eta\rho\mu}{\sigma} - \frac{1}{2}\eta^2 = \delta - \frac{1}{2}\eta^2,$$

we find that G solves

$$\dot{G} = \frac{1}{2}\eta^2 G_{yy}.$$

This is the heat equation, with solution

$$G(\tau, y) = \int_{-\infty}^{\infty} G(0, y + z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz,$$

so

$$g(\tau, y) = e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\int_{-\infty}^{\infty} G(0, y + (\delta - \frac{1}{2}\eta^2)\tau + z) \frac{e^{-\frac{z^2}{2\eta^2\tau}}}{\eta\sqrt{2\pi\tau}} dz \right]^{1/(1-\rho^2)}$$
$$= e^{-\frac{\mu^2\tau}{2\sigma^2}} \left[\mathbb{E}(G(0, y + (\delta - \frac{1}{2}\eta^2)\tau + \eta\sqrt{\tau}N)) \right]^{1/(1-\rho^2)},$$

where N is a standard normal random variable. Using the boundary condition $G(0, y) = e^{-(\lambda\gamma/b)h(e^y)} = e^{-\lambda\gamma(1-\rho^2)h(e^y)}$:

$$V(t, X_t, S_t; \lambda) = -\frac{1}{\gamma} e^{-\gamma X_t - \frac{\mu^2}{2\sigma^2}(T-t)} \times \left[\mathbb{E}\left(\exp(-\lambda\gamma(1-\rho^2) h(S_t e^{\delta(T-t)} e^{\eta\sqrt{T-t}N - \frac{1}{2}\eta^2(T-t)})) \right) \right]^{1/(1-\rho^2)}.$$

Hence we have found the solution provided the expectation above is finite. Note that this is not always true for some important cases such as the short call. To see this, note that $\mathbb{E}U(-(S_T - K)^+) = -\infty$ and the term in the value function above is essentially $\mathbb{E}^0 U(h(S_T))$, so it is also infinite for exponential utility. However, if Assumption 2.1 holds, then the above expectation is finite.

Using the utility indifference argument to obtain the price, we have proved the following result.

THEOREM 5.1. *For h and λ satisfying Assumption 2.1, and $U(x) = -\frac{1}{\gamma} e^{-\gamma x}$, the reservation price for $\lambda h(S_T)$ is*

$$p^e = -\frac{1}{\gamma(1-\rho^2)} \log \mathbb{E}^0 \left[\exp\left\{ -\lambda\gamma(1-\rho^2) h(S_T) \right\} \right]. \tag{5.1}$$

This is an explicit expression for the price under the exponential utility function. Note also that the above price is wealth independent. As mentioned in Section 1, this may not always be desirable for applications.

We now want to find an expansion in terms of small λ which we can compare to our results of the previous section obtained using the power law utility. The expansion is

$$p^e = \lambda \mathbb{E}^0 h(S_T) - \frac{\gamma}{2} \lambda^2 (1-\rho^2) [\mathbb{E}^0 [h(S_T)]^2 - [\mathbb{E}^0 h(S_T)]^2] + O(\lambda^3),$$

which is equivalent to the price found in Davis (2000), although he did not have the representation (5.1). Despite the fact that Davis uses an expansion in ρ, the results are equivalent as both parameters appear in the exponential in (5.1). We find that to leading order the price is precisely the expected value of the claim under the minimal martingale measure. Hence we concentrate on the correction term. Note that the second-order correction is linear in the risk aversion parameter γ. We equate the local absolute risk aversion in the power law and exponential utility models to compare the results. This involves identifying the parameter γ with R/x_0. The price becomes

$$p^e = \lambda \mathbb{E}^0 h(S_T) - \frac{R}{2x_0} \lambda^2 (1-\rho^2) [\mathbb{E}^0 [h(S_T)]^2 - [\mathbb{E}^0 h(S_T)]^2] + O(\lambda^3). \tag{5.2}$$

EXAMPLE 4.1 CONTINUED. We can evaluate (5.2) for the call option with $\lambda > 0$. Calculations give

$$p^e = \lambda \mathbb{E}^0 (S_T - \bar{K})^+ - \frac{R}{2} \frac{\lambda^2}{x} (1-\rho^2) [\mathbb{E}^0 (S_T - \bar{K})^2 I_{(S_T > \bar{K})} - (\mathbb{E}^0 (S_T - \bar{K})^+)^2], \tag{5.3}$$

where

$$\mathbb{E}^0 (S_T - \bar{K})^2 I_{(S_T > \bar{K})} = s^2 e^{(2\delta + \eta^2)(T-t)} N(d_+ + \eta\sqrt{T-t}) + \bar{K}^2 N(d_-) - 2\bar{K} e^{\delta(T-t)} s N(d_+) \tag{5.4}$$

and $\mathbb{E}_t^0 (S_T - \bar{K})^+$ and $d_\pm$ are given in (4.7).

Figure 5.1 graphs the second-order term in (5.3) and the power law utility price (4.20) over values of the strike $\bar{K}$. The numbers are extremely close (since we equated local absolute risk aversion); however, the exponential utility gives a larger correction over the whole range. This is worthy of further investigation.

Now we consider the dependence of the reservation price on the risk aversion parameter with surprising results. Comparing the forms of the exponential and power utilities in the limit as the risk aversion parameter tends to zero, we see that

$$\lim_{\gamma \downarrow 0} \frac{1 - e^{-\gamma x}}{\gamma} = x = \lim_{R \downarrow 0} \frac{x^{1-R}}{1-R},$$

but in the former case the domain of definition is $\mathbb{R}$ whereas in the latter it is $\mathbb{R}^+$. Hence there is no reason to expect identical behavior in the limit as risk aversion decreases to zero.

In Figure 5.2 we graph the second-order price term as a function of R. Note that we plot $-(p^e - \lambda \mathbb{E}^0(S_T - \bar{K})^+)/\lambda^2$ and the equivalent price for the power law case of Section 4. The broken line uses the exponential utility; the solid line uses the power utility. As expected, as risk aversion increases, the reservation price falls. The agent is willing to pay less for the nontraded stock as she becomes less tolerant of risk. However, surprisingly, this relationship reverses for the power utility as R gets very small. For the parameter choices in Figure 5.1 this happens for R below approximately 0.1. As R decreases below this value the agent is prepared to pay less for the risky nontraded asset despite becoming more tolerant of risk.

Recall the optimal strategy given in Theorem 4.1 for the power law utility:

$$\theta^*(t, x, s; \lambda) = \frac{\mu}{\sigma^2 R}(x + \lambda C_t) - \frac{\eta\rho}{\sigma} \lambda s C_t^S + o(\lambda).$$

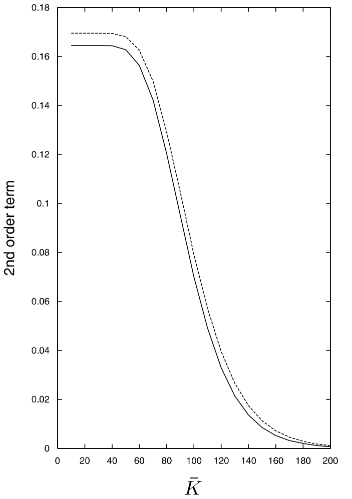

FIGURE 5.1. The second-order term of the reservation price of the claim with payoff $(S_T - \bar{K})^+$ for $0 \le \bar{K} \le 200$. The lower line uses the power law utility, the higher line uses the exponential utility. Parameter values are $\lambda = 0.01,\ s_0 = 100,\ T = 1,\ R = 0.5,\ \mu = 0.04,\ \rho = 0.8,\ \eta = 0.30,\ \sigma = 0.35,\ \delta = 0$, and $x_0 = 500$.

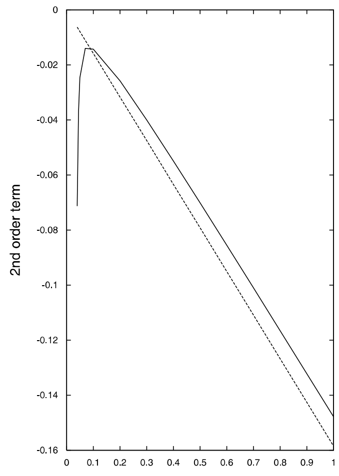

FIGURE 5.2. The second-order term of the reservation price of the claim with payoff $(S_T - \bar{K})^+$ for $0.03 \leq R \leq 1$. Parameter values are $\lambda s_0 = 1.0$, $T = 1$, $\bar{K} = 100$, $\mu = 0.04$, $\rho = 0.8$, $\eta = 0.30$, $\sigma = 0.35$, $\delta = 0$, and $x_0 = 500$. The broken line uses the exponential utility; the solid line uses the power utility.

As $R \downarrow 0$, both terms become large, and fluctuations in the value of P and S are magnified into large fluctuations in the final wealth. Wealth can be close to zero and hence assigned a low utility value, resulting in a lower price.

EXAMPLE 4.2 CONTINUED. For $h(S_T) = S_T^2$

$$p^e = \lambda \mathbb{E}^0 S_T^2 - \frac{R}{2}\frac{\lambda^2}{x}(1-\rho^2)[\mathbb{E}^0 S_T^4 - (\mathbb{E}^0 S_T^2)^2],$$

where $\mathbb{E}^0 S_T^4 = s_0^4 e^{4(\delta + \frac{3}{2}\eta^2)(T-t)}$.

6. CONCLUSION

This paper has studied the utility maximization pricing of claims on nontraded assets, using a close asset to hedge. We extend the Merton investment model to include a nontraded asset. The problem is how to price and hedge payoff $\lambda h(S_T)$ when trading in S is not permissible. This is an incomplete markets problem and the solution depends on the utility function chosen. It is a common practice to use substitute assets to avoid high transactions costs and the results of this paper can be applied to hedge basket options with an index, illiquid shares, or futures contracts, among others.

Our model considers agents with either constant relative risk aversion (power law utility) or constant absolute risk aversion (exponential utility). We use a popular utility argument to find a price in this setup. The reservation price is the amount the investor requires so she is indifferent between the investment problem with zero endowment and the problem with the additional opportunity to buy the claim.

Under the assumption of constant relative risk aversion and exponential Brownian motion, the techniques of duality were used to approximate the hedge and obtain the value function to order λ^2. The results hold under the assumption that the money value in the nontraded asset is small compared to wealth. The use of an expansion enables us to examine the effect of parameter changes on the results.

The problem becomes much simpler when the exponential utility function is chosen. Wealth can be factored out, reducing the PDE to a linear equation that can be solved directly to obtain an explicit price. This has the advantage of enabling an analysis of the sensitivity of the results to the parameters. The price using exponential utility is wealth independent, unlike that obtained using the power utility. Wealth independence may be a desirable feature in some applications, but not so good for others, such as executive stock options.

As expected, the reservation price depends on the drifts of the assets and the level of risk aversion. The examples of a call and power option were analyzed, concentrating on the call. We have shown that the reservation price is increasing in correlation, hence the agent is willing to pay more when she is more likely to have a reasonable hedge. Under our choice of parameters, the second-order correction term was about 0.6% of the first-order term when $\lambda = 1$. Comparison of the reservation price with the exponential utility price showed that, for the call, the corrections were very close. This is reassuring since the price is not that sensitive to the form of the utility, provided the risk aversions match locally. However, they behave very differently as a function of risk aversion as risk aversion tends to zero. This is explained by the fact that the power utility is defined for positive wealth whereas the exponential utility also allows wealth to go negative. If R is close to zero, the wealth could be close to zero which gives very different behavior for the two utilities. Therefore, some care needs to be exercised in choosing between the two utility functions.

A shortcoming of this analysis and others in the literature is that it cannot be used to price a short call position due to the unbounded nature of the payoff. The utility approach is not suitable for short calls and further work could be done in this area.

Finally, an area where these results can be applied is that of executive stock options. These are options on the stock of the company, and are given to executives as part of their compensation package. However, frequently executives are not permitted to trade away the risk using the stock or derivatives on the stock, so that they are essentially receiving options on a nontraded asset; see Henderson (2001).

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