A Multidimensional Exponential Utility Indifference Pricing Model with Applications to Counterparty Risk

Vicky Henderson[†,♯] and Gechun Liang[‡,♯]

[†]Department of Statistics, University of Warwick, Coventry, CV4 7AL, U.K.,
Vicky.Henderson@warwick.ac.uk
[‡]Department of Mathematics, King's College London, London, WC2R 2LS, U.K.,
gechun.liang@kcl.ac.uk
[♯]Oxford-Man Institute, University of Oxford, Oxford, OX2 6ED, U.K.

May 28, 2014

Abstract

This paper considers exponential utility indifference pricing for a multidimensional non-traded assets model subject to inter-temporal default risk, and provides a semigroup approximation for the utility indifference price. The key tool is the splitting method. We apply our methodology to study the counterparty risk of derivatives in incomplete markets.

Keywords: utility indifference pricing, reaction-diffusion PDE with quadratic gradients, splitting method, counterparty risk.

Mathematics Subject Classification (2010): 91G40, 91G80, 60H30.

The authors thank the Oxford-Man Institute for partial support of this research. In addition, the authors would like to thank Damiano Brigo, Rama Cont, David Hobson, Monique Jeanblanc, Lishang Jiang, Eva Lütkebohmert, Andrea Macrina, Shige Peng, Martin Schweizer, Xingye Yue, and Thaleia Zariphopoulou for their helpful discussions.

1 Introduction

The purpose of this article is to consider *exponential* utility indifference pricing in a multidimensional non-traded assets setting subject to inter-temporal default risk, which is motivated by our study of counterparty risk of derivatives in incomplete markets. Our interest is in pricing and hedging derivatives written on assets which are not traded. The market is incomplete as the risks arising from having exposure to non-traded assets cannot be fully hedged. We take a utility indifference approach whereby the utility indifference price for the derivative is the cash amount the investor is willing to pay such that she is no worse off in expected utility terms than she would have been without the derivative.

There has been considerable research in the area of *exponential* utility indifference valuation, but despite the interest in this pricing and hedging approach, there have been relatively few explicit formulas derived. The well known *one dimensional non-traded assets model* is an exception and in a Markovian framework with a derivative written on a single non-traded asset, and partial hedging in a financial asset, Henderson and Hobson [22], Henderson [20], and Musiela and Zariphopoulou [39] used the Cole-Hopf transformation (*or distortion power*) to linearize the non-linear PDE for the value function. This trick results in an explicit formula for the exponential utility indifference price. Subsequent generalizations of the model from Tehranchi [43], Frei and Schweizer [16] and [17] showed that the exponential utility indifference value can still be written in a closed-form expression similar to that known for the Brownian setting, although the structure of the formula can be much less explicit. On the other hand, Davis [14] used the duality to derive an explicit formula for the optimal hedging strategy (see also Monoyios [37]), and Becherer [3] showed that the dual pricing formula exists even in a general semimartingale setting.

As soon as one of the assumptions made in the one dimensional non-traded assets model breaks down, explicit formulas are no longer available. For example, if the option payoff depends also on the traded asset, Sircar and Zariphopoulou [41] developed bounds and asymptotic expansions for the exponential utility indifference price. In an energy context, we may be interested in partially observed models and need filtering techniques to numerically compute expectations (see Carmona and Ludkovski [11] and Chapter 7 of [10]). If the utility function is not exponential, Henderson [20] and Kramkov and Sirbu [32] developed expansions in small quantity for the utility indifference price under power utility.

In this paper, we study exponential utility indifference valuation in a multidimensional setting subject to inter-temporal default risk with the aim of developing a pricing methodology. The main economic motivation for us to develop the multidimensional framework is to consider the counterparty default risk of options traded in over the counter (OTC) markets, often called *vulnerable options*. The recent credit crisis has brought to the forefront the importance of counterparty default risk as there were numerous high profile defaults leading to counterparty losses. In response, there have been many recent studies (see, for example, Bielecki et al [4] and Brigo et al [8, 9]) addressing in particular the counterparty risk of credit default swaps (CDS). In contrast, there is relatively little recent work on counterparty risk for other derivatives, despite OTC options being a sizable fraction of the OTC derivatives market.[1] The option holder faces both price risk arising from the fluctuation of the assets underlying her option and counterparty default risk that the option writer

[1] In fact, OTC options comprised about 10% of the $600 trillion (in terms of notional amounts) OTC derivatives market at the end of June 2010 whilst the CDS market was about half as large at around $30 trillion.

does not honor her obligations. Default occurs either when the assets of the counterparty are below its liabilities at maturity (*following the structural approach*) or when an exogenous random event occurs (*following the reduced-form approach*), so inter-temporal default is considered. In our setting, the assets of the counterparty and the assets underlying the option may be non-traded and thus a multidimensional non-traded assets model naturally arises.

Our use of the utility indifference approach is motivated by its recent use in credit risk modeling where the concern is the default of the reference name rather than the default of the counterparty. Utility based pricing has also been utilized by Bielecki and Jeanblanc [5], Sircar and Zariphopolou [42] and recently Jiao et al [27] [28] in an intensity based setting. Several authors have applied it in modeling of defaultable bonds where the problem remains one dimensional, see in particular Leung et al [33], Jaimungal and Sigloch [26], and Liang and Jiang [34]. In contrast, options subject to counterparty risk are a natural situation where two or more dimensions arise.

Our first contribution is the derivation of a reaction-diffusion partial differential equation (PDE) to characterize the utility indifference price in our multidimensional setting, where we do not rely on dynamic programming principle. Instead, we consider the associated utility maximization problems for utility indifference valuation from a risk-sensitive control perspective. We first transform our utility maximization problems into risk-sensitive control problems, and then employ a PDE comparison principle to derive the pricing PDE for the utility indifference price. See also Theorem 2.3 of Henderson and Liang [24] where we derived the backward stochastic differential equation (BSDE) representation for the utility indifference price in a non-Markovian setting.

Our main contribution is to develop a semigroup approximation for the pricing PDE by the splitting method. In our multidimensional setting, the Cole-Hopf transformation (as in the one dimensional model) cannot be applied directly since the coefficients of the quadratic gradient terms do not match, and due to the existence of the inter-temporal payment term. Motivated by the idea of the splitting method (*or fractional step, prediction and correction*) in numerical analysis, we split the pricing equation into two semilinear PDEs with quadratic gradients and an ordinary differential equation (ODE) with Lipschitz coefficients, such that the Cole-Hopf transformation can be applied to linearize both PDEs, and the Picard iteration can be used to linearly approximate the ODE.

The idea of splitting in our setting is as follows. The time derivative of the pricing equation depends on the sum of semigroup operators corresponding to different factors. For each subproblem corresponding to each semigroup there might be an effective way providing solutions, but for the sum of the semigroups, there may not be an accurate method. The application of splitting method means that we treat the semigroup operators separately. We prove that when the mesh of the time partition goes to zero, the approximated price will converge to the utility indifference price, relying on the *monotone scheme* argument by Barles and Souganidis [2]: any *monotone, stable and consistent* numerical scheme converges to the correct solution provided there exists a comparison principle for the limiting equation. The difficult part of applying the monotone scheme to our problem is the verification of consistency. This is overcome by the *pseudo linear pricing rule* for the utility indifference price introduced by Henderson and Liang [24]. In contrast to the nonlinear expectation, where dominated convergence theorem may not hold, the representation under the pseudo linear pricing rule is linear, so dominated convergence can be employed to verify the commute of limiting processes.

Our third contribution is the application of the splitting method to compute prices of derivatives on a non-traded asset and where the derivative holder is subject to non-traded counterparty default risk. In contrast to the complete market Black-Scholes style formulas obtained by Johnson and Stulz [29], Klein [30] and Klein and Inglis [31], we show the significant impact that non-tradeable risks have on the valuation of vulnerable options and the role played by partial hedging. In particular, our numerical illustrations quantify the effect of non-tradeable price and default risk on option prices. The magnitude of the potential discounts to the complete market price depends upon the likelihood of default and if default is likely, the discount can be extremely high. We also observe that put values can decrease with maturity (in absence of dividends) in situations where the risk of default is significant.

Splitting methods have been used to construct numerical schemes for PDEs arising in mathematical finance (see the review of Barles [1], and Tourin [44] with the references therein). Recently, Nadtochiy and Zariphopoulou [40] applied splitting to the *marginal* Hamilton-Jacobi-Bellman (HJB) equation arising from optimal investment in a two-factor stochastic volatility model with general utility functions. They show their scheme converges to the unique viscosity solution of the limiting equation. Whilst they also apply splitting to an incomplete market problem, their focus is to deal with the lack of a verification theorem in their setting. In contrast, we propose a multidimensional model subject to inter-temporal default risk with constant coefficients and exponential utility. Our contributions include proposing a splitting approach for utility indifference pricing in a multidimensional non-traded assets model with inter-temporal default risk, identifying how to split the resulting pricing PDE and implementing our approximation to consider counterparty risk of derivatives. Other recent works of Halperin and Itkin [18, 19] propose the use of splitting methods to price options on a single illiquid bond via mixed static-dynamic hedging. Our model is instead desgined for multiple non-traded assets, which is necessary for our treatment of counterparty risk in a hybrid structural-reduced form setting. In further contrast to [18, 19], we prove convergence of our splitting method using monotone scheme arguments of Barles and Souganidis [2] and provide a numerical implementation to study counterparty risk.

The paper is organized as follows: In Section 2, we present our multidimensional exponential utility indifference pricing model, and propose a splitting method to solve the pricing equation. In Section 3, we apply the method to study the counterparty risk. Section 4 concludes with the verification theorem of the pricing equation.

2 A Splitting Method for Utility Indifference Valuation

2.1 Model Setup

Let $\mathcal{W} = (W^1, \cdots, W^{n+2})$ be a $(n+2)$-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ satisfying the *usual conditions*, where $\mathcal{F}_t$ is the augmented σ-algebra generated by $(\mathcal{W}_u : 0 \leq u \leq t)$. The market consists of a set of observable but non-traded assets $\mathcal{S} = (S^1, \cdots, S^n)$, whose price processes are driven by

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i dW_t^i + \bar{\sigma}_i dW_t^{n+1} \tag{2.1}$$

for $i = 1, \cdots, n$, and a traded financial index P, whose price process is driven by

$$\frac{dP_t}{P_t} = \mu_P dt + \bar{\sigma}_P dW_t^{n+1} + \sigma_P dW_t^{n+2}. \tag{2.2}$$

The price of each non-traded asset S^i reflects exposure to the traded or market risk W^{n+1} through volatility $\bar{\sigma}_i$ and non-traded idiosyncratic risk W^i through idiosyncratic or undiversifiable volatility σ_i. We define the following parameters for the financial index P:

$$\begin{aligned}
\theta^P &= \frac{(\mu_P)^2}{(\sigma_P)^2}; \quad \bar{\theta}^P = \frac{(\mu_P)^2}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}; \\
\vartheta^P &= \frac{\mu_P \bar{\sigma}_P}{(\sigma_P)^2}; \quad \bar{\vartheta}^P = \frac{\mu_P \bar{\sigma}_P}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}; \\
\kappa^P &= \frac{(\bar{\sigma}_P)^2}{(\sigma_P)^2}; \quad \bar{\kappa}^P = \frac{(\bar{\sigma}_P)^2}{(\sigma_P)^2 + (\bar{\sigma}_P)^2}.
\end{aligned}$$

Assumption 2.1 *All the coefficients are constants, and there exists a risk-free bond or bank account with price $B_t = 1$ for $t \geq 0$.*

Our interest will be in pricing and hedging λ units of a contingent claim written on the non-traded assets $\mathcal{S}$ with maturity T. The payoff is delivered at either a random time τ or maturity T. The random time τ represents an inter-temporal default time, which is constructed in a canonical way. Let e be an independent exponential random variable on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then the random time τ is constructed as follows

$$\tau = \inf\left\{ s \geq 0 : \int_0^s a(\mathcal{S}_u) du \geq e \right\},$$

where $a : \mathbb{R}_+ \to \mathbb{R}_+$ is an intensity function depending on the price processes of the non-traded assets $\mathcal{S}$. The original Brownian filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is enlarged by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for $t \geq 0$ with $\mathcal{H}_t = \sigma(\{\tau \leq u\} : 0 \leq u \leq t)$. Hence τ is the first arrival time of a Cox process (*or doubly stochastic Poisson process*), which satisfies the following enlargement of filtration property: The stochastic process $\mathcal{W}$ is Brownian motion under both filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\mathcal{G}_t\}_{t\geq 0}$ (see Chapter 8 of Bielecki and Rutkowski [6] for the details).

If such a random time τ happens before maturity T, i.e. $\tau \leq T$, the payoff at τ is $R(\mathfrak{C}^\lambda(\mathcal{S}_\tau, \tau))$, where $R : \mathbb{R}_+ \to \mathbb{R}_+$ is a recovery function, and $\mathfrak{C}^\lambda : \mathbb{R}_+^n \times [0, T] \to \mathbb{R}_+$ is the solution of the PDE (2.10). In Theorem 2.4 we shall show that $\mathfrak{C}^\lambda(\cdot, \cdot)$ is actually the utility indifference value function of the contingent claim, so we consider the fractional recovery of market value. If $\tau > T$, the payoff at T is $\lambda g(\mathcal{S}_T)$ where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is some payoff function. Therefore, the total payoff for λ units of this contingent claim is

$$\mathbf{1}_{\{\tau \leq T\}} R(\mathfrak{C}^\lambda(\mathcal{S}_\tau, \tau)) + \mathbf{1}_{\{\tau > T\}} \lambda g(\mathcal{S}_T).$$

Assumption 2.2 *The recovery function $R(\cdot)$, the payoff function $g(\cdot)$ and the intensity function $a(\cdot)$ are Lipschitz continuous. Moreover, $R(x) \leq x$ for $x \geq 0$, and both $g(\cdot)$ and $a(\cdot)$ are bounded.*

The model is applicable in many situations. It might be that these assets are (i) not traded at all, or (ii) that they are traded illiquidly, or (iii) that they are in fact liquidly traded but the investor concerned is not permitted to trade them for some reason. Our main application is to the counterparty risk of derivatives where the final payoff depends upon both the value of the counterparty's assets and the asset underlying the derivative itself. A second potential area of application is to residual or basis risks arising when the asset(s) used for hedging differ from the assets underlying the contract in question (see Davis [14]). Typically this arises when the assets underlying the derivative are illiquidly traded (case (ii) above) and standardized futures contracts are used instead. Contracts may involve several assets, for example, a spread option with payoff $(K - S_T^1 - S_T^2)^+$ or a basket option with payoff $(K - S_T^1 - \cdots - S_T^n)^+$. Such contracts frequently arise in applications to commodity, energy, and weather derivatives. Finally, a one dimensional example of the situation in (iii) is that of employee stock options.

On the other hand, the model also allows for inter-temporal default risk, and the recovery at the prepayment time is mark-to-market value. Hence, the model is well suited to counterparty credit risk, where the inter-temporal payment due to default usually depends on the market value of the contract. Such an inter-temporal default is modeled in a reduced-form, so the corresponding counterparty credit risk model is a hybrid between structural approach and reduced-form approach. Other applications include optimal investment problems with uncertain time horizon (see Blanchet-Scalliet et al [7]), and modeling the prepayment risk of mortgage-backed securities (see Zhou [45]).

Our approach is to consider the utility indifference valuation for such contingent claims. For this we need to consider the optimization problem for the investor both with and without the option. The investor has initial wealth $x \in \mathbb{R}$, and is able to trade the financial index with price P_t (and riskless bond with price 1). This will enable the investor to partially hedge the risks she is exposed to via her position in the claim. Depending on the context, the financial index may be a stock, commodity or currency index, for example.

The holder of the option has an exponential utility function with respect to her terminal wealth:

$$U_T(x) = -e^{-\gamma x} \quad \text{for } \gamma \geq 0.$$

For any time $t \in [0, T]$, the investor holds λ units of the contingent claim, whose price is denoted as $\mathfrak{C}_t^\lambda$ and is to be determined, and invests her remaining wealth $x - \mathfrak{C}_t^\lambda$ in the financial index P. The investor will follow an admissible trading strategy:

$$\pi \in \mathcal{A}_{\mathcal{F}}[t, T] = \{\pi : \pi \text{ is } \mathbb{F}\text{-adapted and uniformly bounded}\},$$

which results in the wealth on the event $\{\tau > t\}$:

$$X_s^{x-\mathfrak{C}_t^\lambda}(\pi) = x - \mathfrak{C}_t^\lambda + \int_t^s \frac{\pi_s}{P_s} dP_s. \tag{2.3}$$

The investor will optimize over such strategies to choose an optimal $\bar{\pi}^*$ by maximizing her expected terminal utility:

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(\mathbf{1}_{\{t < \tau \leq T\}} \left(X_\tau^{x-\mathfrak{C}_t^\lambda}(\pi) + R(\mathfrak{C}^\lambda(\mathcal{S}_\tau, \tau)) \right) + \mathbf{1}_{\{\tau > T\}} \left(X_T^{x-\mathfrak{C}_t^\lambda}(\pi) + \lambda g(\mathcal{S}_T) \right) \right)} | \mathcal{G}_t \right]. \tag{2.4}$$

To define the utility indifference price for the option, we also need to consider the optimization problem for the investor without the option. Her wealth equation is the same as (2.3) but starts from initial wealth x and she will choose an optimal π^* by maximizing

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi)} | \mathcal{G}_t \right]. \tag{2.5}$$

Since the payoff of the optimal portfolio problem (2.5) is $\mathbb{F}$-adapted, (2.5) is equivalent to the standard Merton problem on the event $\{\tau > t\}$:

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi)} | \mathcal{F}_t \right],$$

where the filtration is restricted to $\{\mathcal{F}_t\}_{t \geq 0}$.

The utility indifference price for the option is the cash amount that the investor is willing to pay such that she is no worse off in expected utility terms than she would have been without the option. For a general overview of utility indifference pricing, we refer to the recent monograph edited by Carmona [10] and especially the survey article by Henderson and Hobson [23] therein.

Definition 2.3 *The utility indifference price $\mathfrak{C}_t^\lambda$ of λ units of the derivative with the payoff*

$$\mathbf{1}_{\{t < \tau \leq T\}} R(\mathfrak{C}^\lambda(\mathcal{S}_\tau, \tau)) + \mathbf{1}_{\{\tau > T\}} \lambda g(\mathcal{S}_T)$$

at time $t \in [0,T]$ is defined by the solution to

$$\begin{aligned} & \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} E^{\mathbf{P}} \left[-e^{-\gamma \left(\mathbf{1}_{\{t<\tau\leq T\}} \left(X_\tau^{x-\mathfrak{C}_t^\lambda}(\pi) + R(\mathfrak{C}^\lambda(\mathcal{S}_\tau,\tau)) \right) + \mathbf{1}_{\{\tau>T\}} \left(X_T^{x-\mathfrak{C}_t^\lambda}(\pi) + \lambda g(\mathcal{S}_T) \right) \right)} | \mathcal{G}_t \right] \\ = & \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} E^{\mathbf{P}} \left[-e^{-\gamma X_T^x(\pi)} | \mathcal{G}_t \right]. \end{aligned} \tag{2.6}$$

The hedging strategy for λ units of the derivative at time t on the event $\{\tau > t\}$ is defined by the difference in the optimal trading strategies $\bar{\pi}_t^* - \pi_t^*$.

2.2 Utility Indifference Price

Our main result in this subsection is to show that the utility indifference price is given by $\mathfrak{C}_t^\lambda = \mathbf{1}_{\{\tau>t\}} \mathfrak{C}^\lambda(\mathcal{S}_t, t)$, where $\mathfrak{C}^\lambda(\mathcal{S}_t, t)$ is the solution of the reaction-diffusion PDE (2.10) with quadratic gradients, so it can be interpreted as the (pre-default) utility indifference value function. The function $\mathfrak{C}^\lambda(\mathcal{S}_t, t)$ is the main object that we are working on. Define the following operators:

$$\mathbf{L}^1 = \frac{1}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 \partial_{s_i s_i} + \frac{1}{2} \sum_{i,j=1}^n \bar{\sigma}_i \bar{\sigma}_j s_i s_j \partial_{s_i s_j} + \sum_{i=1}^n \mu_i s_i \partial_{s_i}, \tag{2.7}$$

$$\mathbf{L}^2 = -\sum_{i=1}^n \bar{\vartheta}^P \bar{\sigma}_i s_i \partial_{s_i} - \frac{\gamma}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 (\partial_{s_i})^2 - \frac{\gamma}{2} \sum_{i,j=1}^n (1 - \bar{\kappa}^P) \bar{\sigma}_i \bar{\sigma}_j s_i s_j (\partial_{s_i})(\partial_{s_j}), \tag{2.8}$$

and for $\mathbf{s} = (s_1, \cdots, s_n)$ and $\mathfrak{C}^\lambda(\mathbf{s}, t)$,

$$\mathbf{L}^3 \mathfrak{C}^\lambda(\mathbf{s}, t) = \frac{a(\mathbf{s})}{\gamma} \left[1 - e^{\gamma(\mathfrak{C}^\lambda(\mathbf{s},t) - R(\mathfrak{C}^\lambda(\mathbf{s},t)))} \right]. \tag{2.9}$$

The operator $\mathbf{L}^1$ describes the infinitesimal behavior of the price processes of the non-traded assets $\mathcal{S} = (S^1, \cdots, S^n)$, the operator $\mathbf{L}^2$ reflects the investor's risk aversion, and the operator $\mathbf{L}^3$ reflects the inter-temporal payment which is distorted by the investor's risk aversion.

Theorem 2.4 *(PDE representation for utility indifference price)*

Suppose that Assumptions 2.1 and 2.2 are satisfied. Then the following reaction-diffusion PDE with quadratic gradients:

$$\begin{cases} \partial_t \mathfrak{C}^\lambda(\mathbf{s}, t) + (\mathbf{L}^1 + \mathbf{L}^2 + \mathbf{L}^3)\mathfrak{C}^\lambda(\mathbf{s}, t) = 0, \\ \mathfrak{C}^\lambda(\mathbf{s}, T) = \lambda g(\mathbf{s}) \end{cases} \tag{2.10}$$

on the domain $(\mathbf{s}, t) \in \mathbb{R}_+^n \times [0, T]$ admits a unique bounded classical solution $\mathfrak{C}^\lambda(\mathbf{s}, t)$ which is Lipschitz continuous in $\mathbf{s} \in \mathbb{R}_+^n$. Moreover, the utility indifference price of λ units of the derivative with the payoff $\mathbf{1}_{\{t<\tau\leq T\}} R(\mathfrak{C}^\lambda(\mathcal{S}_\tau, \tau)) + \mathbf{1}_{\{\tau>T\}} \lambda g(\mathcal{S}_T)$ at time $t \in [0, T]$ is given by

$$\mathfrak{C}_t^\lambda = \mathbf{1}_{\{\tau>t\}} \mathfrak{C}^\lambda(\mathcal{S}_t, t),$$

and the hedging strategy for λ units of the option at time t on the event $\{\tau > t\}$ is given by

$$-\frac{\bar{\kappa}^P}{\bar{\sigma}_P} \sum_{i=1}^n \bar{\sigma}_i S_t^i \partial_{s_i} \mathfrak{C}^\lambda(\mathcal{S}_t, t). \tag{2.11}$$

The proof of Theorem 2.4 is provided in Section 4, where we do not rely on the dynamic programming principle. Instead, we transform the optimal portfolio problems (2.4) and (2.5) into risk-sensitive control problems, and derive the PDE (2.10) by the PDE comparison principle. See also Theorem 2.3 of Henderson and Liang [24], where we derived the BSDE representation for the utility indifference price. We note that the number of units λ only appears in the terminal condition. In the following, we present the case $\lambda = 1$, and the price at time $t \in [0, T]$ is simply denoted by $\mathfrak{C}_t = \mathbf{1}_{\{\tau>t\}} \mathfrak{C}(\mathcal{S}_t, t)$.

We first compare to the situation that the market is complete. If the underlying assets $\mathcal{S} = (S^1, \cdots, S^n)$ could be traded, the market would become complete, and the pricing and hedging of the contingent claim with payoff:

$$\mathbf{1}_{\{\tau\leq T\}} R(\bar{\mathfrak{C}}(\mathcal{S}_\tau, \tau)) + \mathbf{1}_{\{\tau>T\}} g(\mathcal{S}_T)$$

falls into the multidimensional Black-Scholes framework with inter-temporal default risk.

Corollary 2.5 *Suppose that Assumptions 2.1 and 2.2 are satisfied, and that $\mathcal{S} = (S^1, \cdots, S^n)$ are traded assets. Let $\bar{\mathfrak{C}}(\mathbf{s}, t)$ be the unique bounded classical solution (which is Lipschitz continuous) of the reaction-diffusion PDE:*

$$\begin{cases} \partial_t \bar{\mathfrak{C}}(\mathbf{s}, t) + (\bar{\mathbf{L}}^1 + \bar{\mathbf{L}}^3)\bar{\mathfrak{C}}(\mathbf{s}, t) = 0, \\ \bar{\mathfrak{C}}(\mathbf{s}, T) = g(\mathbf{s}) \end{cases} \tag{2.12}$$

on the domain $(\mathbf{s}, t) \in \mathbb{R}_+^n \times [0, T]$, where the operators $\bar{\mathbf{L}}^1$ and $\bar{\mathbf{L}}^3$ are given by

$$\bar{\mathbf{L}}^1 = \frac{1}{2} \sum_{i=1}^n \sigma_i^2 s_i^2 \partial_{s_i s_i} + \frac{1}{2} \sum_{i,j=1}^n \bar{\sigma}_i \bar{\sigma}_j s_i s_j \partial_{s_i s_j}$$

$$\bar{\mathbf{L}}^3 \bar{\mathfrak{C}}(\mathbf{s}, t) = a(\mathbf{s})(R(\bar{\mathfrak{C}}(\mathbf{s}, t)) - \bar{\mathfrak{C}}(\mathbf{s}, t)).$$

Then the price of the option with payoff $\mathbf{1}_{\{t<\tau\leq T\}}R(\bar{\mathfrak{C}}(\mathcal{S}_\tau,\tau)) + \mathbf{1}_{\{\tau>T\}}g(\mathcal{S}_T)$ *at time* $t\in[0,T]$ *is given by* $\bar{\mathfrak{C}}_t = \mathbf{1}_{\{\tau>t\}}\bar{\mathfrak{C}}(\mathcal{S}_t,t)$.

The pricing equation (2.10) has an additional nonlinear term $\mathbf{L}^2$ relative to the complete market pricing PDE (2.12), and this $\mathbf{L}^2$ reflects the investor's risk aversion. Moreover, the inter-temporal payment term $\bar{\mathbf{L}}^3$ in (2.12) is distorted to $\mathbf{L}^3$ by the investor's risk aversion in (2.10). We have the following asymptotic result relating the utility indifference price $\mathfrak{C}_t$ to the complete market price $\bar{\mathfrak{C}}_t$ at any time $t\in[0,T]$.

Proposition 2.6 *Assume that*

$$\bar{\vartheta}^P = \mu_i/\bar{\sigma}_i \text{ for } i=1,\cdots,n. \tag{2.13}$$

Then the unit utility indifference value function $\mathfrak{C}(\mathbf{s},t)$ *uniformly converges to the complete market value function* $\bar{\mathfrak{C}}(\mathbf{s},t)$ *as* $\gamma\to 0$ *on any compact subset of* $\mathbb{R}^n_+\times[0,T]$.

Proof. By the condition (2.13), the first-order linear terms in (2.10) become zero:

$$\sum_{i=1}^n \mu_i s_i \partial_{s_i} - \sum_{i=1}^n \bar{\vartheta}^P \bar{\sigma}_i s_i \partial_{s_i} = 0.$$

When $\gamma\to 0$, the terms involving γ in $\mathbf{L}^2$ converge to zero, and $\mathbf{L}^3\mathfrak{C}(\mathbf{s},t)\to\bar{\mathbf{L}}^3\mathfrak{C}(\mathbf{s},t)$. Therefore, by the Arzela-Ascoli compactness criterion, there exists a subsequence $\gamma_n\to 0$ such that the solutions of (2.10), denoted by $\mathfrak{C}(\mathbf{s},t;\gamma_n)$, uniformly converge to $\bar{\mathfrak{C}}(\mathbf{s},t)$ on any compact subset of $\mathbb{R}^n_+\times[0,T]$, where $\bar{\mathfrak{C}}(\mathbf{s},t)$ satisfies (2.12). ■

The restriction (2.13) in fact corresponds to a relation between the Sharpe ratios of the non-traded assets $\mathcal{S}$ and the financial index P. The Sharpe ratio of P is given by $\sqrt{\bar{\theta}^P}$. Similarly, we define the Sharpe ratio of S^i to be $\sqrt{\bar{\theta}^i}$, where $\bar{\theta}^i = \mu_i^2/(\sigma_i^2+\bar{\sigma}_i^2)$. Then (2.13) is equivalent to the relation;

$$\sqrt{\bar{\theta}^i} = \left(\frac{\bar{\sigma}_i\bar{\sigma}_P}{\sqrt{\bar{\sigma}_i^2+\sigma_i^2}\sqrt{\bar{\sigma}_P^2+\sigma_P^2}}\right)\sqrt{\bar{\theta}^P} = \rho_{iP}\sqrt{\bar{\theta}^P},$$

where ρ_{iP} is the correlation between S^i and P. This corresponds to the relation we expect from the capital asset pricing model (CAPM) when assets are traded. Since not all assets are traded here, we would not necessarily expect (2.13) to hold. The intuition is that when the idiosyncratic volatilities disappear, and when assets are traded, there cannot be a difference in using the financial index P or the assets themselves to hedge.

Based on the pricing equation (2.10) and the PDE comparison principle, we present a number of monotone properties of the utility indifference price. Their proofs are similar to Section 4.2, so we omit them.

Proposition 2.7 *The unit utility indifference value function* $\mathfrak{C}(\mathbf{s},t)$ *is increasing with the recovery* $R(\cdot)$*, the payoff* $g(\cdot)$ *and the intensity* $a(\cdot)$*, and is decreasing with the risk aversion parameter* γ*. Moreover, if the condition (2.13) holds, then* $\mathfrak{C}(\mathbf{s},t)$ *is also decreasing in the idiosyncratic volatility of the traded asset* σ_P^2 *(or its proportion of total volatility,* $1-\bar{\kappa}^P$*).*

The last assertion of the above proposition tells us that the higher the idiosyncratic volatility σ_P^2 of the traded asset (*or as a proportion of total volatility*), the worse it is as a hedging instrument, and the lower the price one is willing to pay. This generalizes the monotonicity obtained in the one dimensional non-traded asset model (see, for example, Henderson [21] and Frei and Schweizer [16] in a non-Markovian model with stochastic correlation).

2.3 A Semigroup Approximation by Splitting

For a reaction-diffusion PDE with quadratic gradients like (2.10), it is not possible to obtain an explicit solution. A special case where an explicit solution does exist is the one dimensional version without inter-temporal default. Taking $n = 1$, $\sigma_P = 0$ and $R(x) = x$ in (2.10) recovers the pricing PDE of [22], [20] and [39], which is solved by the Cole-Hopf transformation. However, this transformation does not apply directly to our multidimensional problem (2.10) because the coefficients of the quadratic gradient terms in $\mathbf{L}^2$ do not match, and the existence of the inter-temporal payment term $\mathbf{L}^3$. We note that even applying standard finite difference methods to numerically solve the PDE (2.10) can be troublesome due to the high dimension and nonlinearity. Instead, we will develop a splitting algorithm which will enable us to take advantage again of the Cole-Hopf transformation to linearize the PDEs.

The splitting method (*or fractional step, prediction and correction*) can be dated back to Marchuk [35] in the late 1960's (see also [36]). The application of splitting to nonlinear PDEs such as HJB equations is difficult mainly because of the verification of the convergence for the approximation scheme. This was overcome by Barles and Souganidis [2], who employed the idea of the viscosity solution and proved that any *monotone, stable and consistent* numerical scheme converges provided there exists a comparison principle for the limiting equation.

The idea of splitting in our setting is the following. The time derivative of the pricing PDE (2.10) depends on the sum of semigroup operators (*or the associated infinitesimal operators*) corresponding to the different factors. These semigroups usually are of different nature. For each sub-problem corresponding to each semigroup there might be an effective way providing solutions. For the sum of these semigroups, however, we usually can not find an accurate method. Hence, application of splitting method means that instead of the sum, we treat the semigroup operators separately.

Of course, the tricky part is how to split the equation (*or how to group factors*) effectively. In the following, we separate the pricing PDE (2.10) into three pricing factors. Take $\lambda = 1$. We first make the log-transformation: $x_i = \ln s_i$ for $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and define a new differential operator:

$$\frac{\partial}{\partial \eta} = \sum_{i=1}^{n} \bar{\sigma}_i \frac{\partial}{\partial x_i}. \tag{2.14}$$

The transformation (2.14) is the key step to apply the splitting method for (2.10). Then (2.10) reduces to

$$\partial_t \mathfrak{C} + \left(\hat{\mathbf{L}}^1 + \hat{\mathbf{L}}^2 + \hat{\mathbf{L}}^3 \right) \mathfrak{C} = 0, \tag{2.15}$$

where

$$\hat{\mathbf{L}}^1 = \frac{1}{2}\partial_{\eta\eta} - \frac{\gamma}{2}(1-\bar{\kappa}^P)(\partial_\eta)^2, \tag{2.16}$$

$$\hat{\mathbf{L}}^2 = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\partial_{x_i x_i} + \sum_{i=1}^{n}A_i\partial_{x_i} - \frac{\gamma}{2}\sum_{i=1}^{n}\sigma_i^2(\partial_{x_i})^2 \tag{2.17}$$

$$\hat{\mathbf{L}}^3 = \mathbf{L}^3 \tag{2.18}$$

with

$$A_i = \mu_i - \frac{1}{2}(\sigma_i^2 + \bar{\sigma}_i^2) - \bar{\vartheta}^P\bar{\sigma}_i.$$

For any $0 \le T_1 < T_2 \le T$, and any smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$ for $m = 1, n$, we define the following nonlinear backward semigroup operators $\mathbf{S}^i(T_1,T_2) : \mathcal{C}^\infty(\mathbb{R}^m) \to \mathcal{C}^\infty(\mathbb{R}^m)$ by $\phi(\cdot) \mapsto \mathfrak{C}^i(\cdot, T_1)$ where

$$\partial_t\mathfrak{C}^i + \hat{\mathbf{L}}^i\mathfrak{C}^i = 0; \quad \mathfrak{C}^i(\cdot, T_2) = \phi(\cdot) \tag{2.19}$$

on the domain $[T_1, T_2] \times \mathbb{R}^m$ for $i = 1, 2, 3$.

We observe that (2.19) for $i = 1, 2$ can be linearized by Cole-Hopf transformations. Indeed, by letting $\bar{\mathfrak{C}}^1 = \exp(-\gamma(1-\bar{\kappa}^P)\mathfrak{C}^1)$, then we have $\bar{\mathfrak{C}}^1$ satisfying

$$\partial_t\bar{\mathfrak{C}}^1 + \frac{1}{2}\partial_{\eta\eta}\bar{\mathfrak{C}}^1 = 0. \tag{2.20}$$

By letting $\bar{\mathfrak{C}}^2 = \exp(-\gamma\mathfrak{C}^2)$, then we have $\bar{\mathfrak{C}}^2$ satisfying

$$\partial_t\bar{\mathfrak{C}}^2 + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\partial_{x_i x_i}\bar{\mathfrak{C}}^2 + \sum_{i=1}^{n}A_i\partial_{x_i}\bar{\mathfrak{C}}^2 = 0. \tag{2.21}$$

Moreover, (2.19) for $i = 3$ can be approximated by Picard iterations, since $\hat{\mathbf{L}}^3 = \mathbf{L}^3$ is Lipschitz continuous (see Section 4.2).

Lemma 2.8 *The operators $\mathbf{S}^i(T_1,T_2)$ for $i = 1, 2, 3$ have the following properties:*

- *(i) For any smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$,*

$$\lim_{T_1 \uparrow T_2}\mathbf{S}^i(T_1,T_2)\phi = \phi$$

  uniformly on any compact subset of $\mathbb{R}^m$.

- *(ii) For any $0 \le T_1 < T_2 < T_3 \le T$,*

$$\mathbf{S}^i(T_1,T_3)\phi = \mathbf{S}^i(T_1,T_2)\mathbf{S}^i(T_2,T_3)\phi.$$

- *(iii)*

$$\mathbf{S}^i(T_2,T_2)\phi = \phi.$$

  ((i) (ii) and (iii) ensure that $\mathbf{S}^i(T_1,T_2)$ is indeed a semigroup operator.)

- *(iv) If $\phi \geq \psi$ where ψ is another smooth function, then*

$$\mathbf{S}^i(T_1, T_2)\phi \geq \mathbf{S}^i(T_1, T_2)\psi.$$

- *(v) For any constant $k \in \mathbb{R}$,*

$$\mathbf{S}^i(T_1, T_2)(\phi(\cdot) + k) = \mathbf{S}^i(T_1, T_2)(\phi(\cdot)) + k.$$

- *(vi)*

$$\lim_{T_1 \uparrow T_2} \frac{\mathbf{S}^i(T_1, T_2)\phi - \phi}{T_2 - T_1} + \hat{\mathbf{L}}^i \phi = 0$$

uniformly on any compact subset of $\mathbb{R}^m$.

Proof. (i)-(v) are immediate. We only prove (vi) in the following. The key idea is the *pseudo linear pricing rule* for the utility indifference price introduced by Henderson and Liang [24]. We first prove the case $i = 1$. Note that $\mathbf{S}^1(T_1, T_2) = \mathbf{S}^1(0, T_2 - T_1)$, and it is enough to prove that for any $\phi \in \mathcal{C}^\infty(\mathbb{R})$,

$$\lim_{T_2 - T_1 \downarrow 0} \frac{\mathbf{S}^1(0, T_2 - T_1)\phi - \phi}{T_2 - T_1} = \hat{\mathbf{L}}^1 \phi$$

uniformly on any compact subset of $\mathbb{R}$.

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions, on which supports the Brownian motion W with $\mathbf{P}(W_0 = \eta) = 1$. By Itô's formula, we have

$$\begin{aligned}
&\mathfrak{C}^1(t, W_t) \\
&= \mathfrak{C}^1(T_2 - T_1, W_{T_2 - T_1}) - \int_t^{T_2 - T_1} (\partial_t \mathfrak{C}^1 + \frac{1}{2}\partial_{\eta\eta}\mathfrak{C}^1)(s, W_s)ds - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s)dW_s \\
&= \phi(W_{T_2 - T_1}) - \int_t^{T_2 - T_1} \frac{\gamma}{2}(1 - \bar{\kappa}^P)(\partial_\eta \mathfrak{C}^1)^2(s, W_s)ds - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s)dW_s
\end{aligned}$$

for $t \in [0, T_2 - T_1]$. We note that $\partial_\eta \mathfrak{C}^1(t, \eta)$ is bounded (see Section 4.2 or Theorem 2.9 of Delarue [15]), and therefore,

$$N_t = -\int_0^t \frac{\gamma}{2}(1 - \bar{\kappa}^P)\partial_\eta \mathfrak{C}^1(s, W_s)dW_s, \quad \text{for } t \in [0, T_2 - T_1]$$

is a $\mathbf{P}$-*BMO* martingale, which implies that the Doléans-Dade exponential $\mathcal{E}(N)$ is uniformly integrable. Hence, we can define a new probability measure $\mathbf{Q}$ by $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(N)$. By Girsanov's theorem,

$$B_t = W_t - \langle W, N \rangle_t = W_t + \int_0^t \frac{\gamma}{2}(1 - \bar{\kappa}^P)\partial_\eta \mathfrak{C}^1(s, W_s)ds, \quad \text{for } t \in [0, T]$$

is the Brownian motion under the probability measure $\mathbf{Q}$, and moreover,

$$\begin{aligned}
\mathfrak{C}^1(t, W_t) &= \phi(W_{T_2 - T_1}) - \int_t^{T_2 - T_1} \partial_\eta \mathfrak{C}^1(s, W_s)dB_s \\
&= E^{\mathbf{Q}}[\phi(W_{T_2 - T_1})|\mathcal{F}_t].
\end{aligned}$$

The above formula is under linear conditional expectation, with the probability measure $\mathbf{Q}$ depending on $\mathfrak{C}^1$, and is named *pseudo linear pricing rule* as in [24]. By the above formula and Itô's formula, we further have

$$\begin{aligned}
&\frac{\mathbf{S}^1(0,T_2-T_1)\phi(\eta)-\phi(\eta)}{T_2-T_1}\\
=&\frac{1}{T_2-T_1}\left[E^{\mathbf{Q}}[\phi(W_{T_2-T_1})]-\phi(\eta)\right]\\
=&\frac{1}{T_2-T_1}E^{\mathbf{Q}}\left[\int_0^{T_2-T_1}\partial_\eta\phi(W_s)dW_s+\int_0^{T_2-T_1}\frac{1}{2}\partial_{\eta\eta}\phi(W_s)d\langle W\rangle_s\right]\\
=&\frac{1}{T_2-T_1}E^{\mathbf{Q}}\left[\int_0^{T_2-T_1}\partial_\eta\phi(W_s)dB_s-\int_0^{T_2-T_1}\frac{\gamma}{2}(1-\bar{\kappa}^P)\partial_\eta\phi(W_s)\partial_\eta\mathfrak{C}^1(s,W_s)ds\right.\\
&\left.+\int_0^{T_2-T_1}\frac{1}{2}\partial_{\eta\eta}\phi(W_s)ds\right]\\
=&\frac{1}{T_2-T_1}E^{\mathbf{Q}}\left[\int_0^{T_2-T_1}-\frac{\gamma}{2}(1-\bar{\kappa}^P)\partial_\eta\phi(W_s)\partial_\eta\mathfrak{C}^1(s,W_s)+\frac{1}{2}\partial_{\eta\eta}\phi(W_s)ds\right].
\end{aligned}$$

Note that $\partial_\eta\mathfrak{C}^1(0,\eta)=\partial_\eta\phi(\eta)$ and by dominated convergence theorem,

$$\lim_{T_2-T_1\downarrow 0}\frac{\mathbf{S}^1(0,T_2-T_1)\phi(\eta)-\phi(\eta)}{T_2-T_1}=-\frac{\gamma}{2}(1-\bar{\kappa}^P)(\partial_\eta\phi)^2(\eta)+\frac{1}{2}\partial_{\eta\eta}\phi(\eta).$$

The proof for the case $i=2$ is similar, so we only sketch its proof. We apply Itô's formula to $\mathfrak{C}^2(t,\mathcal{X}_t)$, where $\mathcal{X}=(X^1,\cdots,X^n)$ is given by

$$X_t^i=x^i+\int_0^t A_i ds+\int_0^t\sigma_i dW_s^i.$$

Therefore, by changing the probability measure, we obtain

$$\begin{aligned}
\mathfrak{C}^2(t,\mathcal{X}_t)=&\ \phi(\mathcal{X}_{T_2-T_1})-\int_t^{T_2-T_1}\frac{\gamma}{2}\sum_{i=1}^n\sigma_i^2(\partial_{x_i}\mathfrak{C}^2)^2(s,\mathcal{X}_s)ds\\
&-\int_t^{T_2-T_1}\sum_{i=1}^n\sigma_i\partial_{x_i}\mathfrak{C}^2(s,\mathcal{X}_s)dW_s^i\\
=&\ E^{\mathbf{Q}}[\phi(\mathcal{X}_{T_2-T_1})|\mathcal{F}_t],
\end{aligned}$$

where $\mathbf{Q}$ is defined by the the Doléans-Dade exponential $\mathcal{E}(N)$ with

$$N_t=-\int_0^t\sum_{i=1}^n\frac{\gamma}{2}\sigma_i\partial_{x_i}\mathfrak{C}^2(s,\mathcal{X}_s)dW_s^i,\quad\text{for } t\in[0,T_2-T_1].$$

The rest of the proof follows by the same argument as for the case $i=1$. Finally, the proof for case $i=3$ is trivial, as $\hat{\mathbf{L}}^3$ does not involve any space derivatives. ■

Next we use semigroup operators $\mathbf{S}^i(T_1,T_2)$ for $i=1,2,3$ to give the semigroup approximation for the solution of PDE (2.15) (*or PDE (2.10) with different coordinates*), which is the main result of this section.

Theorem 2.9 *(Semigroup approximation for utility indifference price)*
Suppose that Assumptions 2.1 and 2.2 are satisfied. Let $\Delta: 0=t_0<t_1<\cdots<t_N=T$ be the partition of $[0,T]$ with mesh:

$$|\Delta| := \max_{0\le i\le N-1} |t_{i+1}-t_i|.$$

Then the unit utility indifference value function $\mathfrak{C}(\cdot,0)$ of the derivative with payoff $\mathbf{1}_{\{\tau\le T\}}R(\mathfrak{C}_{\tau-})+\mathbf{1}_{\{\tau>T\}}g(\mathcal{S}_T)$ at the initial time $t=0$ is approximated by

$$\mathfrak{C}^{\Delta}(\cdot,0)=\prod_{i=0}^{N-1}\mathbf{S}^1(t_i,t_{i+1})\mathbf{S}^2(t_i,t_{i+1})\mathbf{S}^3(t_i,t_{i+1})g(\cdot),$$

and

$$\lim_{|\Delta|\to 0}\mathfrak{C}^{\Delta}(\cdot,0)=\mathfrak{C}(\cdot,0).$$

uniformly on any compact subset of $\mathbb{R}^n$.

Proof. The proof is based on the Barles-Souganidis monotone scheme [2], in which they proved that any *monotone, stable and consistent* numerical scheme converges provided there exists a comparison principle for the limiting equation.

In the following, we verify the above conditions. We omit the time arguments T_1 and T_2 in $\mathbf{S}^i(T_1,T_2)$, and simply write $\mathbf{S}^i$. For any $0\le T_1<T_2\le T$ and any smooth functions $\phi\ge\psi$, by (iv) in Lemma 2.8,

$$\mathbf{S}^1\mathbf{S}^2\mathbf{S}^3\phi \ge \mathbf{S}^1\mathbf{S}^2\mathbf{S}^3\psi,$$

so the scheme is monotone. For any $k\in\mathbb{R}$, by (v) in Lemma 2.8,

$$\begin{aligned}\mathbf{S}^1\mathbf{S}^2\mathbf{S}^3(\phi+k) &= \mathbf{S}^1\mathbf{S}^2(\mathbf{S}^3\phi+k)\\ &= \mathbf{S}^1(\mathbf{S}^2\mathbf{S}^3\phi+k)\\ &= \mathbf{S}^1\mathbf{S}^2\mathbf{S}^3\phi+k,\end{aligned}$$

so the scheme is stable. Finally, we verify the scheme is consistent:

$$\begin{aligned}\frac{\mathbf{S}^1\mathbf{S}^2\mathbf{S}^3\phi-\phi}{T_2-T_1}+(\hat{\mathbf{L}}^1+\hat{\mathbf{L}}^2+\hat{\mathbf{L}}^3)\phi &= \frac{\mathbf{S}^1\mathbf{S}^2\mathbf{S}^3\phi-\mathbf{S}^2\mathbf{S}^3\phi}{T_2-T_1}+\mathbf{L}^1\mathbf{S}^2\mathbf{S}^3\phi\\ &\quad+\frac{\mathbf{S}^2\mathbf{S}^3\phi-\mathbf{S}^3\phi}{T_2-T_1}+\mathbf{L}^2\mathbf{S}^3\phi\\ &\quad+\frac{\mathbf{S}^3\phi-\phi}{T_2-T_1}+\mathbf{L}^3\phi\\ &\quad-\mathbf{L}^1\mathbf{S}^2\mathbf{S}^3\phi+\mathbf{L}^1\phi\\ &\quad-\mathbf{L}^2\mathbf{S}^3\phi+\mathbf{L}^2\phi\\ &= I_1+I_2+I_3+I_4+I_5. \end{aligned}\tag{2.22}$$

By (vi) in Lemma 2.8, the terms I_1, I_2 and I_3 in (2.22) converge to 0 when $T_1 \uparrow T_2$. By (i) in Lemma 2.8, the terms I_4 and I_5 in (2.22) also converge to 0 when $T_1 \uparrow T_2$. Therefore the scheme is consistent. ■

3 Application to Counterparty Risk

In this section, we apply our multidimensional non-traded assets model to consider the counterparty risk of derivatives with possible default at maturity. Our concern as the buyer or holder of the option is that the writer or counterparty may default on the option with payoff $h(S_T)$ at maturity T and we will not receive the full payoff. We have in mind several examples. A natural example is that of a commodity producer who is writing options as part of a hedging program (eg. collars). Some of these options may be written on illiquidly traded assets and thus the option holder is subject to basis risk and in addition, is concerned with the default risk of the option writer. A second example is the default risk of a financial institution who has sold options on various underlying assets - stocks, foreign exchange or commodities. In addition to the possibility of basis risk, the buyer of these options does not always have the ability to trade the underlying asset, or perhaps they choose not to (they may be using the derivative as part of a hedge already). A further example may be that of a purchaser of insurance concerned with the default risk of the insurer. Typically the insured party does not trade at all, which motivates our consideration of this special case. Finally, the option holder may be an employee of a company who receives employee stock options if the company remains solvent. She is restricted from trading the stock of the company, but can trade other indices or stocks in the market. In contrast to the other examples, here the assets of the counterparty and the underlying stock are those of the same company.

Consider an option written on an underlying asset with price S^1 with payoff $h(S^1_T)$ at maturity T. Counterparty default is modeled by comparing the value of the counterparty's assets S^2 to a default threshold D at maturity, which depends on the liabilities of the counterparty. Following Klein [30] we consider the situation $D = L$, where L refers to the option writer's liabilities, assumed to be a constant. Generalizations to $D = f(S^1_T)$ are easily incorporated and allow for the option liability itself to influence default, e.g. $f(x) = h(x) + L$ was considered by Klein and Inglis [31] in a risk neutral setting. If the writer defaults, the holder will receive the proportion $h(S^1_T)/L$ of the assets S^2 that her option represents of the writer's liabilities, scaled to reflect a proportional deadweight loss of $\alpha \in [0, 1]$. The payoff of the *vulnerable option* taking counterparty default into account is

$$g(S^1_T, S^2_T) = h(S^1_T) 1_{\{S^2_T \geq L\}} + (1-\alpha)\frac{h(S^1_T)}{L} S^2_T 1_{\{S^2_T < L\}}. \tag{3.1}$$

To guarantee g is bounded, a sufficient condition is that the payoff $h(S^1_T)$ is bounded, for example, a put option. Note that there is a singular point of g at $S^2_T = L$, so we have to approximate g by a sequence of (nondecreasing) Lipschitz continuous functions g^ϵ. For the numerical simulation purpose, we only need to choose one g^ϵ for ϵ small enough.

Note that the above payoff (3.1) of the vulnerable option written on S^1 taking account of the counterparty default can also be regarded as the payoff of a basket option written on (S^1, S^2) without taking account of intertemporal default risk, so it falls into the framework of Section 2. The underlying asset S^1 and the value of the counterparty's assets S^2 are both taken to be non-traded assets so $n = 2$ and prices follow (2.1). The option holder faces some unhedgeable price risk (due to S^1) and some unhedgeable counterparty default risk (due to S^2). She can partially hedge risks by trading the financial index P following (2.2).

Our focus in the implementation on default at maturity enables us to compare our results to several benchmark models - Johnson and Stulz [29], Klein [30] and Klein and Inglis [31]. Each take a structural approach to price vulnerable options in a complete market setting and obtain two dimensional Black-Scholes style formulas. Implicit in this prior literature are the twin assumptions that the asset underlying the option and the assets of the counterparty can be traded, and therefore, can be used to hedge the counterparty risk of derivatives.

Our use of the utility indifference approach is motivated by its recent use in credit risk modeling where the concern is the default of the reference name rather than the default of the counterparty. Utility based pricing has also been utilized by Bielecki and Jeanblanc [5], Sircar and Zariphopolou [42] and recently Jiao et al [27] [28] in an intensity based setting. Several authors have applied it in modeling of defaultable bonds where the problem remains one dimensional, see in particular Leung et al [33], Jaimungal and Sigloch [26], and Liang and Jiang [34]. In contrast, options subject to counterparty risk are a natural situation where two or more dimensions arise.

We were also motivated to study indifference pricing of derivatives subject to counterparty risk by the work of Hung and Liu [25] and Murgoci [38]. These papers take a *good deal bounds* approach to pricing, acknowledging the incompleteness of the market but producing prices which are linear in quantity. Furthermore, the method does not allow for any partial hedging on the part of the investor and can produce bounds which can be quite wide.

Based on Theorem 2.9, we give the following approximation scheme for the unit utility indifference price $\mathfrak{C}(s_1, s_2, 0)$ of the vulnerable option. Following (2.14) we make the change of variable $x = \ln s_1$ and $y = \ln s_2$, and define a new operator:

$$\frac{\partial}{\partial \eta} = \bar{\sigma}_1 \frac{\partial}{\partial x} + \bar{\sigma}_2 \frac{\partial}{\partial y}.$$

- (i) Partition $[0, T]$ into N equal intervals:

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

- (ii) On $[t_{N-1}, t_N]$, predict the solution by solving the following PDE with the given terminal data g^ϵ:

$$\begin{cases} \partial_t \mathfrak{C}^1 + \frac{1}{2}\partial_{\eta\eta}\mathfrak{C}^1 - \frac{\gamma}{2}(1-\bar{\kappa}^P)(\partial_\eta \mathfrak{C}^1)^2 = 0, \\ \mathfrak{C}^1|_{t=t_N} = g^\epsilon. \end{cases}$$

  The above equation can be linearized via the Cole-Hopf transformation:

$$\bar{\mathfrak{C}}^1 = \exp(-\gamma(1-\bar{\kappa}^P)\mathfrak{C}^1).$$

  Thus, we obtain $\mathfrak{C}^1|_{t=t_{N-1}}$ by solving the corresponding linear PDE.

- (iii) On $[t_{N-1}, t_N]$, correct the solution by solving the following PDE with the terminal data

$\mathfrak{C}^1|_{t=t_{N-1}}$:

$$\begin{cases} \partial_t \mathfrak{C}^2 + \frac{1}{2}\sigma_1^2 \partial_{xx}\mathfrak{C}^2 + \frac{1}{2}\sigma_2^2 \partial_{yy}\mathfrak{C}^2 + A_1 \partial_x \mathfrak{C}^2 + A_2 \partial_y \mathfrak{C}^2 \\ \qquad -\frac{\gamma}{2}\sigma_1^2(\partial_x \mathfrak{C}^2)^2 - \frac{\gamma}{2}\sigma_2^2(\partial_y \mathfrak{C}^2)^2 = 0, \\ \mathfrak{C}^2|_{t=t_N} = \mathfrak{C}^1|_{t=t_{N-1}} \end{cases}$$

where A_1, A_2 are given in (2.3) to be $A_i = \mu_i - \frac{1}{2}(\sigma_i^2 + \bar{\sigma}_i^2) - \bar{\vartheta}^P \bar{\sigma}_i;\ \ i = 1, 2$. The above equation can also be linearized by making the exponential transformation:

$$\bar{\mathfrak{C}}^2 = \exp(-\gamma \mathfrak{C}^2).$$

Thus, we obtain $\mathfrak{C}^2|_{t=t_{N-1}}$ by solving the corresponding linear PDE, which is used as the approximation of $\mathfrak{C}|_{t=t_{N-1}}$.

- (iv) Repeat the above procedure on $[t_{N-2}, t_{N-1}]$, and obtain $\mathfrak{C}|_{t=t_{N-2}}$ $\cdots$.

We present results for the European put with payoff $h(S_T^1) = (K - S_T^1)^+$. If S^1 and S^2 are positively correlated, this means when the put option is valuable (in-the-money), the firm's assets S^2 tend to be small, so there is a high risk of default. It is important to take counterparty risk into account for puts in this case, as it will have a relatively large impact on the price. (This would be even more significant when the default trigger involves the option liability). However, for a call, when the call is in-the-money, there is little default risk, and so counterparty risk is less important. Unless otherwise stated, the parameters are: $K = 150$; $T = 1$; $S^1 = 50$; $S^2 = 100$; $L = 1000$; $\alpha = 0.05$; $\gamma = 1$; $\mu_P = 0.1$; $\sigma_P = 0.15$; $\bar{\sigma}_P = 0.2$; $\mu_1 = 0.15$; $\sigma_1 = 0.25$; $\bar{\sigma}_1 = 0.3$; $\mu_2 = 0.1$; $\sigma_2 = 0.3$; $\bar{\sigma}_2 = 0.2$. These parameters result in correlation between the underlying asset and firm's assets of $\rho_{12} = 0.4$; and correlations between each asset and the financial index P of $\rho_{1P} = 0.6$; $\ \rho_{2P} = 0.4$..

In Figure 1 we show how the approximation converges as we increase the number of time steps N. For our parameter values, $N = 11$ steps is sufficient for the prices to converge and we use it in all subsequent figures. We aim to compare the utility indifference price with hedging in the *financial index* with the benchmark risk neutral price in a complete market (computed as in Corollary 2.5 with $n = 2$, as studied in Klein [30]). We also compare to the situation where the financial index is independent of the other assets and thus there is *no hedging* carried out. This price has an explicit formula:

$$-\frac{1}{\gamma} \ln E^{\mathbf{P}} \left[-e^{-\gamma g^\epsilon(S_T^1, S_T^2)} \right].$$

Figure 2 provides a demonstration of the accuracy of the algorithm. We take $\bar{\sigma}_P = 0$ and compare the splitting approximation to the above formula.

Figure 3 shows the vulnerable option price(s) against the underlying asset price S^1. The two panels of Figure 3 are intended to illustrate a "close or likely to default" scenario (the left panel with $S^2 = 500$ relative to $L = 1000$) and a "far or unlikely to default" scenario (the right panel with $S_2 = 1400$ relative to $L = 1000$). In both panels the risk neutral or complete market price is the highest. As the underlying asset price becomes very large, all option prices tend to zero, as the put is worthless, regardless of the default risk. At $S^1 = 0$, in the right panel, the option price is equal to the option strike $K = 150$. In the left panel, the option price is lower due to the risk of counterparty default. As S^1 increases, all option prices decrease, as the moneyness of the put changes. When S^1 is close to zero, we see a dramatic drop in the utility indifference prices (relative to the risk neutral prices) due to the risk aversion towards unhedgeable price and default risks. Recall that since the underlying asset and firm's assets are positively correlated, default risk becomes more important for low values of the underlying asset. The price drop is much more significant in the left panel (and in this extreme case, the option price drops down to zero if no hedging can be carried out), where the likelihood of counterparty default is higher. The option holder's risk aversion causes the utility indifference prices to lie below the risk neutral price (in each default scenario) with the relative discount to the risk neutral price being much greater in the left panel where default is more likely. Assuming the holder can hedge in the financial index, there is a drop of around 75% from the risk-neutral price to the utility indifference price. In the right panel, where the likelihood of default is relatively low, the difference between the utility indifference price(s) and the risk-neutral price is not as dramatic, and is at most around 20% of the risk-neutral price. We also see that the ability to hedge in the correlated financial index (versus no hedging at all) is more important when the default risk is higher (in the left panel).

Figure 4 displays the impact of the option writer's asset value S^2 on the option price for a fixed asset price S^1. We see a dramatic difference in the behavior of the risk neutral price and the utility indifference prices. Under risk neutrality, the option price increases smoothly with S^2. However, under utility indifference, the prices are low and do not change much with values of S^2 below the default trigger of $L = 1000$. This is despite the put being in-the-money. As S^2 increases beyond the default level, the likelihood of default diminishes, and the utility indifference prices start to increase with S^2. Note that the utility price is not always below the risk neutral price. Although Proposition 2.6 tells us that the risk neutral price is obtained as a limiting case of the utility indifference price, it requires condition (2.13) to hold.

Figure 5 compares how the vulnerable option price changes with risk aversion parameter γ, and the idiosyncratic volatilities σ_1, σ_2. The left panel plots vulnerable option prices against maturity T for various values of risk aversion γ. We see that the more risk averse the option holder is, the less she will pay for the option, consistent with Proposition 2.7. The other observation is that option prices for a fixed γ are decreasing with maturity T. The risk neutral price is also decreasing with T, albeit very gradually. This is in contrast to risk neutral prices for non-default European put options which will increase in T provided there are no dividends. The reason is that there is a tradeoff between price and default risk. If the maturity is longer, there is more chance for both S^1 and S^2 to fall - S^1 falling means the put is more valuable, but S^2 falling increases the default risk. For the parameters considered, the default risk is the dominant factor and thus the option price decreases with T. This is also in contrast to the call option, where Klein [30] reports that the risk neutral price increases with maturity.

Recall that we do not expect price monotonicity in terms of the correlations, except in the situation outlined in Proposition 2.7. Here we give an example of prices for various values of the idiosyncratic volatilities σ_1, σ_2. The left panel sets parameters to be $\mu_1 = 0.1$ and $\mu_2 = 0.06$ to satisfy the CAPM restriction on Sharpe ratios. If $\sigma_1 = \sigma_2 = 0$, then we have $\rho_{12} = 1$, $\rho_{1P} = 0.8$ and $\rho_{2P} = 0.8$. Similarly if $\sigma_1 = 0.25$ and $\sigma_2 = 0.3$ then $\rho_{12} = 0.4$, $\rho_{1P} = 0.6$ and $\rho_{2P} = 0.4$. Finally, if $\sigma_1 = \sigma_2 = 1$ then $\rho_{12} = 0.06$, $\rho_{1P} = 0.2$ and $\rho_{2P} = 0.16$. We see that as σ_1, σ_2 increase, the utility indifference price falls. Correspondingly, as the correlations $\rho_{12}, \rho_{1P}, \rho_{2P}$ increase, the option price rises.

4 Proof of Theorem 2.4

4.1 Derivation of Pricing PDE (2.10)

We first transform the optimal portfolio problem (2.4) into a risk-sensitive control formulation, and derive its PDE representation by the PDE comparison principle. The problem (2.4) can be reformulated as

$$
\begin{aligned}
& - e^{-\gamma(x-\mathfrak{C}_t^\lambda)} \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_\mathcal{F}[t,T]} E^{\mathbf{P}}\left[e^{-\gamma\left(\mathbf{1}_{\{t<\tau\le T\}}\left(X_\tau^0(\pi)+R(\mathfrak{C}^\lambda(\mathcal{S}_\tau,\tau))\right)+\mathbf{1}_{\{\tau>T\}}\left(X_T^0(\pi)+\lambda g(\mathcal{S}_T)\right)\right)}|\mathcal{G}_t\right] \\
=& - e^{-\gamma(x-\mathfrak{C}_t^\lambda)} \exp\left\{-\gamma \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_\mathcal{F}[t,T]} \frac{-1}{\gamma}\ln E^{\mathbf{P}}\left[\cdots|\mathcal{G}_t\right]\right\}.
\end{aligned}
$$

For any fixed $\pi\in\mathcal{A}_\mathcal{F}[t,T]$, by using the distribution property of the random time τ (see Chapter 8 of Bielecki and Rutkowski [6] for example), we calculate the above conditional expectation, which is $\mathbf{1}_{\{\tau>t\}}\bar{\mathfrak{W}}(\mathcal{S}_t,t;\pi_t)$ with

$$
\begin{aligned}
& \bar{\mathfrak{W}}(\mathcal{S}_t,t;\pi_t) \\
=& E^{\mathbf{P}}\left[\int_t^T a(\mathcal{S}_s)e^{-\int_t^s a(\mathcal{S}_u)du-\gamma X_s^0(\pi)-\gamma R(\mathfrak{C}^\lambda(\mathcal{S}_s,s))}ds + e^{-\int_t^T a(S_s)ds-\gamma X_T^0(\pi)-\gamma\lambda g(\mathcal{S}_T)}|\mathcal{F}_t\right].
\end{aligned}
$$

Therefore, in order to solve the optimal portfolio problem (2.4), we only need to solve the following risk-sensitive control problem:

$$
\mathfrak{W}(\mathcal{S}_t,t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_\mathcal{F}[t,T]} \mathfrak{W}(\mathcal{S}_t,t;\pi_t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_\mathcal{F}[t,T]} \frac{-1}{\gamma}\ln\bar{\mathfrak{W}}(\mathcal{S}_t,t;\pi_t), \tag{4.1}
$$

and the value process of (2.4) is given by

$$
-e^{-\gamma(x-\mathfrak{C}_t^\lambda)}\times\mathbf{1}_{\{\tau>t\}}e^{-\gamma\mathfrak{W}(\mathcal{S}_t,t)}. \tag{4.2}
$$

In the following, we will first work out the PDE representation for $\mathfrak{W}(\mathcal{S}_t,t;\pi_t)$, and then obtain the PDE representation for $\mathfrak{W}(\mathcal{S}_t,t)$ by PDE comparison principle. For $\pi\in\mathcal{A}_\mathcal{F}[t,T]$, we change the probability measure from $\mathbf{P}$ to $\mathbf{Q}(\pi)$ by defining

$$
\frac{d\mathbf{Q}(\pi)}{d\mathbf{P}} = \mathcal{E}(N(\pi))_T = \mathcal{E}\left(-\int_t^\cdot \gamma\pi_s(\bar\sigma_P dW_s^{n+1}+\sigma_P dW_s^{n+2})\right)_T,
$$

where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential. By Girsanov's theorem, $\mathcal{B}(\pi)=\mathcal{W}-\langle\mathcal{W},N(\pi)\rangle$ is the Brownian motion under the new probability measure $\mathbf{Q}(\pi)$. With the new Brownian motion $\mathcal{B}(\pi)=(B^1(\pi),\cdots,B^{n+2}(\pi))$, the price processes of the non-traded assets $\mathcal{S}=(S^1,\cdots,S^n)$ satisfy

$$
\frac{dS_t^i}{S_t^i} = (\mu_i-\gamma\bar\sigma_i\bar\sigma_P\pi_t)\,dt + \sigma_i dB_t^i(\pi) + \bar\sigma_i dB_t^{n+1}(\pi), \tag{4.3}
$$

and $\bar{\mathfrak{W}}(\mathcal{S}_t, t; \pi_t)$ becomes

$$
\begin{aligned}
&\bar{\mathfrak{W}}(\mathcal{S}_t, t; \pi_t) \\
&= E^{\mathbf{P}}\left[\int_t^T \mathcal{E}(N(\pi))_s e^{-\int_t^s \rho(\mathcal{S}_u;\pi_u)du} a(\mathcal{S}_s) e^{-\gamma R(\mathfrak{C}^\lambda(\mathcal{S}_s,s))} ds \right.\\
&\qquad \left. + \mathcal{E}(N(\pi))_T e^{-\int_t^T \rho(\mathcal{S}_u;\pi_u)du} e^{-\gamma\lambda g(\mathcal{S}_T)} | \mathcal{F}_t\right] \\
&= E^{\mathbf{Q}(\pi)}\left[\int_t^T e^{-\int_t^s \rho(\mathcal{S}_u;\pi_u)du} a(\mathcal{S}_s) e^{-\gamma R(\mathfrak{C}^\lambda(\mathcal{S}_s,s))} ds + e^{-\int_t^T \rho(\mathcal{S}_u;\pi_u)du} e^{-\gamma\lambda g(\mathcal{S}_T)} | \mathcal{F}_t\right],
\end{aligned}
$$

with the stochastic discount factor $\rho(\mathcal{S}_u, \pi_u)$:

$$\rho(\mathcal{S}_u; \pi_u) = a(\mathcal{S}_u) + \gamma\mu_P\pi_u - \frac{\gamma^2}{2}\left(\sigma_P^2 + \bar{\sigma}_P^2\right)(\pi_u)^2.$$

We look for the feedback control $\pi_t = \pi(\mathcal{S}_t, t)$ for some bounded function $\pi(\cdot,\cdot)$. Then the Feynman-Kac formula implies that $\bar{\mathfrak{W}}(\mathcal{S}_t, t; \pi_t)$, with the starting point $\mathcal{S}_t = \mathbf{s} \in \mathbb{R}_+^n$, satisfies the PDE (at least in the viscosity solution sense):

$$
\begin{cases}
\partial_t \bar{\mathfrak{W}}(\mathbf{s}, t; \pi) + \mathbf{L}^1 \bar{\mathfrak{W}}(\mathbf{s}, t; \pi) - \sum_{i=1}^n \gamma \bar{\sigma}_i \bar{\sigma}_P \pi(\mathbf{s}, t) s_i \partial_{s_i} \bar{\mathfrak{W}}(\mathbf{s}, t; \pi) \\
-\rho(\mathbf{s}; \pi(\mathbf{s}, t)) \bar{\mathfrak{W}}(\mathbf{s}, t; \pi) + a(\mathbf{s}) e^{-\gamma R(\mathfrak{C}^\lambda(\mathbf{s},t))} = 0, \\
\bar{\mathfrak{W}}(\mathbf{s}, T; \pi) = e^{-\gamma\lambda g(\mathbf{s})},
\end{cases}
\tag{4.4}
$$

and $\mathfrak{W}(\mathbf{s}, t; \pi) = \frac{-1}{\gamma} \ln \bar{\mathfrak{W}}(\mathbf{s}, t; \pi)$ satisfies the PDE:

$$
\begin{cases}
\partial_t \mathfrak{W}(\mathbf{s}, t; \pi) + \mathbf{L}^1 \mathfrak{W}(\mathbf{s}, t; \pi) + \frac{a(\mathbf{s})}{\gamma}\left[1 - e^{\gamma(\mathfrak{W}(\mathbf{s},t;\pi) - R(\mathfrak{C}^\lambda(\mathbf{s},t)))}\right] \\
-\frac{\gamma}{2}(\sigma_P^2 + \bar{\sigma}_P^2)\pi^2(\mathbf{s}, t) + \mu_P \pi(\mathbf{s}, t) - \sum_{i=1}^n \gamma \bar{\sigma}_i \bar{\sigma}_P s_i \partial_{s_i} \mathfrak{W}(\mathbf{s}, t; \pi) \pi(\mathbf{s}, t) = 0, \\
\mathfrak{W}(\mathbf{s}, T; \pi) = \lambda g(\mathbf{s}).
\end{cases}
\tag{4.5}
$$

Since

$$-\frac{\gamma}{2}(\sigma_P^2 + \bar{\sigma}_P^2)\pi^2(\mathbf{s}, t) + \mu_P \pi(\mathbf{s}, t) - \sum_{i=1}^n \gamma \bar{\sigma}_i \bar{\sigma}_P s_i \partial_{s_i} \mathfrak{W}(\mathbf{s}, t) \pi(\mathbf{s}, t) \le \mathbf{L}^2 \mathfrak{W}(\mathbf{s}, t) + \frac{\bar{\theta}^P}{2\gamma} \tag{4.6}$$

for any bounded $\pi(\cdot,\cdot)$, with

$$\bar{\pi}^*(\mathbf{s}, t) = -\frac{\bar{\kappa}^P}{\bar{\sigma}_P} \sum_{i=1}^n \bar{\sigma}_i s_i \partial_{s_i} \mathfrak{W}(\mathbf{s}, t) - \frac{\bar{\vartheta}^P}{\gamma \bar{\sigma}_P}$$

achieving the equality in (4.6), by the PDE comparison principle, $\mathfrak{W}(\mathcal{S}_t, t) = \sup_{\pi \in \mathcal{A}_{\mathcal{F}}[t,T]} \mathfrak{W}(\mathcal{S}_t, t; \pi_t)$, where $\mathfrak{W}(\mathbf{s}, t)$ satisfies the PDE:

$$
\begin{cases}
\partial_t \mathfrak{W}(\mathbf{s}, t) + \mathbf{L}^1 \mathfrak{W}(\mathbf{s}, t) + \frac{a(\mathbf{s})}{\gamma}\left[1 - e^{\gamma(\mathfrak{W}(\mathbf{s},t) - R(\mathfrak{C}^\lambda(\mathbf{s},t)))}\right] \\
+\mathbf{L}^2 \mathfrak{W}(\mathbf{s}, t) + \frac{\bar{\theta}^P}{2\gamma} = 0, \\
\mathfrak{W}(\mathbf{s}, T) = \lambda g(\mathbf{s}).
\end{cases}
\tag{4.7}
$$

That is, the value process of (2.4) is given by (4.2), and the optimal control of (2.4) on the event $\{\tau > t\}$ is given by

$$\bar{\pi}_t^* = \bar{\pi}^*(\mathcal{S}_t, t) = -\frac{\bar{\kappa}^P}{\bar{\sigma}_P} \sum_{i=1}^n \bar{\sigma}_i S_t^i \partial_{s_i} \mathfrak{W}(\mathcal{S}_t, t) - \frac{\bar{\vartheta}^P}{\gamma \bar{\sigma}_P}.$$

We will verify that $\bar{\pi}^*(\cdot, \cdot)$ is uniformly bounded in $(\mathbf{s}, t)$ in the next subsection.

On the other hand, the optimal portfolio problem (2.5) is a special case of (2.4), whose value process is given by $-e^{-\gamma x} \times \mathbf{1}_{\{\tau > t\}} e^{-\frac{\bar{\vartheta}^P}{2}(T-t)}$, and the optimal control on the event $\{\tau > t\}$ is given by $\pi_t^* = -\frac{\bar{\vartheta}^P}{\gamma \bar{\sigma}_P}$.

By Definition 2.3, the utility indifference price $\mathfrak{C}_t^\lambda$ is such that

$$-e^{-\gamma(x - \mathfrak{C}_t^\lambda)} \times \mathbf{1}_{\{\tau > t\}} e^{-\gamma \mathfrak{W}(\mathcal{S}_t, t)} = -e^{-\gamma x} \times \mathbf{1}_{\{\tau > t\}} e^{-\frac{\bar{\vartheta}^P}{2}(T-t)},$$

from which we obtain

$$\mathfrak{C}_t^\lambda = \mathbf{1}_{\{\tau > t\}} \left(\mathfrak{W}(\mathcal{S}_t, t) - \frac{\bar{\theta}^P}{2\gamma}(T-t) \right) = \mathbf{1}_{\{\tau > t\}} \mathfrak{C}^\lambda(\mathcal{S}_t, t).$$

Since $\partial_{s_i} \mathfrak{W}(\mathcal{S}_t, t) = \partial_{s_i} \mathfrak{C}^\lambda(\mathcal{S}_t, t)$, by Definition 2.3, the hedging strategy $\bar{\pi}_t^* - \pi_t^*$ on the event $\{\tau > t\}$ is given by (2.11).

4.2 Well posedness of the Pricing PDE (2.10)

We first show that the solution $\mathfrak{C}^\lambda(\mathbf{s}, t)$ to the PDE (2.10) is bounded. It is obvious that the solution is nonnegative: $\mathfrak{C}^\lambda(\mathbf{s}, t) \geq 0$. By Assumption 2.2, $R(x) \leq I(x)$ where $I(\cdot)$ is the identity function $I(x) = x$. The solution of the corresponding pricing PDE with the full recovery $I(\cdot)$ is denoted by $\mathfrak{C}^\lambda(\mathbf{s}, t; I)$, i.e., $\mathfrak{C}^\lambda(\mathbf{s}, t; I)$ satisfies the following PDE with quadratic gradients:

$$\begin{cases} \partial_t \mathfrak{C}^\lambda(\mathbf{s}, t; I) + (\mathbf{L}^1 + \mathbf{L}^2) \mathfrak{C}^\lambda(\mathbf{s}, t; I) = 0, \\ \mathfrak{C}^\lambda(\mathbf{s}, T; I) = \lambda g(\mathbf{s}). \end{cases} \tag{4.8}$$

Note that $(\partial_t + \mathbf{L}^1 + \mathbf{L}^2 + \mathbf{L}^3) \mathfrak{C}^\lambda(\mathbf{s}, t; I) \leq 0$, since

$$\mathbf{L}^3 \mathfrak{C}^\lambda(\mathbf{s}, t; I) \leq \frac{a(\mathbf{s})}{\gamma} \left[1 - e^{\gamma(\mathfrak{C}^\lambda(\mathbf{s}, t; I) - I(\mathfrak{C}^\lambda(\mathbf{s}, t; I)))} \right] = 0,$$

so that $\mathfrak{C}^\lambda(\mathbf{s}, t; I)$ is the supersolution to (2.10). On the other hand, $\mathfrak{C}^\lambda(\mathbf{s}, t)$ is the (sub)solution to (2.10), and $\mathfrak{C}^\lambda(\mathbf{s}, T; I) = \mathfrak{C}^\lambda(\mathbf{s}, T)$. By the PDE comparison principle, we conclude that $\mathfrak{C}^\lambda(\mathbf{s}, t) \leq \mathfrak{C}^\lambda(\mathbf{s}, t; I)$ on $\mathbb{R}_+^n \times [0, T]$.

The solution $\mathfrak{C}^\lambda(\mathbf{s}, t; I)$ to PDE (4.8) is interpreted as the utility indifference price without intertemporal default, and we claim that $\mathfrak{C}^\lambda(\mathbf{s}, t; I)$ is bounded from above by $\bar{\mathfrak{C}}^\lambda(\mathbf{s}, t; I)$, which is the solution to PDE (4.8) with $\gamma = 0$, that is, $\bar{\mathfrak{C}}^\lambda(\mathbf{s}, t; I)$ satisfies the following linear PDE:

$$\begin{cases} \partial_t \bar{\mathfrak{C}}^\lambda(\mathbf{s}, t; I) + \mathbf{L}^1 \bar{\mathfrak{C}}^\lambda(\mathbf{s}, t; I) - \sum_{i=1}^n \bar{\vartheta}^P \bar{\sigma}_i s_i \partial_{s_i} \bar{\mathfrak{C}}^\lambda(\mathbf{s}, t; I) = 0, \\ \bar{\mathfrak{C}}^\lambda(\mathbf{s}, T; I) = \lambda g(\mathbf{s}) \end{cases} \tag{4.9}$$

Indeed, note that $(\partial_t+\mathbf{L}^1+\mathbf{L}^2)\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)\le 0$, since the terms involving γ in $\mathbf{L}^2$ can be regrouped as

$$-\frac{\gamma}{2}\sum_{i=1}^{n}\sigma_i^2 s_i^2\left(\partial_{s_i}\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)\right)^2-\frac{\gamma}{2}(1-\bar{\kappa}^P)\left(\sum_{i=1}^{n}\bar{\sigma}_i s_i\partial_{s_i}\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)\right)^2\le 0$$

so that $\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)$ is the supersolution to (4.8). On the other hand, $\mathfrak{C}^\lambda(\mathbf{s},t;I)$ is the (sub)solution to (4.8), and $\bar{\mathfrak{C}}^\lambda(\mathbf{s},T;I)=\mathfrak{C}^\lambda(\mathbf{s},T;I)$. By PDE comparison principle, we conclude that $\mathfrak{C}^\lambda(\mathbf{s},t;I)\le\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)$ on $\mathbb{R}_+^n\times[0,T]$.

It is well known that the linear PDE (4.9) admits a unique classical bounded solution since the terminal data $g(\cdot)$ is bounded. Hence, there exists a constant K such that

$$0\le\mathfrak{C}^\lambda(\mathbf{s},t)\le\mathfrak{C}^\lambda(\mathbf{s},t;I)\le\bar{\mathfrak{C}}^\lambda(\mathbf{s},t;I)\le K.$$

Next, we show that $\mathfrak{C}^\lambda(\mathbf{s},t)$ is the unique classical solution to (2.10), and $\sum_{i=1}^n s_i\partial_{s_i}\mathfrak{C}^\lambda(\mathbf{s},t)$ is uniformly bounded in $(\mathbf{s},t)$. This will give us the uniform boundedness of $\bar{\pi}^*(\cdot,\cdot)$ and therefore the optimal trading strategy. Indeed, since $R(\cdot)$ is Lipschitz continuous, $a(\cdot)$ is bounded, and $\mathfrak{C}^\lambda(\mathbf{s},t)\in[0,K]$, $\mathbf{L}^3(x)$ is Lipschitz continuous in $x\in[0,K]$. On the other hand, $\mathbf{L}^2$ has at most quadratic growth in gradients. We can apply Theorem 2.9 of Delarue [15] to conclude that the solution $\mathfrak{C}^\lambda(\mathbf{s},t)$ is indeed the unique classical solution to (2.10) and $\sum_{i=1}^n s_i\partial_{s_i}\mathfrak{C}^\lambda(\mathbf{s},t)$ is uniformly bounded.

References

[1] Barles, G., The convergence of approximation schemes for parabolic equations arising in finance theory, *Numerical methods in finance, edited by L. C. G. Rogers and D. Talay, Cambridge University Press*, (1997), 1–21.

[2] Barles, G. and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Analysis*, 4(3), (1991), 271–283.

[3] Becherer, D., Rational hedging and valuation of integrated risks under constant absolute risk aversion, *Insurance: Mathematics and Economics*, 33(1), (2003), 1–28.

[4] Bielecki, T. R., Crepey, S., Jeanblanc, M. and B. Zagari, Valuation and hedging of CDS counterparty exposure in a markov copula model, *International Journal of Theoretical and Applied Finance*, 15(1), (2012), 1–39.

[5] Bielecki, T. R. and M. Jeanblanc, Indifference pricing of defaultable claims, *Indifference Pricing, edited by R. Carmona, Princeton University Press*, (2009), 211–240.

[6] Bielecki, T. R. and M. Rutkowski, Credit risk: Modelling, valuation and hedging, *Springer Finance. Springer-Verlag, Berlin*, (2002).

[7] Blanchet-Scalliet, C., El Karoui, N., Jeanblanc, M. and L. Martellini, Optimal investment decisions when time-horizon is uncertain, *Journal of Mathematical Economics*, 44(11), (2008), 1100–1113.

[8] Brigo, D., Capponi, A. and A. Pallavicini, Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps, *Mathematical Finance*, 24, (2014), 125–146.

[9] Brigo, D. and K. Chourdakis, Counterparty risk for credit default swaps: Impact of spread volatility and default correlation, *International Journal of Theoretical and Applied Finance*, 12, (2009), 1007–1026.

[10] Carmona, R. (editor), Indifference pricing, theory and applications, *Princeton University Press*, (2009).

[11] Carmona, R. and M. Ludkovski, Pricing commodity derivatives with basis risk and partial observations, *Preprint*, (2006).

[12] Crépey, S., Bilateral counterparty risk under funding constraints part I: Pricing, *Math. Finance.*, to appear.

[13] Crépey, S., Bilateral counterparty risk under funding constraints part II: CVA, *Math. Finance.*, to appear.

[14] Davis, M. H. A., Optimal hedging with basis risk, *Second Bachelier Colloquium on Stochastic Calculus and Probability, edited by Y. Kabanov, R. Lipster, and J. Stoyanov, Springer*, (2006), 169–187.

[15] Delarue, F., Estimates of the solutions of a system of quasi-linear PDEs: a probabilistic scheme, *Séminaire de Probabilités XXXVII, Lecture Notes in Math., Spring*, 1832, (2003), 290–332.

[16] Frei, C. and M. Schweizer, Exponential utility indifference valuation in two Brownian settings with stochastic correlation, *Advances in Applied Probability*, 40, (2008), 401–423.

[17] Frei, C. and M. Schweizer, Exponential Utility Indifference Valuation in a General Semimartingale Model, *Optimality and Risk: Modern Trends in Mathematical Finance, edited by F. Delbaen, M. Rásonyi and C. Stricker, Springer*, (2009), 49-86.

[18] Halperin, I. and A. Itkin, Pricing illiquid options with $N+1$ liquid proxies using mixed-dynamic static hedging, *International Journal of Theoretical and Applied Finance*, 16, 7, (2013).

[19] Halperin, I. and A. Itkin, Pricing options on illiquid assets with liquid proxies using utility indifference and dynamic-static hedging, *Quantitative Finance*, 14, 3, (2014), 427-442.

[20] Henderson, V., Valuation of claims on nontraded assets using utility maximization, *Mathematical Finance*, 12, (2002), 351–373.

[21] Henderson, V., The impact of the market portfolio on the valuation, incentives and optimality of executive stock options, *Quantitative Finance*, 5(1), (2005), 1–13.

[22] Henderson, V. and D. Hobson, Substitute hedging, *RISK*, 15(5), (2002), 71–75. (Reprinted in Exotic Options: The Cutting Edge Collection, RISK Books, London, 2003.)

[23] Henderson, V. and D. Hobson, Utility indifference pricing: an overview, *Indifference Pricing, edited by R. Carmona, Princeton University Press*, (2009), 44–74.

[24] Henderson, V. and G. Liang, Pseudo linear pricing rule for utility indifference valuation, *Finance and Stochastics*, to appear.

[25] Hung, M. and Y. Liu, Pricing vulnerable options in incomplete markets, *Journal of Futures markets*, 25, (2005), 135–170.

[26] Jaimungal, S. and G. Sigloch, Incorporating risk and ambiguity aversion into a hybrid model of default, *Mathematical Finance*, 22(1), (2012), 57–81.

[27] Jiao, Y. and H. Pham, Optimal investment with counterparty risk: a default-density modeling approach, *Finance and Stochastics*, 15(4), (2011), 725–753.

[28] Jiao, Y., Kharroubi, I. and H. Pham, Optimal investment under multiple defaults risk: a BSDE-decomposition approach, *Annals of Applied Probability*, 23(2), (2013), 427–857.

[29] Johnson, H. and R. Stulz, The pricing of options with default risk, *Journal of Finance*, 42, (1987), 267–280.

[30] Klein, P., Pricing Black-Scholes options with correlated credit risk, *Journal of Banking and Finance*, 20, (1996), 1211–1229.

[31] Klein, P. and M. Inglis, Pricing vulnerable European options when the option's payoff can increase the risk of financial distress, *Journal of Banking and Finance*, 25, (2001), 993–1012.

[32] Kramkov, D. and M. Sirbu, Asymptotic analysis of utility-based hedging strategies for small number of contingent claims, *Stochastic Processes and Their Applications*, 117(11), (2007), 1606-1620.

[33] Leung, T., S. Sircar and T. Zariphopoulou, Credit derivatives and risk aversion, *Advances in Econometrics, edited by T. Fomby, J.-P. Fouque and K. Solna, Elsevier Science*, 22, (2008), 275–291.

[34] Liang, G. and L. Jiang, A modified structural model for credit risk, *IMA Journal of Management Mathematics*, 23(2), (2012), 147–170.

[35] Marchuk, G. I., Some application of splitting-up methods to the solution of mathematical physics problems, *Aplikace Matematiky*, 13, (1968), 103–132.

[36] Marchuk G. I., Splitting and alternating direction methods, *Handbook of Numerical Analysis*, 35, (1990), 435–461.

[37] Monoyios, M., Malliavin calculus method for asymptotic expansion of dual control problems, *SIAM J. Finan. Math.*, 4 (2013), 884–915.

[38] Murgoci, A., Vulnerable options and good deal bounds: a structural model, *Applied Mathematical Finance*, 20(3), (2013), 246–263.

[39] Musiela, M. and T. Zariphopoulou, An example of indifference prices under exponential preferences, *Finance and Stochastics*, 8, (2004), 229–239.

[40] Nadtochiy, S. and T. Zariphopoulou, An approximation scheme for the solution to the optimal investment problem in incomplete markets, *SIAM J. Finan. Math.*, 4(1), (2013), 494–538.

[41] Sircar, R. and T. Zariphopoulou, Bounds and asymptotic approximations for utility prices when volatility is random, *SIAM Journal on Control and Optimization*, 43(4), (2005), 1328–1353.

[42] Sircar, R. and T. Zariphopoulou, Utility valuation of multi-name credit derivatives and application to CDOs, *Quantitative Finance*, 10(2), (2010), 195–208.

[43] Tehranchi, M., Explicit solutions of some utility maximization problems in incomplete markets, *Stochastic Processes and Their Applications*, 114(1), (2004), 109–125.

[44] Tourin, A., Splitting methods for Hamilton-Jacobi equations, *Numer. Methods Partial Differential Equations*, 22(2), (2006), 381–396.

[45] Zhou, T., Indifference valuation of mortgage-backed securities in the presence of prepayment risk, *Mathematical Finance*, 20, (2010), 479–507.

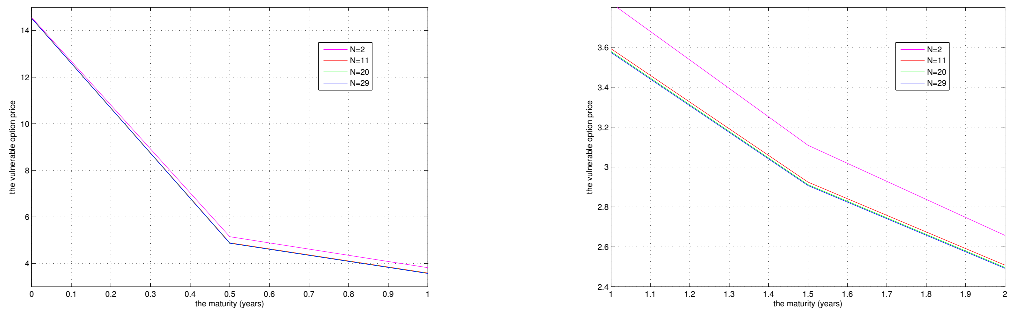

Figure 1: Approximation of the option price for various time steps N. The left panel takes $T \in [0, 1]$; the right panel takes $T \in [1, 2]$.

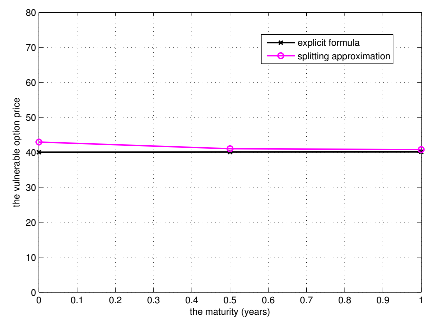

Figure 2: Comparison of the explicit price and the approximated price via splitting when $\bar{\sigma}^P = 0$, $S^2 = 1400$, and $N = 11$.

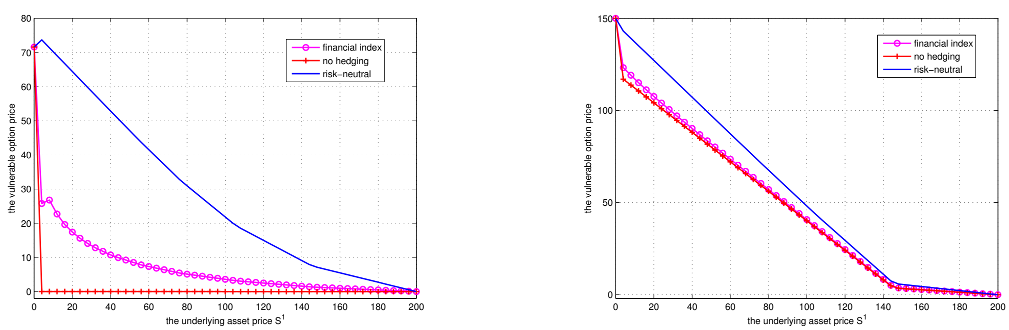

Figure 3: Vulnerable option price against the underlying asset price S^1. The left panel takes $S^2 = 500$; the right panel takes $S^2 = 1400$.

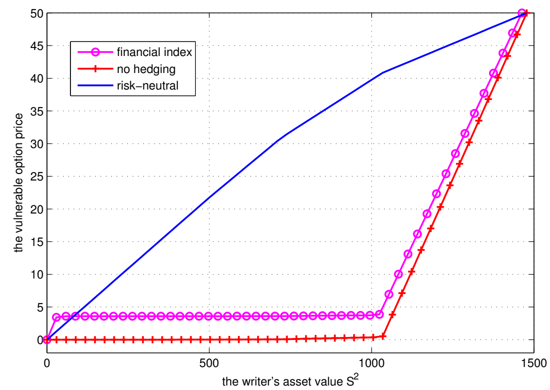

Figure 4: Vulnerable option price against the writer's asset value S^2.

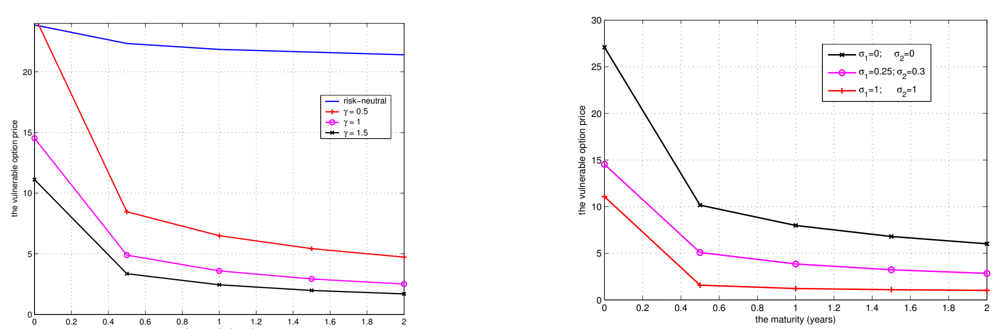

Figure 5: Impact of risk aversion and correlation. The left panel gives the option price against maturity for various risk aversion parameters γ. The right panel gives the price against various correlation parameters. We set $\mu_1 = 0.1$ and $\mu_2 = 0.06$ to satisfy the parameter restriction of Proposition 2.6.