Partial Liquidation under Reference-Dependent Preferences[†]

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We propose a multiple optimal stopping model whereby an investor can sell a divisible asset position at times of her choosing. Investors have S-shaped reference-dependent preferences whereby utility is defined over gains and losses relative to a reference level, and is concave over gains and convex over losses. For a price process following a time-homogeneous diffusion, we employ the constructive potential-theoretic solution methods developed by Dayanik and Karatzas (2003). As an example, we revisit the single optimal stopping model of Kyle, Ou-Yang and Xiong (2006) to allow for partial liquidation. In contrast to the extant literature, we find that the investor may partially liquidate the asset at distinct price thresholds above the reference level. Under other parameter combinations, the investor sells the asset in a block, either at, or above the reference level.

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1 Introduction

Prospect theory was proposed by Kahneman and Tversky (1979) and extended by Tversky and Kahneman (1992). Under prospect theory, utility is reference-dependent so is defined over gains and losses relative to a reference level, rather than over final wealth. The utility function exhibits concavity in the domain of gains and convexity in the domain of losses, and the function is steeper for losses than for gains, a feature known as loss aversion. Prospect theory was originally developed to better fit decision making behavior observed in experimental studies.

In recent years, optimal stopping models employing reference-dependent preferences have been developed in order to understand dynamic behavior of individuals with such preferences and to see to what extent the theory can be used to explain both experimental and empirically observed behavior. A strand of this literature, beginning with Kyle, Ou-Yang and Xiong (2006), has considered problems of optimal sale timing of risky assets under reference-dependent preferences. In this paper we will extend the model of Kyle et al (2006) to consider the question of partial liquidation of assets. Indeed Kyle et al (2006) remark *" ...it would be of interest to incorporate partial liquidation in our model" (p284).*

We propose an infinite horizon optimal stopping model whereby an investor with S-shaped reference-dependent preferences can sell her divisible asset position at times of her choosing in the future. She derives utility from gains and losses relative to a reference level and utility is realized at the time when she sells her last tranche of asset. We first give a general result which allows for a multiple stopping problem (where stopping times are allowed to coincide) to be viewed as a sequence of standard optimal stopping problems. This result is then applied to derive results for a model where utility is given by piecewise exponential functions, steeper for losses than for gains, and the asset price follows a Brownian motion with drift. These explicit calculations enable us to compare to the paper of Kyle et al (2006) who solve the block-sale case under the same modeling assumptions.

Our main finding is in showing that in the extended Kyle et al (2006) model, the investor may engage in partial sales. This is interesting because it is the first time it has been shown that partial liquidation can occur under an S-shaped value function. It is in contrast to the finding in Henderson (2012) where under the Kahneman-Tversky S-shaped value function and exponential Brownian motion, the agent did not choose to partially sell an asset. In addition, the finding that in some circumstances the investor liquidates at exactly break-even captures the spirit of the break-even effect of Thaler and Johnson (1990). In an experimental setting, Thaler and Johnson find that in the presence of prior losses, gambles that offer a chance to break even are very attractive.

Researchers are interested in modeling investor trading behavior under S-shaped reference-dependent preferences (of prospect theory) to see if it can better explain stylized facts in the empirical and experimental data. In particular, reference-dependence is a long standing explanation of why individual investors tend to sell winners too early and ride losers too long, a behavior called the disposition effect (Shefrin and Statman (1985)). In this vein, Kyle et al (2006), Henderson (2012), Barberis and Xiong (2012) and Ingersoll and Jin (2013) contribute optimal stopping models for an investor with reference-dependent preferences under differing assumptions. Kyle et al (2006) and Henderson (2012) treat one-shot or block sale optimal stopping problems under alternative assumptions on the S shaped utility and price processes. In particular, Henderson (2012) (see also Ingersoll and Jin (2013)) contributed a model whereby the investor sells at a loss voluntarily. This provided a better match to the disposition effect (the tendency to sell more readily at a gain than at a loss, see Odean (1998)). Henderson (2012) also considers partial liquidation but finds under the Kahneman-Tversky S-shaped value function and exponential Brownian motion, the agent did not choose to partially sell.

Recent laboratory experiments of Magnani (2017) have been designed to test predictions of S-shaped reference-dependent preferences in a dynamic setting - that decision makers delay realizing disappointing outcomes but rush to realize outcomes that are better than expected. In his experiment, subjects choose when to stop an exogenous stochastic process and most tend to stop at a lower level than the risk-neutral upper threshold and delay capitulating until the process reaches a point significantly below the risk-neutral lower threshold. Imas (2016) studies how realized and paper losses affect behavior in an experiment where subjects make a sequence of investment decisions. In one of the treatments of this experiment, subjects decide whether to realize the outcome of the investment in the middle of the sequence and are found to be more likely to realize gains than losses.

Barberis and Xiong (2012), Ingersoll and Jin (2013) (and also He and Yang (2016)) consider realization utility models whereby investor's treat their investing experience as a series of investment episodes, and receive utility from each individual sale at the time of sale. Mathematically, they sum up the utility of each individual sale. Barberis and Xiong (2012) assume a piecewise linear utility function and they find that the investors never voluntarily sell a stock at a loss. Ingersoll and Jin (2013) extend the model by assuming an S-shaped utility function and find that the investors voluntarily sell a stock both at a gain and at a loss. Recently, He and Yang (2016) extend to include an adaptive reference point which adapts to the stock's prior gain or loss. However, each of these models is separable, in that multiple identical units of assets would be sold simultaneously at the same threshold. None address the question of partial liquidation.

Our aim in this paper is to give a simple, tractable optimal stopping model with S-shape reference-dependent preferences where partial sales *do* arise as an optimal solution. We employ the constructive potential-theoretic solution methods developed by Dayanik and Karatzas (2003) for optimal stopping of linear diffusions. This approach will be particularly useful for our problem as the smooth-fit principle does not apply because of the non-differentiability of the utility function, making the usual variational approach more challenging to apply. One-dimensional optimal stopping problems have been analysed by exploiting the relationship between functional concavity and r-excessivity (Dynkin (1965), Dynkin and Yushkevich (1969)) which has been applied by Dayanik and Karatzas (2003). See also Alvarez (2001, 2003) for related techniques. Carmona and Dayanik (2008) extend the methodology to consider a optimal multiple stopping problem for a regular diffusion process posed in the context of American options when the holder has a number of exercise rights. To make the problem non-trivial it is assumed that the holder chooses the consecutive stopping times with a strictly positive break period (otherwise the holder would use all his rights at the same time). It is difficult to explicitly determine the solution and Carmona and Dayanik (2008) describe a recursive algorithm. In contrast, here in our problem we do not wish to impose any breaks between stopping times, but rather, formulate a model setting where it may be *optimal* to have such breaks. Finally, direct methods for optimal stopping have also been used in stochastic switching problems (Bayraktar and Egami (2010)) and similar ideas are employed by Henderson and Hobson (2012) to solve a problem involving a perfectly divisible tranche of options on an asset with diffusion price process.

One strand of the recent literature has concerned itself with portfolio optimization (optimal control) under prospect theory and examples of this work include Jin and Zhou (2008) and Carassus and Rasonyi (2015). Another focus of the recent literature is on the probability weighting of prospect theory. However, probability weighting leads to a time-inconsistency and thus a difference in behaviour of naive and sophisticated agents, see Barberis (2012). Henderson, Hobson and Tse (2017) (building on seminal work of Xu and Zhou (2008)) study agents who can precommit to a strategy and show that under some assumptions (satisfied by the models of interest including the Kahneman and Tversky (1979, 1992) specification) it consists of a stop-loss threshold together with a continuous distribution on gains. However, recent results (Ebert and Strack (2015), also Henderson, Hobson and Tse (2017)) have shown that naive prospect theory agents never stop gambling. We focus in this paper on reference-dependent S shaped preferences in the absence of probability weighting and extend the literature in the direction of holding a quantity of asset rather than just one unit.

2 General Framework

2.1 The Partial Liquidation Problem

Consider an investor who is holding $N \geq 1$ units of claim on an asset with current price Y_t. The investor is able to liquidate or sell the position in the asset at any time in the future. She can choose times τ_i; $i = 1, ..., N$ at which to liquidate her N units of the claim, and hence is able to partially liquidate her divisible position. We will write $\tau_1 \geq ... \geq \tau_N$ so τ_i denotes the sale time when there are i units *remaining* in the portfolio. For each unit i, the investor receives payoff $h^i(Y_{\tau_i})$ where the $h^i(\cdot)$ are non-decreasing functions, and compares this amount to a corresponding reference level h^i_R. As is often the case in the literature, an interpretation of h^i_R is the breakeven level or the amount paid for the claim on the asset itself, and we will later specialize to this choice.

We would like a formulation in which the potential partial sales are not independent (so delaying a partial sale will impact on future sales) and so our investor considers her position as an investment episode which is closed and valued once the final partial sale takes place. This might be appropriate for institutional investors who are more likely to view investments in terms of overall portfolio position. Under this interpretation, the investor's problem can be written as:

$$V_N(y,0) = \sup_{\tau_1 \geq \cdots \geq \tau_N} \mathbb{E}\left[U\left(\sum_{i=1}^{N}(h^i(Y_{\tau_i}) - h_R^i)\right)|Y_0 = y\right] \tag{1}$$

where utility function U is an increasing function. Later we will specialize to the reference-dependent S-shaped U given in the next section.

Finally, to close this section, we comment on alternative specifications. First note that in common with Kyle et al (2006), we do not include a discount term in the specification above. Although it would be possible to do so, it is questionable whether it is desirable when losses are involved. Second, as shown in Henderson (2012) and Barberis and Xiong (2012), if the investor instead considered each partial sale as an independent investment episode then she would optimize:

$$\sup_{\tau_1 \geq \cdots \geq \tau_N} \mathbb{E}\left[\sum_{i=1}^{N} U\left(h^i(Y_{\tau_i}) - h_R^i\right)|Y_0 = y\right] \tag{2}$$

Whilst this captures the spirit of Barberis and Xiong's (2012) realization utiity whereby investors consider a series of investing "episodes", mathematically, this formulation splits into N independent stopping problems and thus does not capture the interdependency we desire.

2.2 Reference-Dependent Preferences

When we present results for a specific model, we shall take the two piece exponential utility function used by Kyle et al (2006):

$$U(x) = \begin{cases} \phi_1(1 - e^{-\gamma_1 x}), & \text{if } x \geq 0 \\ \phi_2(e^{\gamma_2 x} - 1), & \text{if } x < 0 \end{cases} \tag{3}$$

where $\phi_1, \phi_2, \gamma_1, \gamma_2 > 0$. Above the reference point, the agent's utility function is a concave exponential function, with γ_1 measuring the local absolute risk aversion. Below the reference point, the value function is a convex exponential function, with γ_2 measuring the local absolute risk loving level. In addition, we assume $\phi_1\gamma_1 < \phi_2\gamma_2$ to ensure that the agent is loss averse, that is, more sensitive to losses than to gains around the reference point, ie. $U'(0-) > U'(0+)$.

An alternative specification proposed by Kahneman and Tversky (1979) uses piecewise power functions, however this choice involves infinite marginal utility at the origin which may have less desirable consequences for some applications of prospect theory, see the discussion in De Giorgi and Hens (2006) and recent work of Azevedo and Gottlieb (2012).

2.3 The Price Process

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting a Brownian Motion $W = (W_t)_{t>0}$ and let $Y = (Y_t)_{t\geq 0}$ be a one dimensional time-homogeneous diffusion process solving:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \tag{4}$$

for Borel functions[1] $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to \mathbb{R}^+$ where $\mathcal{I} = (a_\mathcal{I}, b_\mathcal{I}) \subseteq \mathbb{R}$ is the state space of Y_t with endpoints $-\infty \leq a_\mathcal{I} < b_\mathcal{I} \leq \infty$. Let $\tau^Y_{(a,b)} = \inf\{t :\ Y_t \notin (a, b)\}$, ie. the first time Y leaves the interval (a, b). Consider the scale function $S(\cdot)$ of the diffusion Y_t satisfying:

$$\mathcal{D}f(x) = \frac{1}{2}\sigma^2(x)\frac{d^2 f}{dx^2}(x) + \mu(x)\frac{df}{dx}(x) = 0 \qquad x \in \mathcal{I} \tag{5}$$

and ensuring the process $S(Y_{t\wedge\tau})$ is a local martingale, see Revuz and Yor (1999).

In Section 3.2 we specialize to the model used in Kyle et al. (2006) and hence take:

$$dY_t = \mu dt + \sigma dW_t \tag{6}$$

where μ and $\sigma > 0$ are constants and $\mathcal{I} = (-\infty, \infty)$. Under these assumptions the scale function is given by $S(y) = e^{\eta y}$ if $\mu < 0$, $-e^{\eta y}$ if $\mu > 0$, and $S(y) = y$ if $\mu = 0$ where $\eta = -2\mu/\sigma^2$. Note that η depends solely on the return-for-risk ratio μ/σ^2, closely related to the Sharpe ratio.

[1]We assume that $\mu(\cdot)$ and $\sigma(\cdot)$ are sufficiently regular so there exists a weak solution to the SDE so that the scale function $S(\cdot)$ exists. See Revuz and Yor (1999).

3 Solution to the Partial Liquidation Problem

3.1 The General Problem

To solve (1) we break down the stopping problem into N sub-problems. Denote initial wealth by x (total gains or losses from previous sales) and define:

$$\begin{aligned} V_N(y,x) &= \sup_{\tau_1 \geq \cdots \geq \tau_N} \mathbb{E}\left[U\left(x + \sum_{i=1}^{N} (h^i(Y_{\tau_i}) - h_R^i) \right) |Y_0 = y \right] \\ &= \sup_{\tau_1 \geq \cdots \geq \tau_N} \mathbb{E}\left[\mathbb{E}\left[U\left(x + \sum_{i=1}^{N} (h^i(Y_{\tau_i}) - h_R^i) \right) |\mathcal{F}_{\tau_N} \right] |Y_0 = y \right] \end{aligned} \tag{7}$$

We are primarily interested in (1), ie. $x = 0$. The following result will facilitate the decomposition of (7) into N sub-problems.

In order to be able to solve the problem in (7) we assume that the problem satisfies:

$$\mathbb{E}\left[\sup_{0 \leq t_N \leq \cdots \leq t_1 \leq \infty} |U\left(x + \sum_{i=1}^{N} (h^i(Y_{t_i}) - h_R^i) \right)| \right] < \infty \tag{8}$$

Proposition 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Y be an Ito diffusion process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and let $U(\cdot)$ be an increasing continuous function, with $U(0) = 0$. Given:*

$$\mathbb{E}\left[\sup_{0 \leq t_n \leq \cdots \leq t_1 \leq \infty} |U\left(\sum_{i=1}^{n} h^i(Y_{t_i}) \right)| \right] < \infty \tag{9}$$

then it follows that:

$$\begin{aligned} &\sup_{\tau_n \leq \cdots \leq \tau_1} \mathbb{E}\left[\mathbb{E}\left[U\left(\sum_{i=1}^{n} h^i(Y_{\tau_i}) \right) |\mathcal{F}_{\tau_n} \right] |Y_0 = y \right] \\ &\qquad = \sup_{\tau_n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau_{n-1} \leq \cdots \leq \tau_1 \,:\, \tau_{n-1} \geq \tau_n} \mathbb{E}\left[U\left(\sum_{i=1}^{n} h^i(Y_{\tau_i}) \right) |\mathcal{F}_{\tau_n} \right] |Y_0 = y \right] \end{aligned}$$

Proof. The result follows if we show:

$$\begin{aligned} &\sup_{\tau_n \leq \cdots \leq \tau_1} \mathbb{E}\left[\mathbb{E}\left[U\left(\sum_{i=1}^{n} h^i(Y_{\tau_i}) \right) |\mathcal{F}_{\tau_n} \right] |Y_0 = y \right] \\ &\qquad \geq \sup_{\tau_n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau_{n-1} \leq \cdots \leq \tau_1 \,:\, \tau_{n-1} \geq \tau_n} \mathbb{E}\left[U\left(\sum_{i=1}^{n} h^i(Y_{\tau_i}) \right) |\mathcal{F}_{\tau_n} \right] |Y_0 = y \right] \end{aligned} \tag{10}$$

since the reverse inequality is trivial. Given an arbitrary stopping time τ_n consider the random variable:

$$Z^{(\tau_{n-1},\ldots,\tau_1)} = \mathbb{E}\left[U\left(\sum_{i=1}^{n} h^i(Y_{\tau_i})\right)|\mathcal{F}_{\tau_n}\right]$$

and consider the family $\Gamma = \{Z^\alpha : \alpha \in \mathcal{I}\}$ where $\mathcal{I}$ is the set of all $(n-1)$ tuples of $\{F_t\}$-measurable stopping times $(\xi_{n-1},\ldots,\xi_1)$ satisfying $\tau_n \leq \xi_{n-1} \leq \cdots \leq \xi_1$ almost surely. As shown in Lemma 6.1, the family Γ has the lattice property and hence there exists a countable subset $\mathcal{J} \subseteq \mathcal{I}$ where $\mathcal{J} = \{\alpha_j \ : \ j \in \mathbb{N}\}$ and:

$$Z^* = \operatorname*{ess\,sup}_{\alpha\in\mathcal{I}} Z^\alpha = \lim_{j\to\infty} Z^{\alpha_j} \qquad \text{with} \qquad Z^{\alpha_1} \leq Z^{\alpha_2} \leq \ldots \quad \mathbb{P}-a.s.$$

Using (9) and Jensen's inequality we get $\mathbb{E}\left[|Z^*|\right] < \infty$ and hence by MON the right hand side of (10) becomes:

$$\begin{aligned}\sup_{\tau_n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\alpha\in\mathcal{I}} Z^\alpha\right] &= \sup_{\tau_n} \lim_{j\uparrow\infty} \mathbb{E}\left[Z^{\alpha_j}\right] \\ &\leq \sup_{\tau_n} \sup_{\tau_{n-1}\leq\cdots\leq\tau_1 \ : \ \tau_{n-1}\geq\tau_n} \mathbb{E}\left[Z^{(\tau_{n-1},\ldots,\tau_1)}\right]\end{aligned}$$

□

Assuming the condition in (8) applies, from Proposition 3.1 it follows that for $1 \leq n \leq N$:

$$\begin{aligned}V_n(y,x) &= \sup_{\tau_n\leq\cdots\leq\tau_1} \mathbb{E}\left[U\left(x + \sum_{i=1}^{n}(h^i(Y_{\tau_i}) - h^i_R)\right)|Y_0 = y\right] \\ &= \sup_{\tau_n} \mathbb{E}\left[\sup_{\tau_{n-1}\leq\cdots\leq\tau_1 \ : \ \tau_{n-1}\geq\tau_n} \mathbb{E}\left[U\left(x + \sum_{i=1}^{n}(h^i(Y_{\tau_i}) - h^i_R)\right)|\mathcal{F}_{\tau_n}\right]|Y_0 = y\right] \\ &= \sup_{\tau_n} \mathbb{E}\left[V_{n-1}(Y_{\tau_n}, x + h^n(Y_{\tau_n}) - h^n_R)|Y_0 = y\right] \qquad (11)\end{aligned}$$

where $V_0(y,x) = U(x)$.

Given the time-homogeneity of the problem the structure of the solution must be to stop when the price process Y exits an interval. Thus, the approach is to consider stopping times of this form and choose the “best” interval. We first transform the problem into natural scale - this simplifies calculations as we then work with (local) martingales. Define $\Theta_t = S(Y_t)$ to transform the process into natural scale and let $\Theta_0 = \theta_0 = S(Y_0)$. We can map exit times of price Y from an interval to exit times of Θ from a transformed interval.

ie. $\tau^Y_{(a,b)} = \inf\{t: \ Y_t \notin (a,b)\} \equiv \inf\{t: \ \Theta_t \notin (S(a), S(b))\} = \inf\{t: \ \Theta_t \notin (\phi,\psi)\} = \tau^\Theta_{(\phi,\psi)}$, where $S(a) = \phi, S(b) = \psi$.

Define $g_n(\theta, x)$ to be the value of the game with $1 \le n \le N$ units remaining, wealth x, and plan to sell one unit immediately. Then by definition,

$$g_n(\theta, x) = V_{n-1}(S^{-1}(\theta), x + h^n(S^{-1}(\theta)) - h^n_R). \tag{12}$$

We now give some intuitive arguments to describe the solution when we consider a bounded interval and then follow this with more cases in Proposition 3.2. Consider first *any* fixed interval $(a,b) \in \mathcal{I}$ such that $(S(a), S(b))$ is bounded, and with n units remaining,

$$\begin{aligned}
&\mathbb{E}\left[V_{n-1}(Y_{\tau^Y_{(a,b)}}, x + h^n(Y_{\tau^Y_{(a,b)}}) - h^n_R)|Y_0 = y\right] \\
&= \mathbb{E}\left[g_n(\Theta_{\tau^\Theta_{(\phi,\psi)}}, x)|\Theta_0 = \theta\right] \\
&= g_n(\phi, x)\frac{\psi-\theta}{\psi-\phi} + g_n(\psi, x)\frac{\theta-\phi}{\psi-\phi}
\end{aligned} \tag{13}$$

where we use the probabilities of the (bounded) martingale $(\Theta_t)_{t\le\tau^\Theta_{(\phi,\psi)}}$ hitting the ends of the interval. We then choose the "best" interval (ϕ,ψ):

$$\sup_{\phi<\theta<\psi}\{g_n(\phi, x)\frac{\psi-\theta}{\psi-\phi} + g_n(\psi, x)\frac{\theta-\phi}{\psi-\phi}\} = \bar{g}_n\left(\theta, x\right) \tag{14}$$

where $\bar{g}_n(\theta, x)$ is the smallest concave majorant of the function $g_n(\theta, x)$.

Figure 1 gives a stylized representation of $g_n(\theta, x)$ as a function of θ. We can use the graph to explain intuitively that the solution to the "best" interval problem above is indeed the smallest concave majorant. We want to choose endpoints ϕ, ψ to maximize the expression in brackets in (14). If we start at the point θ_A on the graph, then the expression in the curly brackets in (14) is maximized by taking $\phi = \psi = \theta_A$ (all other pairs of endpoints give values beneath $g_n(\theta_A, x)$). This corresponds to immediate stopping since we stop when we exit the interval (ϕ,ψ). However, if we start at the point θ_B, the quantity in brackets is maximized if we take $\phi = \phi_B$ and $\psi = \psi_B$. In fact, for any starting point in the interval (ϕ_B, ψ_B), the endpoints ϕ_B, ψ_B are best. Thus, for any $\theta \in (\phi_B, \psi_B)$, the solution is to stop when the transformed price Θ_t reaches either endpoint of the interval. Outside the interval (ϕ_B, ψ_B), the solution is to stop immediately. The solution is to take the smallest concave majorant, which is equal to the function g_n itself for θ outside the interval (ϕ_B, ψ_B) and the

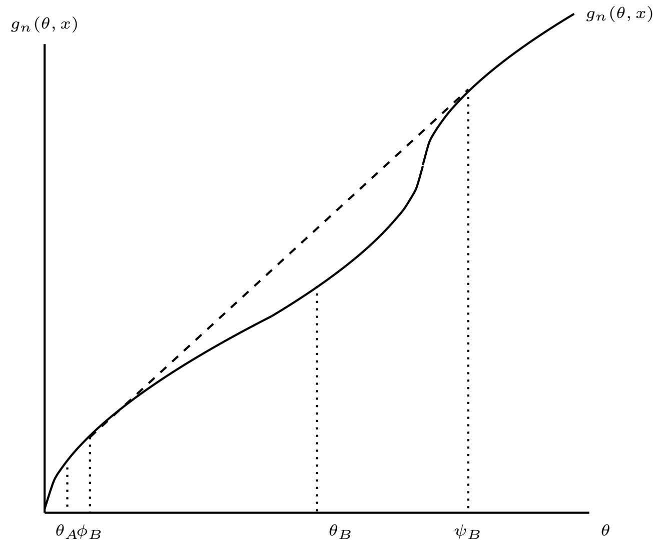


Figure 1: Stylized representation of the function $g_n(\theta, x)$ as a function of transformed price θ, for fixed x, where $\theta = S(y)$. The function $g_n(\theta, x)$ represents the value of the game to the investor holding $1 \le n \le N$ units of asset if she sells immediately. The smallest concave majorant $\bar{g}_n(\theta, x)$ is formed by taking the straight dashed line for $\theta \in (\phi_B, \psi_B)$ and the function g_n itself for $\theta \le \phi_B$ and $\theta \ge \psi_B$.

dashed straight line joining the endpoints for values of θ inside the interval. This intuition lies behind the following result.

Proposition 3.2. *Let $Y = (Y_t)_{t\ge 0}$ be a one dimensional time-homogeneous diffusion process with dynamics given in (4) and state space $\mathcal{I} = (a_\mathcal{I}, b_\mathcal{I}) \subseteq \mathbb{R}$. On the interval $(S(a_\mathcal{I}), S(b_\mathcal{I}))$, let $\bar{g}_n(\theta, x)$ be the smallest concave majorant of $g_n(\theta, x)$ where $1 \le n \le N$.*
(i) Suppose $S(a_\mathcal{I}) = -\infty$. Then $V_n(y, x) = V_{n-1}(b_\mathcal{I}, x + h^n(b_\mathcal{I}) - h^n_R)$; $\quad y \in (a_\mathcal{I}, b_\mathcal{I})$.
(ii) Suppose $S(a_\mathcal{I}) > -\infty$. Then $V_n(y, x) = \bar{g}_n(S(y), x)$; $\quad y \in (a_\mathcal{I}, b_\mathcal{I})$.

Proof of Proposition 3.2. Whilst this follows directly from optimal stopping theory (see Peskir and Shiryaev (2006) and Dayanik and Karatzas (2003)) we prove the result specifically for our problem. Both results are proven for $n = 1$ as the same arguments can be easily extended for general n.

(i) Trivially $V_1(y, x) \le U(h^1(b_\mathcal{I}) - h^1_R + x)$. Consider a sequence $b_n \uparrow b_\mathcal{I}$ and let $\tau_n = \tau^Y_{(a_\mathcal{I}, b_n)}$. The local martingale Θ leaves the interval $(S(a_\mathcal{I}), S(b_n)) = (-\infty, S(b_n))$ almost surely on

the right and hence $V_1(y,x) \geq U(h^1(b_n) - h^1_R + x) \uparrow U(h^1(b_\mathcal{I}) - h^1_R + x)$ giving the result. The same argument can be repeated for general n due to representation (11).
(ii) Holding x constant, let $\tilde{g}_1(\cdot, x)$ be *any* increasing, concave majorant of $g_1(\cdot, x)$. By definition,

$$V_1(y,x) = \sup_\tau \mathbb{E}[U(h^1(Y_\tau) - h^1_R + x)|Y_0 = y] = \sup_\tau \mathbb{E}[g_1(\Theta_\tau, x)|\Theta_0 = \theta]$$

But for every stopping rule τ:

$$\mathbb{E}[g_1(\Theta_\tau, x)|\Theta_0 = \theta] \leq \mathbb{E}[\tilde{g}_1(\Theta_\tau, x)|\Theta_0 = \theta] \leq \tilde{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta], x)$$

where we use the fact $\tilde{g}_1(\cdot, x)$ is a concave majorant of $g_1(\cdot, x)$ and Jensen's inequality. Finally given that a local martingale bounded below is a super-martingale we obtain:

$$\tilde{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta], x) \leq \tilde{g}_1(\theta, x)$$

and hence $V_1(y,x) \leq \bar{g}_1(\theta, x)$ where $\bar{g}_1$ is the smallest concave majorant of g_1. Thus we have shown we cannot do better than $\bar{g}_1$, allowing for general stopping rules. It remains to show that there is a stopping rule which attains this bound.
Suppose first that $\limsup_{\psi \uparrow S(b_\mathcal{I})} \frac{g_1(\psi,x)}{\psi - S(a_\mathcal{I})} = \infty$. In this case $\bar{g}_1(\theta, x) = \infty$ for $\theta \in (S(a_\mathcal{I}), S(b_\mathcal{I}))$. Then there exists $b_n \uparrow b_\mathcal{I}$ such that $\frac{\bar{g}_1(S(b_n),x)}{S(b_n) - S(a_\mathcal{I})} \uparrow \infty$. Then, for any $S(a_\mathcal{I}) \leq \hat{\phi} < \theta$ we have for $\tau^*_n = \tau^\Theta_{(\hat{\phi}, S(b_n))} = \tau^Y_{(S^{-1}(\hat{\phi}), b_n)}$

$$\mathbb{E}[U(h^1(Y_{\tau^*_n}) - h^1_R + x)|Y_0 = S^{-1}(\theta)] \geq \bar{g}_1(\hat{\phi}, x)\frac{S(b_n) - \theta}{S(b_n) - \hat{\phi}} + \bar{g}_1(S(b_n), x)\frac{\theta - \hat{\phi}}{S(b_n) - \hat{\phi}} \uparrow \infty.$$

Now suppose $\limsup_{\psi \uparrow S(b_\mathcal{I})} \frac{g_1(\psi,x)}{\psi - S(a_\mathcal{I})} < \infty$. In this case $\bar{g}_1(\cdot, x)$ is a finite function. Fix θ and let $\Upsilon = \{\upsilon : \bar{g}_1(\upsilon, x) = g_1(\upsilon, x)\}$. Suppose $\theta \in \Upsilon$. Then with $\tau = 0$, $\mathbb{E}[U(h^1(Y_0) - h^1_R + x)|\Theta_0 = \theta] = g_1(\theta, x) = \bar{g}_1(\theta, x)$ and we are done. Otherwise define $\phi^* = \sup\{\xi < \theta : \xi \in \Upsilon\}$, $\psi^* = \inf\{\xi > \theta : \theta \in \Upsilon\}$ and set $\phi^* = S(a_\mathcal{I})$ if the set $\{\xi < \theta : \theta \in \Upsilon\}$ is empty, and $\psi^* = \infty$ if $\{\xi > \theta : \theta \in \Upsilon\}$ is empty.
Suppose $\psi^* < \infty$. Then $\bar{g}_1(\theta, x)$ is linear in θ on the interval $\theta \in (\phi^*, \psi^*)$ and $\Theta_{t \wedge \tau_{(\phi^*, \psi^*)}}$ is a martingale. Then

$$\mathbb{E}[U(h^1(Y_{\tau^\Theta_{(\phi^*,\psi^*)}}) - h^1_R + x)|\Theta_0 = \theta] = \mathbb{E}[g_1(\Theta_{\tau^\Theta_{(\phi^*,\psi^*)}}, x)] = \mathbb{E}[\bar{g}_1(\Theta_{\tau^\Theta_{(\phi^*,\psi^*)}}, x)] = \bar{g}_1(\theta, x).$$

Conversely, if $\psi^* = \infty$, then choose $\theta_n > \theta$ so that

$$\bar{g}_1(\phi^*, x)\frac{\theta_n - \theta}{\theta_n - \phi^*} + \bar{g}_1(\theta_n, x)\frac{\theta - \phi^*}{\theta_n - \phi^*} > \bar{g}_1(\theta, x) - \epsilon.$$

Then, for the stopping time $\tau^* = \tau^{\Theta}_{(\phi^*, \theta_n)} = \tau^{Y}_{(S^{-1}(\phi^*), S^{-1}(\theta_n))}$ we get that $\mathbb{E}[U(h^1(Y_{\tau^*}) - h^1_R + x)|Y_0 = S^{-1}(\theta)] \geq \bar{g}_1(\theta, x) - \epsilon$. □

Note that Dayanik and Karatzas (2003) (Proposition 5.10) use alternative conditions of:

$$\lambda_{a_{\mathcal{I}}} = \limsup_{y \downarrow a_{\mathcal{I}}} \; g_n^+(S(y), x) < \infty \qquad \text{and} \qquad \lambda_{b_{\mathcal{I}}} = \limsup_{y \uparrow b_{\mathcal{I}}} \frac{g_n^+(S(y), x)}{S(y)} < \infty \tag{15}$$

to characterize non-degenerate solutions in their single stopping problem.

Following from Proposition 3.2, by denoting the smallest concave majorant of $g_n(\theta, x)$ by $\bar{g}_n(\theta, x)$, it follows that for $2 \leq n \leq N$,

$$g_n(\theta, x) = \bar{g}_{n-1}(\theta, x + h^n(S^{-1}(\theta)) - h^n_R) \tag{16}$$

This implies that the value function at the n^{th} step can be obtained by first solving for the value function at the $(n-1)^{th}$ step.

3.2 Solution and Discussion

Having obtained such a characterization for the value function under partial liquidation, we shall apply the above methodology to the price process and preference function defined in Sections 2.3 and 2.2 respectively. We shall limit our discussion to the case when $N = 2$. The solutions for $N \geq 2$ can then be obtained through the same approach but will become rather unwieldy. Since our aim is to show that the investor may partially liquidate, we only need consider $N = 2$ to show this.

We specialize to the case when the investor is selling or liquidating the asset itself, so consider $h^i(y) = y$ for $i = 1, \ldots, N$, with the common reference price $h_R = y_R$ for $i = 1, \ldots, N$. We also interpret the reference price y_R as the price at which the asset was purchased in the past. Before stating the main result described above, we shall first re-state the results obtained by Kyle et al.(2006) and Henderson (2012) for the case of $N = 1$; that is, when only block sales are allowed. Kyle et al (2006) use a variational approach which is challenging due to the S-shaped utility function. Note that Kyle et al (2006) do not include case (ii) as they rule it out by an additional assumption on parameter values, perhaps because it assisted in the ease of their calculations.

Proposition 3.3. *(Kyle et al (2006), Henderson (2012))*
Consider the optimal liquidation problem in (1) with $N = 1$, $h^1(y) = y$ *and* $h^1_R = y_R$ *and suppose that the price process* $(Y_t)_{t\geq 0}$ *is given by a Brownian Motion with drift* $dY_t = \mu dt + \sigma dW_t$ *(see (6)) and the utility function* U *is the* S*-shaped piecewise exponential given by (3). Define* $\eta = -2\mu/\sigma^2$*. The solution is given by the following four cases:*

(i) If $\eta \leq 0$*, the investor waits indefinitely and never liquidates.*

(ii) If $0 < \eta < ((\phi_1/\phi_2)\gamma_1)$ *the investor always sells at a level* $\bar{y}^{(1)}_u > y_R$ *or above. The level* $\bar{y}^{(1)}_u$ *is given by:*

$$\bar{y}^{(1)}_u = y_R - \frac{1}{\gamma_1}\left(\ln\left[\left(\frac{\phi_1+\phi_2}{\phi_1}\right)\left(\frac{2\mu}{2\mu-\gamma_1\sigma^2}\right)\right]\right) \tag{17}$$

(iii) If $0 < ((\phi_1/\phi_2)\gamma_1) < \eta < \gamma_2$ *the investor sells everywhere at or above break-even level* y_R.

(iv) If $\eta \geq \gamma_2$*, investor sells immediately at any price level.*

We see from the above proposition for the block sale problem that there are four cases depending upon the relative parameter values. Two cases are degenerate. If $\mu \geq 0$, as in case (i), the investor waits indefinitely regardless of price. Conversely, if the Sharpe ratio is sufficiently large and negative compared to the risk seeking parameter γ_2, the investor sells immediately. The interesting situations arise in between these extremes where we have two possibilities depending on where η lies in the interval $0 < \frac{\gamma_1\phi_1}{\phi_2} < \gamma_2$. In (iii), when the Sharpe ratio is relatively poor, such that η is in $(\frac{\gamma_1\phi_1}{\phi_2}, \gamma_2)$. the investor waits below the breakeven level and liquidates at the breakeven level y_R itself. The width of the interval $(\frac{\gamma_1\phi_1}{\phi_2}, \gamma_2)$ reflects the strength of loss aversion in this model. If the investor is more loss averse, then this interval is larger, and so a greater range of Sharpe ratios will fall into this case. In (ii), when the Sharpe ratio is better, and $\eta \in (0, \frac{\gamma_1\phi_1}{\phi_2})$, the investor waits beyond the breakeven level and will not liquidate until the investment is in gains. Note in (iii) and (iv), given our interpretation of the reference level as the price paid, liquidation will only occur at the breakeven level itself. Effectively, the investor never holds the asset in these scenarios. Note also that the investor never waits to sell at a loss in this model.

Our interest in this paper is how the above generalizes to partial liquidation. We can now state our extension to $N = 2$ units.

Proposition 3.4. *Consider the optimal partial liquidation problem in (1) with $N = 2$, $h^2(y) = h^1(y) = y$ and $h^2_R = h^1_R = y_R$ and suppose that the price process $(Y_t)_{t\geq 0}$ is given by a Brownian Motion with drift $dY_t = \mu dt + \sigma dW_t$ (see (6)) and the utility function U is the S-shaped piecewise exponential given by (3). Define $\eta = -2\mu/\sigma^2$. The solution consists of five cases depending on the relative parameter values:*

(i) If $\eta \leq 0$, the investor waits indefinitely and never liquidates.

(ii) If $0 < \eta < \gamma_1\phi_1/\phi_2$, the investor sells first at $\bar{y}_u^{(2)}$ and then at $\bar{y}_u^{(1)}$. (If $\bar{y}_u^{(2)} \geq \bar{y}_u^{(1)}$ then the investor sells both assets at $\bar{y}_u^{(2)}$.)

(iii) The investor sells both units of asset at a price level $\bar{y}_u^{(2)} > y_R$ if:

(a) $0 < \eta/2 < \gamma_2 \leq \eta$ and $\eta/2 < \gamma_1\phi_1/\phi_2$ or;

(b) $0 < \eta < \gamma_2$ and $\eta/2 < \gamma_1\phi_1/\phi_2 < \eta$,

(iv) If $0 < \eta/2 < \gamma_2$ and $\eta/2 \geq \gamma_1\phi_1/\phi_2$, the investor sells both units of asset at y_R.

(v) If $0 < \gamma_2 \leq \eta/2$, the investor sells immediately at all price levels.

The threshold $\bar{y}_u^{(2)}$ in (ii) and (iii) is given by:

$$\bar{y}_u^{(2)} = y_R - \frac{1}{2\gamma_1}\left(\ln\left[\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\left(\frac{\mu}{\mu - \gamma_1\sigma^2}\right)\right]\right) \tag{18}$$

Similar to the case when only block sales are allowed (Proposition 3.3), the above proposition shows that under partial liquidation, the behaviour if the investor still depends on where the value of η lies in comparison with scaled versions of the key quantities $\frac{\gamma_1\phi_1}{\phi_2}$ and γ_2. Thus again, decisions rely on the Sharpe ratio of the asset together with the investor's risk aversion, risk seeking and loss aversion measures.

A first observation is that whilst we require $\frac{\gamma_1\phi_1}{\phi_2} < \gamma_2$ for loss aversion to hold, we may have either ordering of γ_2 and $\frac{2\gamma_1\phi_1}{\phi_2}$, with the ordering $\frac{2\gamma_1\phi_1}{\phi_2} < \gamma_2$ reflecting stronger loss aversion. We observe that the extreme cases where the investor waits indefinitely (see (i)) and sells immediately (see (v)) are still present when the asset can be partially sold. However, the Sharpe ratio of the asset needs to be worse compared to the block sale model to put the investor in the "sell immediately" case, since we now require $\eta > 2\gamma_2$.

The situation we are most interested in occurs when $0 < \eta < \frac{\gamma_1\phi_1}{\phi_2}$ (see (ii) above). This is the parameter regime when the Sharpe ratio, whilst negative, is better than all the other cases with $\eta > 0$. In this parameter regime, the investor sells one unit of asset at threshold $\bar{y}_u^{(2)}$ and the other unit of asset at threshold $\bar{y}_u^{(1)}$. We have the following result.

Corollary 3.5. *Let $0 < \eta < \gamma_1\phi_1/\phi_2$. If parameters are such that*

$$\frac{\phi_2}{\phi_1}\left[\frac{2\eta}{\gamma_1}\left(1+\frac{\eta}{2\gamma_1}\right)\right] < 1 \tag{19}$$

we have $\bar{y}_u^{(2)} < \bar{y}_u^{(1)}$ and the asset will be sold at two distinct thresholds. Otherwise, $\bar{y}_u^{(2)} > \bar{y}_u^{(1)}$ and both units of asset are sold together at threshold $\bar{y}_u^{(2)}$.

Figure 2 illustrates results from case (ii). All panels use a reference level (per unit) of $y_R = 1$. Panels (a)-(c) use parameters: $\phi_1 = 0.5, \phi_2 = 0.9,\ \gamma_1 = 3, \gamma_2 = 2,\ \eta = 0.66$. Condition (19) is satisfied, giving $\bar{y}_u^{(2)} = 1.213 < \bar{y}_u^{(1)} = 1.227$. Panels (d)-(f) use parameters: $\phi_1 = 0.2, \phi_2 = 0.9, \gamma_1 = 3, \gamma_2 = 1$ and $\eta = 0.66$. Condition (19) is not satisfied for this choice, hence both units are sold at $\bar{y}_u^{(2)} = 1.101$.

The remaining cases (iii) and (iv) involve selling both units of asset simultaneously, either at a gain threshold $\bar{y}_u^{(2)}$ (case (iii)) or at the breakeven level y_R (case (iv)). Figure 3 illustrates results from cases (iii)(a) and (iv). Again, all panels use a reference level (per unit) of $y_R = 1$. Panels (a) and (b) illustrate case (iii)(a). Parameters are: $\phi_1 = 0.5, \phi_2 = 1.3, \gamma_1 = 2.5, \gamma_2 = 1$ and $\eta = 1.65$. We see both units are sold at $\bar{y}_u^{(2)} = 1.022 > y_R$. Panels (c)-(e) illustrate case (iv). Parameters are: $\phi_1 = 0.5, \phi_2 = 1.3, \gamma_1 = 1, \gamma_2 = 2,\ \eta = 1.64$. Both units are sold at $\bar{y}_u^{(1)} = \bar{y}_u^{(2)} = 1 = y_R$.

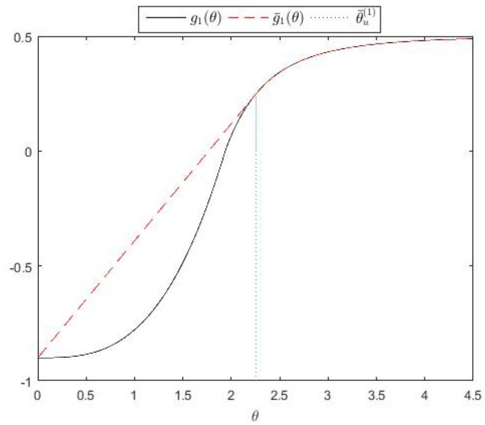
To demonstrate the difference between the solutions in the block sale and partial sale models, consider an example where $\eta \in (\frac{\gamma_1\phi_1}{\phi_2}, \gamma_2)$. If only block sales were allowed, Proposition 3.3

(iii) says the investor sells everywhere at or above the breakeven level y_R and hence, under our assumptions, effectively never enters the game in this parameter regime. Once we allow partial sales, there are two possibilities. The investor may still sell at y_R (and thus effectively never enter) but only if loss aversion is relatively strong, in that we have $\frac{2\gamma_1\phi_1}{\phi_2} < \eta < \gamma_2$ (see Proposition 3.4 case (iv)). The other possibility is that the investor sells both units of asset at a gain, at threshold $\bar{y}_u^{(2)}$, see Proposition 3.4 case (iii) (b).

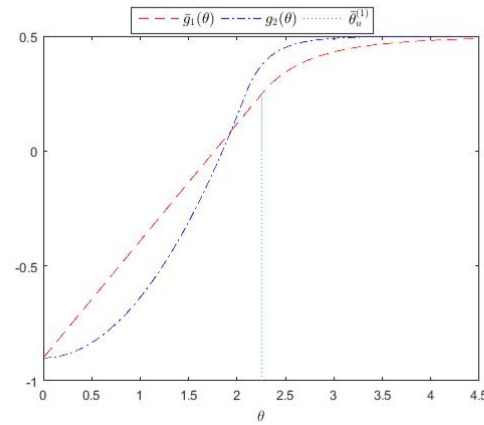
4 Discussion and Conclusions

Researchers have studied multiple optimal stopping problems under standard concave utility functions in other settings. For example, Grasselli and Henderson (2009), Leung and Sircar (2009) and Henderson and Hobson (2011) consider the exercise of American options under concave utilities and demonstrate that the optimal solution involves exercising a tranche of (identical) options over different asset price thresholds. Intuitively, a risk averse investor wants to spread the risk of continuing to hold the options by exercising them separately. Similarly, intuition would tell us that an investor who is risk seeking with convex utility, would prefer to engage in a block sale. What might we expect from an S-shaped reference dependent utility? Since there are concave and convex parts to the utility, we could reasonably expect that either might be dominant, depending on parameters. Somewhat surprisingly, Henderson (2012) showed that under Tversky and Kahneman (1992) S shaped function and exponential Brownian motion, the investor's optimal strategy, when not degenerate, *always* involved selling both units of asset together. In this paper we demonstrate that it is indeed possible to obtain a situation whereby the investor chooses to sell her asset gradually rather than in a block.

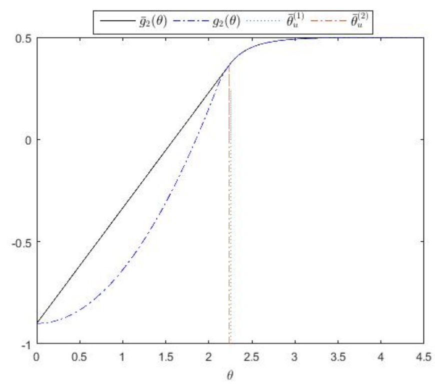
Our results suggest that it would be worthwhile for experimental tests of optimal stopping under reference-dependent preferences to extend their focus to consider the question of how individuals sell a divisible quantity of asset. For example, in the context of Magnani (2017)'s laboratory test, do subjects with a quantity of asset still stop once (before the risk neutral upper threshold B^*) or do they sometimes stop more than once (and where in relation to



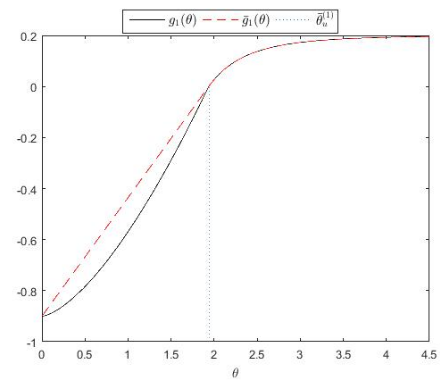
(a) Case (ii). $g_1(\theta)$ vs $\bar{g}_1(\theta)$.



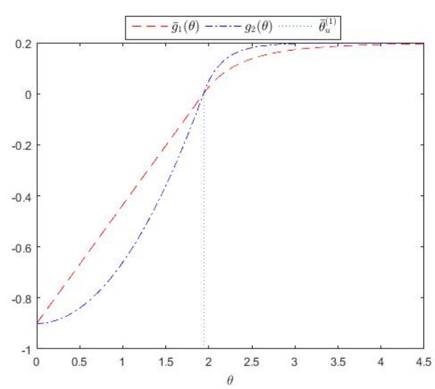
(b) Case (ii). $\bar{g}_1(\theta)$ vs $g_2(\theta)$.



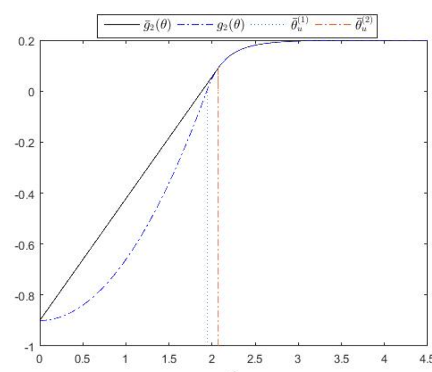
(c) Case (ii). $g_2(\theta)$ vs $\bar{g}_2(\theta)$. ($\bar{\theta}_u^{(1)} = 2.251$, $\bar{\theta}_u^{(2)} = 2.230$, $\bar{y}_u^{(1)} = 1.227$ and $\bar{y}_u^{(2)} = 1.213$)



(d) Case (ii). $g_1(\theta)$ vs $\bar{g}_1(\theta)$.

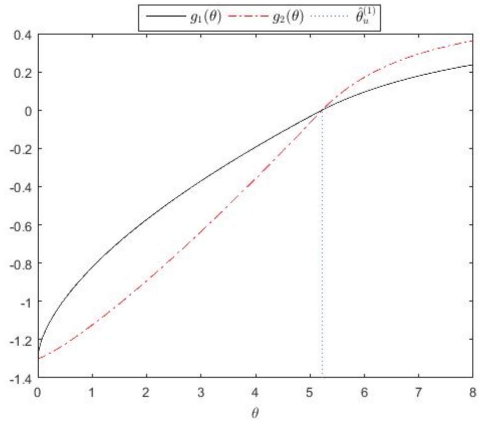


(e) Case (ii). $\bar{g}_1(\theta)$ vs $g_2(\theta)$.

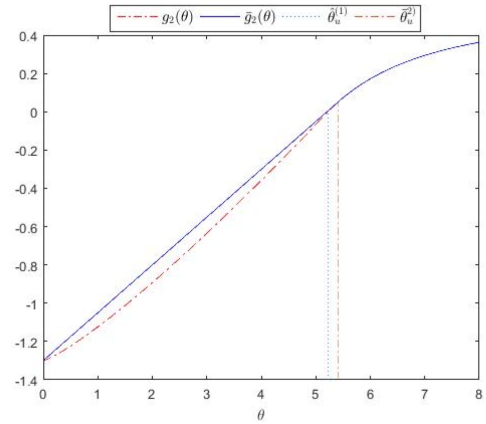


(f) Case (ii). $g_2(\theta)$ vs $\bar{g}_2(\theta)$. ($\bar{\theta}_u^{(1)} = 1.940$, $\bar{\theta}_u^{(2)} = 2.071$, $\bar{y}_u^{(1)} = 1.002$ and $\bar{y}_u^{(2)} = 1.101$)

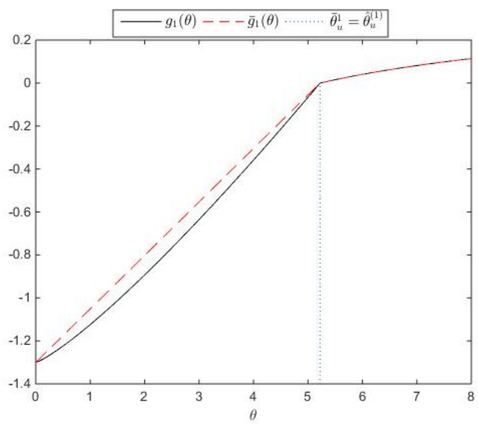
Figure 2: Optimal Liquidation of Two Units of Asset under the model of Proposition 3.4, case (iii). Panels (a)-(c) use parameters: $\phi_1 = 0.5, \phi_2 = 0.9, \gamma_1 = 3, \gamma_2 = 2$, $\eta = 0.66$. Condition (19) is satisfied, giving $\bar{y}_u^{(2)} = 1.213 < \bar{y}_u^{(1)} = 1.227$. Panels (d)-(f) use parameters: $\phi_1 = 0.2, \phi_2 = 0.9, \gamma_1 = 3, \gamma_2 = 1$ and $\eta = 0.66$. Condition (19) is not satisfied for this choice, hence both units are sold at $\bar{y}_u^{(2)} = 1.101$. All panels use a reference level (per unit) of $y_R = 1$. Set $x = 0$.



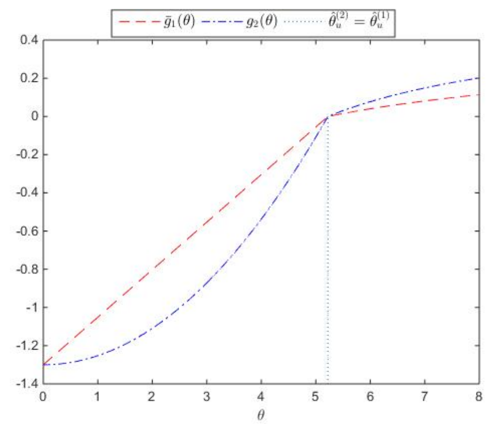
(a) Case (iii)(a). $g_1(\theta)$ vs $g_2(\theta)$.



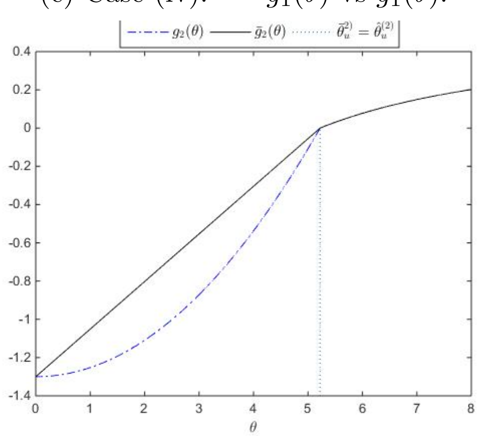
(b) Case (iii)(a). $g_2(\theta)$ vs $\bar{g}_2(\theta)$. ($\bar{\theta}_u^{(2)} = 5.418$ and $\bar{y}_u^{(2)} = 1.022$)



(c) Case (iv). $g_1(\theta)$ vs $\bar{g}_1(\theta)$.



(d) Case (iv). $\bar{g}_1(\theta)$ vs $g_2(\theta)$.



(e) Case (iv). $g_2(\theta)$ vs $\bar{g}_2(\theta)$. ($\bar{\theta}_u^{(1)} = \bar{\theta}_u^{(2)} = 5.22$, $\bar{y}_u^{(1)} = \bar{y}_u^{(2)} = 1$)

Figure 3: Optimal Liquidation of Two Units of Asset under the model of Proposition 3.4, cases (iii)(a) & (iv). Panels (a) and (b) illustrate case (iii)(a) with $\phi_1 = 0.5, \phi_2 = 1.3, \gamma_1 = 2.5, \gamma_2 = 1$ and $\eta = 1.65$. Both units are sold at $\bar{y}_u^{(2)} = 1.022 > y_R$. Panels (c)-(e) illustrate case (iv) with $\phi_1 = 0.5, \phi_2 = 1.3, \gamma_1 = 1, \gamma_2 = 2, \eta = 1.64$. Both units are sold at $\bar{y}_u^{(1)} = \bar{y}_u^{(2)} = 1 = y_R$. All panels use a reference level (per unit) of $y_R = 1$. Set $x = 0$.

B^*)?

Potential further theoretical work may examine the additional feature of an exogenous end-of-game whereby the asset is liquidated upon arrival of the first jump of a Poisson process (see Kyle et al (2006), Barberis and Xiong (2012) for examples). Whilst injecting realism, this addition would be at the expense of the tractability of the solution method and for this reason, we do not pursue it here.

5 References

Alvarez, L. H. R., 2001, "Singular stochastic control, linear diffusions, and optimal stopping: A class of solvable problems", *SIAM Journal on Control and Optimization*, 39, 6, 1697-1710.

Alvarez, L. H. R., 2003, "On the properties of r-excessive mappings for a class of diffusions", *Annals of Applied Probability*, 13, 4, 1517-1533.

Azevedo E. and D. Gottlieb, 2012, "Risk Neutral Firms can Extract Unbounded Profits from Consumers with Prospect Theory Preferences", *Journal of Economic Theory*, 147, 1291-1299.

Barberis N., 2012, "A Model of Casino Gambling", *Management Science*, 58, 35-51.

Barberis N. and W. Xiong, 2012, "Realization Utility", *Journal of Financial Economics*, 104, 251-271.

Bayraktar E. and M. Egami, 2010, "On the One-Dimensional Optimal Switching Problem", *Mathematics of Operations Research*, 35 (1), 140-159.

Carassus L. and M. Rasonyi, 2015, "On optimal investment for behavioural investors in discrete-time multiperiod incomplete markets", *Mathematical Finance*, 25, 115–153.

Calvet L.E., Campbell J.Y. and P. Sodini, 2009, "Fight or Flight? Portfolio Rebalancing by Individual Investors", *Quarterly Journal of Economics*, February, 301-348.

Carmona R. and S. Dayanik, 2008, "Optimal Multiple Stopping of Linear Diffusions", *Mathematics of Operations Research*, 33, 2, 446-460.

Dayanik S. and I. Karatzas, 2003, "On the Optimal Stopping Problem for One-Dimensional Diffusions", *Stochastic Processes and Their Applications*, 107, 2, 173-212.

Dayanik S., 2008, "Optimal stopping of linear diffusions with random discounting", *Mathematics of Operations Research*, 33, 3, 645-661.

De Giorgi E., T. Hens, 2006, "Making Prospect Theory fit for Finance", *Financial Markets and*

Portfolio Management, 20, 3, 339-360.

Dynkin, E.B., 1965, *Markov processes*, Vol. II, Academic Press Inc., Publishers, New York.

Dynkin E.B. and A.A. Yushkevich, 1969, *Markov Processes: Theorems and Problems*, Plenum Press, New York.

Ebert S. and Strack P., 2015, "Until the Bitter End: On Prospect Theory in a Dynamic Context‘', *American Economic Review*, 105(4), 1618-33.

Grasselli M. and V. Henderson, 2009, "Risk aversion and block exercise of executive stock options", *Journal of Economic Dynamics and Control*, 33, 1, 109-127.

He X. and L. Yang, 2016, "Realization Utility with Adaptive Reference Points", Working paper.

Henderson V., 2012, "Prospect Theory, Liquidation and the Disposition Effect", *Management Science*. 58, 2, 445-460.

Henderson V. and D. Hobson, 2011, "Optimal Liquidation of Derivative Portfolios", *Mathematical Finance*, 21, 3, 2011, 365-382.

Henderson V, Hobson D, and A.S.L. Tse, 2017, Randomized Strategies and Prospect Theory in a Dynamic Context, *Journal of Economic Theory*, 168, 287-300.

Ingersoll J.E. and L.J. Jin, 2013, "Realization Utility with Reference Dependent Preferences", *Review of Financial Studies*, 26, 3, 723-767.

Imas A., 2016, "The realization effect: Risk-taking after realized versus paper losses", *The American Economic Review*, 106(8):2086?2109, 2016.

Jin, H. and X.Y.Zhou, 2008, "Behavioral Portfolio Selection in Continuous Time," *Mathematical Finance*, 18, 385-426.

Kahneman, D. and A. Tversky, 1979, "Prospect Theory: An Analysis of Decision Under Risk," *Econometrica*, 46, 171-185.

Kyle, A.S., Ou-Yang, H. and W. Xiong, 2006, "Prospect Theory and Liquidation Decisions," *Journal of Economic Theory*, 129, 273-288.

Leung T. and R. Sircar, 2009, "Accounting for Risk Aversion, Vesting, Job Termination Risk and Multiple Exercises in Valuation of Employee Stock Options", *Mathematical Finance*, 19(1), 99-128.

Lien J. and J. Zheng, 2015, "Deciding when to quit: Reference-dependence over slot machine outcomes", *The American Economic Review*, 115(5):366?370, 2015.

Magnani J., 2015a, "Testing for the Disposition Effect on Optimal Stopping Decisions", *American Economic Review*, 105(5), 371-375.

Magnani J., 2015b, "Reference Dependent Preferences and Optimal Stopping Decisions: A Laboratory Investigation", *Working paper*.

Peskir G. and A.N. Shiryaev, 2006, *Optimal stopping and free-boundary problems*, Lectures in Mathematics, ETH Zurich, Birkhauser.

Revuz D. and M. Yor, 1999, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin.

Shefrin, H. and M. Statman, 1985, "The Disposition to Sell Winners too Early and Ride Losers too long: Theory and Evidence," *Journal of Finance*, 40, 777-790.

Thaler R. and E. Johnson,1990, "Gambling with the house money and trying to breakeven: The effects of prior outcomes on risky choice, *Management Science*, 36, 643–660.

Tversky, A. and D. Kahneman, 1992, "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty*, 5, 297-323.

Xu Z. and X.Y. Zhou, 2013, "Optimal stopping under probability distortion", *Annals of Applied Probability*, 23, 1, 251-282.

6 Appendix

Lemma 6.1. Let τ_n be an $\{\mathcal{F}_t\}$ measurable stopping time and U a monotonic-increasing continuous-function satisfying $U(0) = 0$. Then the family of $\mathcal{F}_{\tau_n}$ measurable random variables $\Gamma = \{Z^\alpha : \alpha \in \mathcal{I}\}$ as defined in the Proof of Proposition 3.1 has the lattice property.

Proof. Let $\alpha,\ \xi \in \mathcal{I}$, where $\alpha = (\alpha_{n-1}, \dots, \alpha_1)$ and $\xi = (\xi_{n-1}, \dots, \xi_1)$ satisfying:

$$\tau_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1 \quad \text{and} \quad \tau_n \leq \xi_{n-1} \leq \cdots \leq \xi_1$$

respectively. Furthermore define:

$$Z^\alpha = \mathbb{E}\big[U\big(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}\big)|\mathcal{F}_{\tau_n}\big]$$
$$Z^\xi = \mathbb{E}\big[U\big(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\xi_i}\big)|\mathcal{F}_{\tau_n}\big]$$

Consider $\upsilon = (\upsilon_{n-1}, \dots, \upsilon_1) \in \mathcal{I}$, defined by[2]:

$$\upsilon_i = \alpha_i \mathbb{I}_{\{Z^\alpha \geq Z^\xi\}} + \xi_i \mathbb{I}_{\{Z^\alpha < Z^\xi\}}$$

Defining Z^υ analogously to Z^α and Z^ξ it follows that:

$$\begin{aligned} Z^\upsilon &= \mathbb{E}\big[U\big(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}\big)\mathbb{I}_{\{Z_\alpha \geq Z_\xi\}} + U\big(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\xi_i}\big)\mathbb{I}_{\{Z^\alpha < Z^\xi\}}|\mathcal{F}_{\tau_n}\big] \\ &\geq \mathbb{E}\big[U\big(Y_{\tau_n} + \sum_{i=1}^{n-1} Y_{\alpha_i}\big)|\mathcal{F}_{\tau_n}\big] \\ &= Z^\alpha \end{aligned}$$

and similarly $Z^\upsilon \geq Z^\xi$.

□

Proof of Proposition 3.3.

Consider first the case when $\mu > 0$ ($\eta < 0$). The state-space $\mathcal{I} = (-\infty, \infty)$ of Y has natural endpoints and $S(\mathcal{I}) = (-\infty, 0)$. By Proposition 3.2, $V_1(y, x) = U(x + b_\mathcal{I} - h^1_R) = \infty$. This gives case (i).

Now consider the case when $\mu < 0$ ($\eta > 0$). We have $S(\mathcal{I}) = (0, \infty)$, $S(y_R) = \exp(-2\mu/\sigma^2 y_R)$ and hence $S^{-1}(\theta) = -(\sigma^2/(2\mu))\ln(\theta)$ for $\theta \in (0, \infty)$. Fix x. Hence the stopping problem translates

[2] The fact each υ_i is a stopping time follows from the fact that Z^α and Z^ξ are $\mathcal{F}_{\tau_n}$ measurable.

into finding a function $\bar{g}_1(\cdot, x)$ corresponding with the smallest concave majorant of the function $g_1(\theta, x)$, where:

$$g_1(\theta,x) = \begin{cases} \phi_1(1-\exp(\gamma_1 y_R)\theta^{\gamma_1\frac{\sigma^2}{2\mu}}) & \text{for } \theta \geq \exp(\eta y_R) \\ \phi_2(\exp(\gamma_2 y_R)\theta^{-\gamma_2\frac{\sigma^2}{2\mu}} - 1) & \text{for } \theta < \exp(\eta y_R) \end{cases} \tag{20}$$

By differentiating $g_1(\theta, x)$ and observing the behaviour of $g_1'(\theta, x)$ at $\theta = 0$ it follows that for $\gamma_2 \leq \eta$, $g_1(\cdot)$ is concave on both $(0, \exp(\eta y_R))$ and $(\exp(\eta y_R), 0)$ implying that the smallest concave majorant of $g_1(\theta, x)$ is the function itself and hence the stopping region coincides with the whole state-space $\mathcal{I}$. This gives case (iv).

Furthermore when $\gamma_2 > \eta$, it follows that $g_1(\theta, x)$ is convex in θ for $\theta < \exp(\eta y_R)$ and concave otherwise. This implies that the smallest concave majorant $\bar{g}_1(\theta, x)$ of $g_1(\theta, x)$ is obtained by considering the chord from the point $(0, -\phi_2)$ to a point $(\bar{\theta}_u, g_1(\bar{\theta}_u, x))$ at which the slope of the chord is equal to the slope of $g_1(\theta, x)$ for $\theta \geq S(y_R)$, giving:

$$\bar{y}_u^{(1)} = y_R - \frac{1}{\gamma_1}\ln\left(\left(\frac{\phi_1+\phi_2}{\phi_1}\right)\left(\frac{2\mu}{2\mu - \gamma_1\sigma^2}\right)\right) \tag{21}$$

Since $\bar{y}_u^{(1)} \geq y_R$, from (21) it follows that $\bar{y}_u^{(1)} > y_R$ for $0 < \eta < ((\phi_1/\phi_2)\gamma_1)$ (Case(ii)) and $\bar{y}_u^{(1)} = y_R$ otherwise (Case (iii)). □

Proof of Proposition 3.4.

Suppose $\mu > 0$ ($\eta < 0$). Similar arguments given for Case (i) in Proposition 3.3 show that under these assumptions the investor will wait indefinitely and never liquidate his position (Case (i)).

Henceforth we shall assume that $\eta > 0$. From the proof of Proposition 3.3 we know that:

$$g_1(\theta,x) = \begin{cases} \phi_1(1-\exp(\gamma_1(y_R - x))\theta^{\gamma_1\frac{\sigma^2}{2\mu}}) & \text{for } \theta \geq \exp(\eta(y_R - x)) \\ \phi_2(\exp(\gamma_2(y_R - x))\theta^{-\gamma_2\frac{\sigma^2}{2\mu}} - 1) & \text{for } \theta < \exp(\eta(y_R - x)) \end{cases} \tag{22}$$

Suppose $\eta \geq \gamma_2$. From Proposition 3.3 we know that these assumptions imply $\bar{g}_1(\theta, x) = g_1(\theta, x)$ and hence $g_2(\theta, x) = g_1(\theta, x + s^{-1}(\theta) - y_R)$ gives:

$$g_2(\theta,x) = \begin{cases} \phi_1(1-\exp(\gamma_1(2y_R - x))\theta^{\gamma_1\frac{\sigma^2}{\mu}}) & \text{for } \theta \geq \hat{\theta}^{(2)}(x) \\ \phi_2(\exp(\gamma_2(2y_R - x))\theta^{-\gamma_2\frac{\sigma^2}{\mu}} - 1) & \text{for } \theta < \hat{\theta}^{(2)}(x) \end{cases} \tag{23}$$

where $\hat{\theta}^{(2)}(x) = \exp(\frac{\eta}{2}(2y_R - x))$.

By differentiating $g_2(\cdot, x)$ w.r.t. θ and setting $x = 0$, $g_2'(\infty-, 0) \downarrow 0$ and:

1. $g_2'(0+, 0) \uparrow \infty$ if $\eta/2 > \gamma_2$

2. $g_2'(0+,0) \uparrow -\frac{\phi_2\gamma_2\sigma^2}{\mu}e^{-\gamma_2(2y_R-x)}$ if $\eta/2 = \gamma_2$

3. $g_2'(0+,0) \downarrow 0$ if $\eta/2 < \gamma_2$

In the case when $\eta/2 \geq \gamma_2$, the results above suggest that $g_2(\cdot)$ is concave for $\theta \in (0, \hat{\theta}^{(2)}(x)]$ and $\theta \in [\hat{\theta}^{(2)}(x), \infty)$ implying that $\bar{g}_2(\theta, x) = g_2(\theta, x)$. Thus the stopping region coincides again with all of $\mathcal{I}$ giving case (v).

If $\eta/2 < \gamma_2$, (3) implies that for constant x, $g_2(\theta, x)$ is convex for $\theta < \hat{\theta}^{(2)}(x)$ and concave otherwise. This implies that $\bar{g}_2(\theta, 0)$ is obtained by considering a chord from $(0, -\phi_2)$ to a point $(\bar{\theta}_u^{(2)},\ g_2(\bar{\theta}_u^{(2)}, 0))$, where the slope of the chord is equal to the slope of $g_2(\theta, 0)$. One should also note that given $g_2(\theta, 0)$ is convex for $\theta < \hat{\theta}^{(2)}(0)$ then $\bar{\theta}_u^{(2)} \geq \hat{\theta}^{(2)}(0)$. Since $S^{-1}(\hat{\theta}^{(2)}(0)) = y_R$, a similar argument to the one given for the proof of Cases (ii) and (iii) of Proposition 3.3 gives:

1. $\bar{y}_u^{(2)} > y_R$ for $\frac{\phi_1\gamma_2}{\phi_2} > \frac{\eta}{2}$ and;

2. $\bar{y}_u^{(2)} = y_R$ for $\frac{\phi_1\gamma_2}{\phi_2} \leq \frac{\eta}{2}$,

where $\bar{y}_u^{(2)} = S^{-1}(\bar{\theta}_u^{(2)})$. Now (1) gives case (iii)a in the statement of the proposition since from above the investor first stops at $\bar{y}_u^{(2)} > y_R$ and then sells the second asset immediately (since $\eta \geq \gamma_2$). A similar argument partially gives case (iv) from (2) above. (In order to complete case (iv) we also have to show that the investor also sells both assets at y_R when $0 < \eta < \gamma_2$.)

Having gone through all the cases for $\eta \geq \gamma_2$, consider the instance when $0 < \eta < \gamma_2$ and suppose $\eta < \frac{\phi_1\gamma_1}{\phi_2}$. From Proposition 3.3 we know that:

$$\bar{g}_1(\theta, x) = \begin{cases} \phi_1(1 - \exp(\gamma_1(y_R - x))\theta^{\gamma_1\frac{\sigma^2}{2\mu}}) & \text{for } \theta \geq \bar{\theta}^{(1)}(x) \\ (\phi_1(1-\delta) + \phi_2)\delta^{-(2\mu/\gamma_1\sigma^2)}\exp(\frac{2\mu}{\sigma^2}(2y_R - x))\theta - \phi_2 & \text{for } \theta < \bar{\theta}^{(1)}(x)) \end{cases} \tag{24}$$

where:

$$\delta = \left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\left(\frac{2\mu}{2\mu - \gamma_1\sigma^2}\right) \qquad \text{and} \qquad \bar{\theta}^{(1)}(x) = \exp\left(\frac{-2\mu}{\sigma^2}(y_R - x)\right)\delta^{(2\mu/\gamma_1\sigma^2)} \tag{25}$$

We have already seen in Proposition 3.3 that (24) gives a stopping region $\Gamma = \left[\bar{y}_u^{(1)}, \infty\right)$ where $\bar{y}_u^{(1)}$ is given by (17). By letting $g_2(\theta, x) = \bar{g}_1(\theta, x + S^{-1}(\theta) - y_R)$ we obtain:

$$g_2(\theta, x) = \begin{cases} \phi_1(1 - \exp(\gamma_1(2y_R - x))\theta^{\gamma_1\frac{\sigma^2}{\mu}}) & \text{for } \theta \geq \hat{\theta}^{(2)}(x) \\ K\exp(\frac{2\mu}{\sigma^2}(2y_R - x))\theta^2 - \phi_2 & \text{for } \theta < \hat{\theta}^{(2)}(x)) \end{cases} \tag{26}$$

where:

$$K = (\phi_1(1-\delta)+\phi_2)\delta^{-(2\mu/\gamma_1\sigma^2)} \qquad \text{and} \qquad \hat{\theta}^{(2)}(x) = \exp\left(\frac{-\mu}{\sigma^2}(2y_R - x)\right)\delta^{(\mu/\gamma_1\sigma^2)} \tag{27}$$

From our assumption that $\mu < 0$ we get $g_2'(0+,0) \downarrow 0$ and $g_2'(\infty-,0) \downarrow 0$ and hence $g_2(\cdot,0)$ is concave for $\theta > \hat{\theta}^{(2)}(x)$ and convex for $\theta < \hat{\theta}^{(2)}(x)$. The smallest concave majorant $\bar{g}_2(\cdot,0)$ can hence be obtained by replicating the same methodology used in the previous case and hence the stopping region is given by $\left[\bar{y}_u^{(2)}, \infty\right)$ where:

$$\bar{y}_u^{(2)} = y_R - \frac{1}{2\gamma_1}\ln\left(\left(\frac{\phi_1+\phi_2}{\phi_1}\right)\left(\frac{\mu}{\mu-\gamma_1\sigma^2}\right)\right). \tag{28}$$

Furthermore it is possible to show that since $\eta < (\phi_1\gamma_1/\phi_2)$ then $\bar{y}_u^{(2)} > y_R$.

The above discussion suggests that the investor will sell the first asset at the level $\bar{y}_u^{(2)}$ and the second at any level at or above $\bar{y}_u^{(1)}$ proving case (ii). We now compare $\bar{y}_u^{(2)}$ and $\bar{y}_u^{(1)}$. It is straightforward to show that the condition

$$\frac{\phi_2}{\phi_1}\left[\frac{2\eta}{\gamma_1}\left(1+\frac{\eta}{2\gamma_1}\right)\right] < 1$$

is equivalent to the ordering: $\bar{y}_u^{(2)} < \bar{y}_u^{(1)}$, and hence in this case the asset is sold at two distinct thresholds. If $\bar{y}_u^{(2)} > \bar{y}_u^{(1)}$, then both units are sold together at $\bar{y}_u^{(2)}$. This proves Corollary 3.5.

Finally we examine the case when $0 < \eta < \gamma_2$ and $\frac{\phi_1\gamma_1}{\phi_2} \le \eta$. By deriving $\bar{g}_1(\theta,x)$ from Proposition 3.3 and following a similar approach to the one outlined in the previous case the following results are obtained.

1. If $0 < \eta < \gamma_2$ and $\frac{\eta}{2} < \frac{\gamma_1\phi_1}{\phi_2} < \eta$ the investor liquidates the first asset at a level $\bar{y}_u^{(2)} > y_R$ where $\bar{y}_u^{(2)}$ is given by (18) and the second asset at any level at or above y_R. This implies that both assets are sold at $\bar{y}_u^{(2)}$ (Case (iii)b).

2. If $0 < \eta < \gamma_2$ and $\frac{\gamma_1\phi_1}{\phi_2} \le \frac{\eta}{2}$ the investor liquidates both assets at the reference level y_R, completing the proof for case (iv).

□