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Is there an informationally passive benchmark for option pricing incorporating maturity?

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Figlewski proposed testing the incremental contribution of the Black–Scholes model by comparing its performance against an 'informationally passive' benchmark, which was defined to be an option pricing formula satisfying static no-arbitrage constraints. In this paper we extend Figlewski's analysis to include options of more than one maturity. Once maturity has been included in the model, any 'informationally passive' call pricing function is consistent with some 'active' model. In this sense, the notion of a passive model cannot be extended to pricing formulas incorporating option maturity. We derive the index dynamics of the active model implicit in Figlewski's implied G example. These dynamics are far more complicated than the dynamics of the Samuelson–Black–Scholes or Bachelier models. The main implication of our analysis is that an appropriate benchmark for assessing option pricing models should in fact have simple dynamics, such as those of Bachelier or the Black–Scholes models. This is despite the fact that the maturity extension of Figlewski's model gives as good a fit as the Black–Scholes model.

1. Introduction

The Black and Scholes (1973) model for option pricing is the industry standard and won its inventors a Nobel prize. Despite its widespread use, the theoretical underpinnings of the model are often violated in practice. Volatility is not constant, and is widely documented to exhibit smiles and skews, see Rubinstein (1985). Nonetheless it is frequently used as a benchmark model against which other models are compared.

The Black–Scholes option pricing formula is based on the assumption of log-normal dynamics, as proposed by Samuelson (1965), and on the principles of a perfect, frictionless market. The concept of dynamic replication then leads to the specification of a call pricing formula. However, this formula is only valid if the model is correct, both in the sense that the asset follows an exponential Brownian motion, and that the market is frictionless.

Given that neither of these assumptions holds true, Figlewski (2002) proposed that the Black–Scholes model should itself be tested against other models. If dynamic replication breaks down, then the Black–Scholes call pricing formulae can be viewed purely as a formula which satisfies certain natural static no-arbitrage conditions, and should be tested against other formula satisfying these same constraints. Figlewski (2002, p. 88) proposed testing the Black–Scholes model or formula against other 'informationally passive' models, where an informationally passive model was defined as a call pricing formula which 'satisfies portfolio dominance constraints, but has no economic content beyond that'. As an example of this philosophy, Figlewski (2002) proposed such a model which he called the 'implied G' model. In his paper the implied G (or IG for short) model played the role of a new benchmark.

One of the uses of an option pricing model is to infer an option price from market prices of 'nearby' options, perhaps involving a similar strike or time to maturity. In the Black and Scholes (1973) model this is accomplished via implied volatility. For example, yesterday's implied volatility might be used to compute an option price today or an option price might be calculated from interpolation between implied volatilities of two options with strikes spanning the strike of interest. The paper of Figlewski (2002) recognizes that this usage of the Black–Scholes option pricing formula does not rely on its precise form. In fact any function of the right shape could be

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used in its place. Figlewski's example, the IG model, is of roughly the right shape. The point is that the IG model is not chosen to provide a best fit, but rather is a simplest attempt at finding a pricing function with approximately the right characteristics†. Figlewski's main thesis is that the contribution of any model, and in particular the Black–Scholes model, should be considered in the light of its incremental contribution over a passive benchmark such as the IG model.

There are two distinct applications to which a trader can put the Black–Scholes model. In the first usage the trader calculates the implied volatility from a single option and uses that volatility to calculate the price of a related security. (For each different security the trader wishes to price he might calibrate with a different option.) In the second usage the trader calculates the best-fit implied volatility from a set of traded options of different strikes, and uses that volatility to give prices for each of a different set of options. The first of these approaches recognizes that market data admits smiles and skews and allows the trader to account for this. However, in doing this the trader is being inconsistent in his use of Black–Scholes. The second situation suffers no such inconsistency, but then the trader cannot match his model to market data, he can only give a best fit.

Figlewski (2002) tests his alternative model against Black–Scholes in both of these usages. In the first case he uses today's option value to predict the price tomorrow of an option with the same strike and maturity. In the second case he uses today's prices of all the options of a given maturity to calculate a best fit volatility, which is then used to predict tomorrow's prices for those same options.

Figlewski finds that his example of an informationally passive model provides roughly as good a fit to the data as the Black–Scholes model. In the first case, when the Black–Scholes model is used inconsistently, it tends to outperform the passive alternative, whereas when Black–Scholes is used consistently, the IG model provides a better fit.

Figlewski's IG model does not explicitly incorporate a time parameter. This means that it cannot readily be used to compare options with different times to maturity. As Figlewski (2002, p. 95) states, one natural extension of his work is to 'add structure across option maturities'. Our primary aim in this article is to consider such an extension. The main contribution of this paper is to extend his example IG, or rather a modification of his example which we label MIG‡, to explicitly include a time-parameter. This extension leads us to reconsider the notion of an 'informationally passive formula': once maturity has been included it is no longer clear that an option formula can be divorced from underlying dynamics. Indeed, under some reasonable and simplifying assumptions, and using the framework of Dupire (1993, 1994), if option prices are specified for all maturities then the risk-neutral dynamics of the underlying are also specified. Even if we return to the original Figlewski (2002) case with a single maturity, it is no longer clear that there is such an entity as an informationally passive model—the use of any pricing formula is equivalent to restricting attention to a small class of models which are consistent with the associated call prices.

Put another way, Figlewski (2002) distinguishes between an 'informationally passive model', which is essentially an option pricing formula satisfying portfolio dominance constraints, and a model-based or 'active' alternative where the pricing formulae are derived via the dynamics of the underlying. The option prices which arise from the model (under the assumptions of a perfect market) must satisfy the portfolio dominance constraints. However, once maturity is included, this distinction between passive and active models disappears. In that case, any informationally passive model by Figlewski's definition is consistent with some dynamics for the underlying, and if we search amongst the class of diffusion models then there is a unique active model associated with each supposedly passive option pricing formula. In this sense, there is no distinction between an informationally passive model and an active alternative. On the contrary, one way to choose a benchmark model is to choose the option pricing formula which is consistent with the simplest price dynamics. This simplicity is achieved by the Samuelson (1965)/Black–Scholes (1973) and Bachelier (1900) models.

The second contribution of this paper is to test the time-extended model, which we label IGT (and MIGT for the modified version) against the Black–Scholes model (and the Bachelier or arithmetic Brownian motion model). For the empirical testing we use the same dataset as Figlewski (2002), namely traded options on the S&P 500 over the period 2 January 1991 to 29 December 1995. This is so that we can compare our results to his directly. The tests we use are broadly similar to those used by Figlewski (2002), and our choice of tests is motivated to a large extent by his results. Again we find that the IG, IGT, MIG and MIGT models all perform remarkably well (especially considering that no attempt is made to optimize the form of these models).

The third contribution of this paper is to attempt to understand and explain why these models all give such a good fit. It turns out that the good performance of these models is linked to the fact that they generate implied volatility smiles which closely match those found in the dataset that we use for testing.

Unfortunately Figlewski's primary example of an informationally passive model admits simple arbitrage. For market parameters (based on the data used both by Figlewski and in this paper) the Figlewski model would

†As Figlewski (2002, p. 90) states: 'No effort has been made to tweak the model in any way to improve its performance.'
‡Modified implied G.

give a price ranging from 50 cents to \$2 for a put option with zero strike, which must by necessity be worthless. With this in mind, the fourth contribution of our paper is to propose a 'modified informationally passive' model MIG, satisfying static arbitrage constraints†.

We find that the modified model MIG gives very similar performance to Figlewski's original model‡. However, MIG does not admit static arbitrage. In general MIG outperforms the Black–Scholes model in exactly the same situations as the original Figlewski model. More especially if an implied volatility is calculated for each option then Black–Scholes outperforms both the IG and MIG models. However if Black–Scholes is used consistently then IG and MIG both outperform the Black–Scholes pricing function.

We find that both IG and MIG fit best for low strike options (in-the-money calls and out-of-the-money puts). This is the exact opposite of the behaviour reported by Figlewski (2002)§. We give a plausible explanation, using the implied volatility smile and skew of the data, of why the results we report are to be expected.

The paper is organized as follows. Static no-arbitrage criteria and the Black and Scholes (1973) option pricing model are given in section 2. In section 3, we describe the Figlewski (2002) model and our proposed modified model. These models are extended to allow for maturity dependence in section 4. Section 5 describes the data used in the empirical testing of the pricing models. Our modified models are tested against the Black and Scholes (1973) pricing formula, the Bachelier (1900) arithmetic Brownian motion model, and the passive model of Figlewski (2002) in section 6, and the results reported in terms of RMSE. In the penultimate section we discuss these results and give explanations, before we conclude in a final section. The origins of the form of the proposed passive option pricing functions are given in the Appendix.

2. Static no-arbitrage properties

Let C denote the price of a European call option on the stock index level S_t¶, with strike X, maturity T and riskless rate r. We can write

$$C = C(t, S_t, T, X) = \mathrm{e}^{-r(T-t)}\mathbb{E}_t(S_T - X)^+, \qquad (1)$$

where t is the current time, S_T is the realized value of the index at maturity and expectations are taken with respect to the risk-neutral measure. When we think of a fixed option with given strike and maturity, perhaps when deriving a pricing equation for C, it is usual to think of $C = C_{T,X}(t, S_t)$ as a function of current time and index level. Conversely when we think of the market prices of a family of traded options at a fixed moment in time we should think of $C = C_{t,S_t}(T, X)$ as a function of strike and maturity. The same ideas apply to the price $P = P(t, S_t, T, X) = P_{T,X}(t, S_t) = P_{t,S_t}(T, X)$ of a put option.

There are a number of important properties that option prices must satisfy in order to exclude simple static arbitrages. Merton (1973) derives these properties for stock options. In order to exclude static arbitrages we must have (i)–(iv):

(i) $C_{t,S_t}(T, X)$ is a decreasing, convex function of the strike.

(ii) The current price of a call option with zero strike is equal to the stock price

$$C_{t,S_t}(T, 0) = \lim_{X\downarrow 0} C_{t,S_t}(T, X) = S_t.$$

(iii) The call value is increasing in maturity: for $T \geq \hat{T} \geq t$

$$C_{t,S_t}(T, X\mathrm{e}^{r(T-t)}) \geq C_{t,S_t}(\hat{T}, X\mathrm{e}^{r(\hat{T}-t)}).$$

(iv) Put–call parity holds

$$C_{t,S_t}(T, X) - P_{t,S_t}(T, X) = S_t - X\mathrm{e}^{-r(T-t)}.$$

It is also the case that for any model for which option prices are consistent with (1):

(v) Far out-of-the-money call prices approach zero

$$\lim_{X\uparrow\infty} C_{t,S_t}(T, X) = 0.$$

(vi) At-the-money options have positive time value, for $T > 0$

$$C_{t,S_t}(T, S_t\mathrm{e}^{r(T-t)}) > 0.$$

Many of these properties have analogous forms for the call price function $C_{T,X}(t, S_t)$‖. In particular,

(vii) $C_{T,X}(t, S_t)$ is an increasing, convex function of the asset price.

(viii) Far out-of-the-money call prices approach zero

$$C_{T,X}(t, 0) = \lim_{S_t\downarrow 0} C_{T,X}(t, S_t) = 0.$$

†In defense of Figlewski's (2002) original model, although the IG model misprices a put with zero strike (whereas MIG prices it at zero) over the range of traded options, the differences between the two models are very small. To this extent, the fact that IG admits static arbitrage can be viewed as a theoretical problem that has little impact in practice. Provided that the IG model is only used 'locally' then no mispricing problems arise. Indeed, we also find many circumstances in which IG provides a better fit to data than MIG.

‡There is good reason to believe that both of the models IG and MIG would fit option price data with a symmetric smile, such perhaps as currency option data, better than index option data for which implied volatilities display a skew. We discuss this reasoning in section 7, when we attempt to explain why the IG and MIG models give such a good fit to data.

§It seems that entries in some of the tables in Figlewski (2002, Exhibit V) have been accidentally reversed.

¶Throughout we assume that S has been adjusted for dividends.

‖In order to even state these properties it is necessary to assume a certain amount of regularity. For instance, we need to assume that S_t is a Markov process.

(ix) Far in-the-money call prices approach the value of a long forward contract with the same strike

$$\lim_{S_t \uparrow \infty} \left\{ C_{T,X}(t, S_t) - (S_t - \mathrm{e}^{-r(T-t)} X) \right\} = 0.$$

The Black and Scholes (1973) call option pricing formula, which satisfies properties (i)–(ix) is given by

$$\begin{aligned} BS(t, S_t, T, X) &= BS^{\sigma^2}_{t, S_t}(T, X) \\ &= S_t N(d_+) - X\mathrm{e}^{-r(T-t)} N(d_-), \end{aligned} \quad (2)$$

where σ is the volatility of the stock price and

$$d_\pm = \frac{\ln(S_t \mathrm{e}^{r\tau}/X) \pm (\sigma^2 \tau/2)}{\sigma\sqrt{\tau}}.$$

Here $\tau = T - t$ is the time to maturity. The put price is given via put–call parity in (iv).

3. The Figlewski formula and arbitrage-free modifications

In the previous section we wrote down a minimal list of conditions that an option pricing function must satisfy in order to preclude arbitrage. In this section we describe the informationally passive IG model, show that it fails to satisfy some of these conditions and propose a modification MIG which satisfies all properties (i)–(ix) of section 2. MIG is also informationally passive, in that it simply satisfies the static no-arbitrage conditions.

Let $f^{\mathcal{G}}_{t,S_t}(T, X)$ denote the time t price of a call on stock index level S_t, with strike X, maturity T, riskless rate r, and parameterized by $\mathcal{G}$. This call pricing function must satisfy (i)–(iii). Suppose $f^{\mathcal{G}}_{t,S_t}(T, X)$ is increasing in $\mathcal{G}$ so that $\mathcal{G}$ plays the role of an implied volatility parameter. Then, given a market call price we can infer $\mathcal{G}$ and substitute this back into the formula to price a related option. For example given today's market price of an option we can infer the implied value of $\mathcal{G}$ and use it to give a price tomorrow for an option with the same strike and maturity. (Of course the value of the index may have changed during this period.)

Figlewski (2002) uses the function

$$IG^G_{t,S_t}(T, X) = \sqrt{G + \frac{(S_t - X\mathrm{e}^{-r(T-t)})^2}{4}} + \frac{(S_t - X\mathrm{e}^{-r(T-t)})}{2}, \quad (3)$$

which we refer to as the IG model. Put–call parity in (iv) defines the put price to be

$$IG^G_{t,S_t}(T, X) - (S_t - X\mathrm{e}^{-r(T-t)}).$$

However, notice that if the strike approaches zero in the IG model (3),

$$IG^G_{t,S_t}(T, 0) = \lim_{X \downarrow 0} IG^G_{t,S_t}(T, X) = \sqrt{G + \frac{S_t^2}{4}} + \frac{S_t}{2} > S_t$$

for $G > 0$†. Thus this choice of function is inadmissible as a call price function as property (ii) is violated and the IG model admits arbitrage. In particular, there is a difference between the price of a call option on the stock with strike zero and a unit of the stock itself‡.

We propose instead to use the modified function

$$\begin{aligned} MIG^g_{t,S_t}(T, X) = &\sqrt{gS_t + \frac{(S_t - X\mathrm{e}^{-r(T-t)} - g)^2}{4}} \\ &+ \frac{S_t - X\mathrm{e}^{-r(T-t)} - g}{2}, \end{aligned} \quad (4)$$

which we refer to as the MIG model. Here g plays the role of the implied volatility parameter§. For this function

$$MIG^g_{t,S_t}(T, 0) = \lim_{X \downarrow 0} MIG^g_{t,S_t}(T, X) = S_t$$

and if put prices are given by put–call parity then MIG satisfies all the necessary conditions for no-arbitrage.

In general it is quite difficult to construct option pricing functions satisfying the no-arbitrage properties (i)–(iii). A motivation and origin for the choice of both of the functions IG and MIG is explained in the Appendix. These ideas allow us to construct a family of candidate pricing functions. However, as Figlewski is careful to point out, the aim is not to find a best-fit model, but rather to compare the Black–Scholes model against the simplest alternative model satisfying static no-arbitrage.

4. Extension to time dependence and implied index dynamics

Unlike the Black and Scholes (1973) pricing formula (2), the IG model of Figlewski (2002) and our modified MIG models do not explicitly depend on the option maturity, apart from in the discounting terms. We can extend both models to include a maturity dependence. One reason for doing this is to allow us to compare options with different times to expiry, and as Figlewski (2002, p. 95) suggests

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†Note that if we think instead of the call price as a function of current time and the index level S_t, then property (viii) does not hold for the IG model. We have

$$IG^G_{T,X}(t, 0) = \lim_{S_t \downarrow 0} IG^G_{T,X}(t, S_t) > 0.$$

‡The problem with a model with call prices given by the function IG is that it is consistent with a price process which can go negative.

§Figlewski (2002) interprets the parameter G in his model in this way. However the parameters G and g might more correctly be interpreted as analogs of implied squared volatility. See also footnote on p. 8, where $\sqrt{G}$ takes the place of a constant multiple of σ.

'add structure across option maturities'. The second and more fundamental reason is to better understand the model.

In his paper Figlewski (2002, p. 81) states that 'the Black–Scholes model and others like it, must assume market conditions that rule out profitable dynamic arbitrage opportunities'. In contrast, by implication, a passive model makes no assumptions about either the market, or about the underlying dynamics. However, as we shall see, once maturity has been introduced into the pricing model then to a large extent the dynamics for the underlying have been specified. Even if maturity is not introduced into the model, then to be consistent the pricing function must have an extension to include maturities, and hence must belong to a severely restricted class of candidate price processes.

The simplest way to incorporate maturity is to replace the constant parameters G and g with the functions $(T-t)G$ and $(T-t)g$ which are proportional in time†. We obtain

$$IGT^G_{t,S_t}(T,X) = \sqrt{G(T-t) + \frac{(S_t - X\mathrm{e}^{-r(T-t)})^2}{4}} + \frac{(S_t - X\mathrm{e}^{-r(T-t)})}{2} \tag{5}$$

and

$$MIGT^g_{t,S_t}(T,X) = \sqrt{gS_t(T-t) + \frac{(S_t - X\mathrm{e}^{-r(T-t)} - g(T-t))^2}{4}} + \frac{S_t - X\mathrm{e}^{-r(T-t)} - g(T-t)}{2}. \tag{6}$$

We refer to the time-extended versions (5) and (6) as the models IGT and MIGT, respectively.

In the subsequent analysis we follow Dupire (1993, 1994). Under the assumption that the underlying price process is a diffusion‡, and given European call prices $C_{t,S_t}(T,X)$, then the risk neutral price process for the spot is fully determined. There is a unique diffusion coefficient $a_c(S_u,u)$ such that the index level follows the stochastic differential equation

$$\mathrm{d}S_u = rS_u\mathrm{d}u + a_c(S_u,u)\mathrm{d}W_u \tag{7}$$

under the risk neutral probability measure. In fact, the implied dynamics depend on the current call prices, and we can write

$$a_c(x,u) = \sqrt{2\left(\frac{\frac{\partial C_{t,S_t}(T,X)}{\partial T} + rX\frac{\partial C_{t,S_t}(T,X)}{\partial X}}{\frac{\partial^2 C_{t,S_t}(T,X)}{\partial X^2}}\right)}\Bigg|_{X=x,\,T=u}, \tag{8}$$

where $C_{t,S_t}(T,X)$ is the call price function, thought of as a function of strike.

The dynamics for the index level under the time modified IGT model can be calculated using (8) and (5) as the call pricing function. The index follows (7) with

$$a_c(S_u,u) = \sqrt{\frac{4}{u-t}\left(G(u-t)\mathrm{e}^{2r(u-t)} + \frac{(S_t\mathrm{e}^{r(u-t)} - S_u)^2}{4}\right)}. \tag{9}$$

Similarly, the index level under the time modified MIGT model follows (7) with§

$$a_c(S_u,u) = \sqrt{\frac{2D^2\mathrm{e}^{2r(u-t)}(S_t + S_u\mathrm{e}^{-r(u-t)} + g(u-t) - 2D)}{S_t(u-t)}}, \tag{10}$$

where $D^2 = gS_t(u-t) + [(S_t - S_u\mathrm{e}^{-r(u-t)} - g(u-t))^2/4]$.

Notice first the contrast between the diffusion coefficients driving the index level under these two models and the Samuelson (1965) model used for the Black and Scholes (1973) option pricing equation. Under the Samuelson model,

$$\mathrm{d}S_u = rS_u\mathrm{d}u + \sigma S_u\mathrm{d}W_u$$

so $a_c(S_u,u) = \sigma S_u$ with σ constant. In both time extended models, the diffusion coefficients (9) and (10) depend upon the constant parameters G and g, current index

†More generally we could have used any increasing functions of time to maturity, but our aim is to give the simplest possible extension to the time varying case. For an at-the-money option, under our choice $(T-t)G$, $IGT^G_{t,S_t}(T, S_t\mathrm{e}^{r(T-t)}) = \sqrt{G}\sqrt{T-t}$ whereas in the Black–Scholes model $BS^{\sigma^2}_{t,S_t}(T, S_t\mathrm{e}^{r(T-t)}) \sim \sigma\sqrt{T-t}(X/\sqrt{2\pi})$ to leading order. Hence our choice of linear scaling in time to maturity is the most natural.

‡The assumption that the underlying follows a diffusion process is a combination of two assumptions, firstly that the underlying is a Markov process, and secondly that it is continuous. Both of these assumptions can be challenged as unrealistic, and both are clearly idealizations. However, making these assumptions leads to what, in many cases, is the simplest possible model consistent with the given call prices.

§Compare the dynamics for the index under the two time-extended models IGT and MIGT. Notice that for the IGT model,

$$a_c(0,u) > 0,$$

whereas for the MIGT model,

$$a_c(0,u) = 0.$$

In particular, in the IGT model, when the index hits zero its diffusion coefficient is non-zero and the price process can and does go negative. Conversely, in the modified model, MIGT, when the index first hits zero, the diffusion coefficient is also zero and the process stops. This explains why the IG model gives positive value to put options with zero strike, whereas MIG correctly gives a zero value to these options.

level S_u, but also on current time, and the initial index level S_t. Hence, although the option pricing functions (5) and (6) are not too complicated, the implied index dynamics consistent with these functions are much more complicated than the lognormal model of Samuelson (1965) and Black and Scholes (1973).

Although Figlewski (2002, p. 81) comments that his example, and indeed any informationally passive model, 'does not involve the theoretical apparatus that Black–Scholes and other active pricing models require to rule out dynamic arbitrage', once maturity has been included his example is consistent with the exactly one diffusion model, and incorporating even simple time dependence results in quite complicated dynamics for the index.

These observations concerning the incompatibility of the notion of a passive model and a formula incorporating option maturity are the main message of this paper. The remainder of the paper undertakes empirical testing of IG, MIG and the time-extended versions and interprets the results obtained.

5. The data

The data used in this study is daily data on S&P 500 index options taken from the Berkeley Options Database. Option prices correspond to the average of the last bid and ask quotes reported before 3:00PM CST. We also use values of the S&P500 index, dividend payout on the S&P 500 over the remaining option life and riskless interest rates. The data runs from 2 January 1991 to 29 December 1995.

The data is the same as used by Figlewski (2002). This allows us to make a direct comparison with his results.

We construct a dividend adjusted index value by subtracting the present value of the dividends over the remaining option life from the raw series. This dividend adjusted series is used in place of the raw series. The interest rate is LIBOR obtained from the British Bankers Association, interpolated between adjacent months. We disregarded any observations where option prices violated arbitrage bounds or where implied volatilities were unable to be calculated. There were also a very small number of observations with an obvious recording error, for instance, the option maturity was before the current date, and these were also eliminated. Following this, there are 181 601 observations remaining in the dataset.

6. Testing benchmark pricing models against Black–Scholes and Bachelier

Our analysis is guided by many of Figlewski's findings. Firstly, he finds that regression and the R^2 statistic is not an illuminating tool when assessing the ability of candidate option pricing models to fit option price data. Instead he uses the root-mean-squared-error† (RMSE) in comparing out-of-sample model predictions to the observed market prices. Secondly, he finds only minimal differences in his numerical results (and no differences in his conclusions) when he treats puts and calls separately or as a single class. Thirdly, he finds that historical volatility provides poor estimates for option prices when used with the Black–Scholes model.

Motivated by the above we consider three usages of the Black–Scholes, IG and MIG models. In the first usage (Figlewski's Model 7) an implied volatility is calculated for each traded option and used to predict the option price one day later. We call this situation Model O. The second situation is when a single implied parameter is calculated for all strikes for a given maturity, and used to price options. We call this Model M and it corresponds to Figlewski's Model 4. The third usage is when a single implied parameter is calculated at the close of each day. This parameter applies to all strikes and all maturities. We call this Model D‡. Since Figlewski (2002) does not explicitly include time in his model, there is no direct comparison. We compare these predicted prices with the observed prices and calculate the RMSE of the prices.

In this way we repeat the analysis of Models 4 and 7 of Figlewski (2002) for the Black–Scholes model and FIG. His results were as given in table 1. Figlewski (2002) concluded that his model $IG^G_{t,S_t}(X,T)$ fitted market prices better than a theoretically consistent usage of the Black–Scholes model allowing only a single implied volatility for each option maturity. However, when he allowed for different implied volatilities for each option Black–Scholes outperformed his model.

We examine the performance of the models IG^G_{t,S_t}, MIG^g_{t,S_t}, IGT^G_{t,S_t} and $MIGT^g_{t,S_t}$ in fitting the options data. These are compared against the standard Black and Scholes (1973) model, and also against a Brownian motion or Bachelier model (labelled Ba), and a modified Brownian motion model (MBa) in which the non-negativity of the stock-price process is respected and the Brownian motion is assumed to be absorbing at zero. Our results are given in table 2§.

†RMSE is defined as

$$\frac{1}{N}\sum_{i=1}^{N}(C^{(i)}_{\text{model}} - C^{(i)}_{\text{market}})^2,$$

where $C^{(i)}_{\text{model}}$ and $C^{(i)}_{\text{market}}$ are the predicted model price and the observed market price respectively of the ith option.

‡The terminology O/M/D refers to one volatility per option/maturity/day.

§Note that the results for the Bachelier and modified Bachelier models are indistinguishable to the given level of accuracy. One needs to take 10 decimal places before the numerical performance of the models differs. This is because for market parameters, the event that the stock price hits zero is several standard deviations from the mean, and therefore the probability of this event is so small as to leave the model fits unaffected.

Table 1. RMSE for option prices. Extract from Figlewski's Exhibit 4.

Model	O One per option	M One per maturity
Black–Scholes	0.466	1.398
FIG	0.501	1.280

Table 2. Root mean squared pricing error for Models O, M and D. Implied parameters for day t are used to calculate option prices on day $t+1$.

Model	O One per option	M One per maturity	D One per day
No. obs.	175742	180590	181307
Black–Scholes	0.4225	1.2954	1.3874
Ba	0.4217	1.0679	1.1297
MBa	0.4217	1.0679	1.1297
IG	0.4528	1.2073	4.1414
MIG	0.4524	1.2568	4.1843
IGT	0.4442	1.2054	1.3099
MIGT	0.4436	1.2547	1.3840

The numerical values in table 2 are broadly similar to those in table 1 (at least for Black–Scholes and IG and Models O and M) but typically about 10% smaller, perhaps because we have been more zealous in eliminating suspect data. We find that when taking one implied volatility per option, the Bachelier and Black–Scholes models outperform the implied G class of models. In this case, the models IG and MIG give almost indistinguishable results. IGT and MIGT are also indistinguishable, but give a marginally better fit than IG and MIG. This follows from the fact that they give a slightly more sophisticated treatment of maturity which changes from the time the implied parameter is calculated to the time the option is repriced. The best fit of all is provided by the Brownian motion or Bachelier model. If we take one implied parameter per maturity, we still find that the arithmetic Brownian motion models provide the best fit. The relative performance of the other models changes—now IG and IGT outperform MIG and MIGT which in turn outperform Black–Scholes. Finally, if we take one implied parameter per day then IG and MIG perform very badly. (This is almost inevitable as these models are not designed to cope with more than one maturity at once.) Although the performances of the Black–Scholes, Bachelier (and modified Bachelier), IGT and MIGT models are comparable, we observe that the Bachelier model provides the best fit, followed by IGT, with MIGT just outperforming Black–Scholes.

Now consider the problem, instead of using implied parameters to fit tomorrow's option price, rather using the price of a neighbouring option (i.e. an option with nearby strike) to predict the current option price. For each option, we choose two options with the same maturity—the one with the next highest strike and the one with the next lowest strike. We calculate the implied volatility for the pair of neighbouring options and take an average which we then use to derive a predicted price for the original option. (If the original option corresponds to the smallest or largest strike traded then we just use the implied volatility of the option with the next smallest or largest strike.) The RMSE are given in table 3.

Table 3. Root mean squared pricing error for Models O. Implied parameters calculated from options with neighbouring strikes are used to give a predicted model price, which is then compared with market prices.

	Smallest strike	Non-extreme strike	Largest strike
Model O: Calls			
No. obs.	7055	78742	7055
Black–Scholes	0.7061	0.2902	0.7045
Ba/MBa	0.5467	0.2831	0.5193
IG	0.7960	0.3614	1.2279
MIG	0.6920	0.3496	1.2753
Model O: Puts			
No. obs.	6954	74686	6954
Black–Scholes	0.7496	0.3704	0.8694
Ba/MBa	0.5389	0.3738	0.7247
IG	0.5641	0.3723	1.3299
MIG	0.4481	0.3601	1.3783

Since the same time is used throughout these calculations, there is no difference between the models IG and IGT or between MIG and MIGT so we report only one set of numbers. The numerical difference between the fit of the Bachelier and modified Bachelier models is also so tiny that we just report the numbers once. For the calls with non-extreme strikes, the Bachelier model gives the best predictions, followed by the Black–Scholes model. The modified model MIG performs better than IG, but only by a small margin. If we look at extreme option values (deep in-the-money or out-of-the-money) then all models do worse. The Bachelier model continues to provide the best fit (and substantially outperforms Black–Scholes). For calls with the lowest strike, MIG outperforms Black–Scholes and outperforms IG by a larger margin. This is the part of the data where the fact that IG permits arbitrage is most likely to have an effect. For calls with the highest strike, IG gives a better fit than MIG, but both models perform poorly relative to the Black–Scholes and Bachelier models.

For puts the pattern of behaviour is similar, but with some important differences. (Note that put–call parity should mean that we find little difference between the tables). The performance of all four models for non-extreme strikes is very similar with MIG just outperforming the other models and Bachelier being worst. MIG also provides the best fit for low strike options. All models do worst for options with the highest strike, but the deterioration in fit is most marked for the implied G and its modified version, to the extent that for the most in-the-money puts, the Bachelier and Black–Scholes models outperform the implied G class by some margin.

Now we return to the situation where today's volatility is used to predict tomorrow's option price and consider

Table 4. Root mean squared pricing error for Model M. Implied parameters from day t are used to calculate option prices on day $t+1$.

	$X \ll S$	$X < S$	$X \sim S$	$X > S$	$X \gg S$
Model M: Calls					
No. obs.	14132	22565	28925	17380	9374
Black–Scholes	1.0158	1.6231	1.0926	1.4411	0.2481
Ba/BaT	0.9270	1.3023	0.9132	1.1616	0.1690
IG	0.6881	0.8126	1.0599	2.0340	1.1339
MIG	0.5720	0.8550	1.1090	2.1326	1.1628
IGT	0.6790	0.8395	1.0993	1.9929	1.1101
MIGT	0.5679	0.8874	1.1435	2.0907	1.1385
Model M: Puts					
No. obs.	16091	22105	29336	16073	4539
Black–Scholes	0.8623	1.6520	1.1204	1.4815	0.6881
Ba/BaT	0.7568	1.2953	0.9204	1.2011	0.6953
IG	0.5107	0.6621	0.9950	2.0247	1.0742
MIG	0.3650	0.7327	1.0591	2.1332	1.1045
IGT	0.4881	0.6890	1.0250	1.9788	1.0550
MIGT	0.3439	0.7656	1.0835	2.0865	1.0845

the relative fit of the seven models in Model M, where a single implied parameter is calculated for each maturity. We divide the options into puts and calls and into five moneyness classes: strike far below spot, strike below spot, strike near spot, strike above spot and strike far above spot. Like Figlewski (2002) we consider the moneyness

$$M = \frac{1}{\sqrt{T-t}} \ln \frac{S}{X e^{-r(T-t)}} \tag{11}$$

and divide options into five categories $M < -1.5\sigma_{BS}$, $-1.5\sigma_{BS} < M < -0.5\sigma_{BS}$, $-0.5\sigma_{BS} < M < 0.5\sigma_{BS}$, $0.5\sigma_{BS} < M < \sigma_{BS}1.5$, $1.5\sigma_{BS} < M$. Here σ_{BS} is the Black–Scholes implied volatility of the option. The results are given in table 4.

Observe that the pairs IG and IGT and MIG and MIGT give almost identical results. In fact the time-extended versions generally outperform the original models (though not by much, and not always), presumably because of the fact that time is incorporated in a consistent fashion into the model. Note also that the qualitative features are the same for both puts and calls, in the sense that the relative performance of models is typically the same whether we consider puts or calls, provided that, as in the table, we work with high or low strike options rather than in or out-of-the-money options.

The fit (in terms of the RMSE) of both the IG and MIG models deteriorates as strike increases (until for options of very high strike performance improves). In contrast, the performance of the Bachelier and Black–Scholes models shows no consistent pattern across strikes, except that again for high strike options the fit is improved. As a result, the IG and MIG models fit better than the Bachelier and Black–Scholes models for low strikes, but fit worse for high strikes.

For comparison purposes table 5 includes (in modified form) the equivalent entries from Figlewski (2002, Exhibit V). In fact it turns out that there is little correspondence between our numbers and the numbers in Figlewski's Exhibit V. However, if we reverse the labelling on the rows of Exhibit V, so that Figlewski's out-of-the-money calls (i.e. high strike calls) are relabelled as low strike calls $X \ll S$, and similarly Figlewski's out-of-the-money puts (i.e. low strike puts) are relabelled as high strikes puts then we find the results as given in table 5. Now there is excellent correspondence between tables 4 and 5, both in terms of the numbers of puts and calls in each category, and also for the RMSE reported in each cell. Thus it appears that the entries in Exhibit V of Figlewski (2002) have accidentally been reversed.

Table 5. Modified extract from Exhibit V of Figlewski (2002). The modification is that the entries labelled in Figlewski (2002) as being high-strike are labelled as low strike in this table. Subject to this relabelling, there is excellent correspondence between the numbers in Exhibit V and table 4.

	$X \ll S$	$X < S$	$X \sim S$	$X > S$	$X \gg S$
Model M: Calls					
No. obs.	13196	22293	29734	17862	9534
Black–Scholes	1.105	1.852	1.242	1.494	0.250
IG	0.696	0.936	1.195	2.132	1.159
Model M: Puts					
No. obs.	16575	22398	29769	15997	5222
Black–Scholes	0.857	1.693	1.240	1.818	0.766
IG	0.505	0.712	1.089	2.267	0.865

This also explains why the findings that the IG and MIG models fit best for low strikes, and that they outperform Black–Scholes for options of this type, is the reverse of the findings reported in Figlewski (2002). Figlewski (2002, p. 92) states that 'the [IG] model... does much better than Black Scholes for OTM calls and ITM puts [high strikes], and does much worse for ITM calls and OTM puts [low strikes]'.

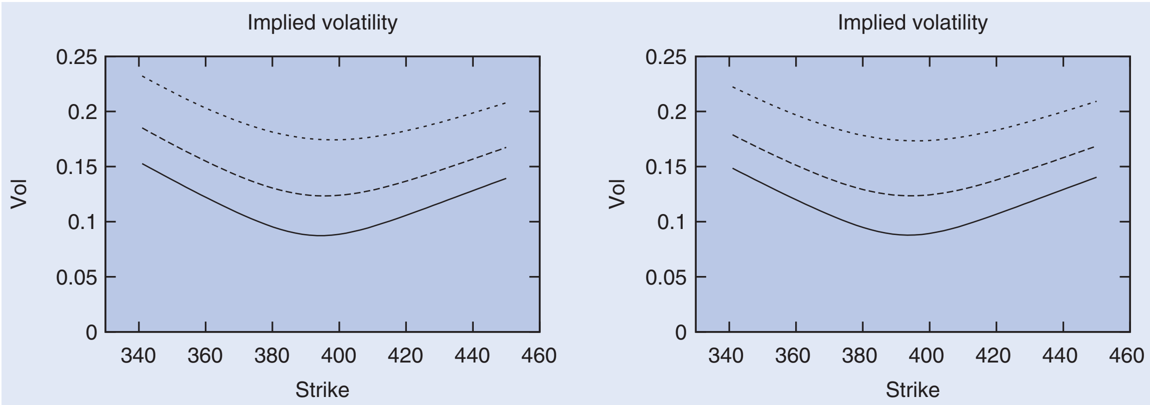

Figure 1. Black–Scholes implied volatilities calculated from options prices generated from the IG (left graph) and MIG (right) models. The three lines correspond to three different values of the parameter G or g, which increases by a factor of 2, as we move up from line to line.

One of the main aims of the next section is to use the basic properties of the IG and MIG models to explain why they give superior fit for low strike options.

7. Interpretation of the empirical results

Consider first the difference between the IG and MIG models and the Black–Scholes model. We can illustrate these differences by considering the Black–Scholes implied volatilities of option prices generated from the alternative models. Figure 1 shows these implied volatilities for a range of values of parameters G and g.

The key observations are that for both IG and MIG models, the implied volatility curves are convex and roughly symmetric, and the effect of increasing G (or g) is a parallel vertical shift. Changing G (or g) does not change the shape of the volatility surface. When we introduce time explicitly into the models, then we find that convexity decreases with time.

Now consider the options data and consider families of options traded on the same day with the same maturity. There are 7054 such day/maturity pairs (approximately 1400 days with 5 maturities per day) with more than two traded strikes. On each day, and for each maturity we can fit an implied volatility smile

$$I = a + bM + cM^2 + \epsilon_i,$$

where M represents the moneyness as defined in (11). The mean values we find for a, b, c are given in table 6. We conclude that the data for individual days and maturities has a negative skew and is generally convex, but occasionally has a frown.

Figure 2 shows a synthetic dataset using implied volatilities from the equation

$$I = 0.135 - 0.181\ M + 0.271M^2.$$

The range of strikes are those X for which the moneyness given by (11) satisfies $|M| \leq \sigma/4$, where $\sigma = 0.135$, and we take 20 evenly spaced values.

Table 6. Regression values based on the 7054 day/maturity pairs.

	a	b	c
Mean	0.1352	−0.1813	0.2714
Std Dev.	0.0281	0.0676	0.3300

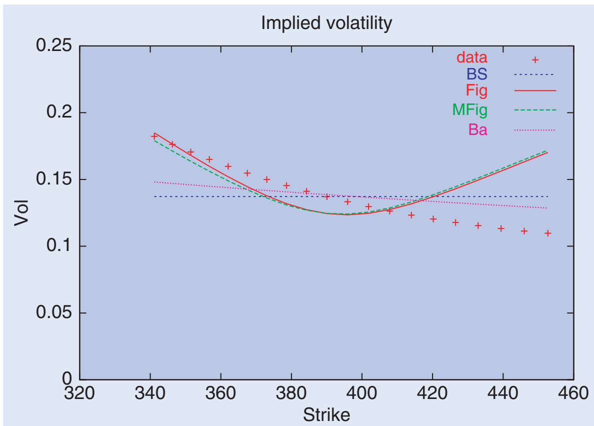

Figure 2. Synthetic implied volatility data, and the corresponding best fit model implied volatilities.

For comparison, figure 3 shows the implied volatilities of calls and puts traded on 1991/06/03 with a maturity date 1991/12/03. (The index level was 381.7 and the interest rate $r = 0.0629$, so that the at-the-money strike is 393.9). The same spot, maturity and interest rate were used for figure 2. Note that in figure 3 there are two data points per strike—one corresponding to a put and the other to a call. (If put–call parity held perfectly these values would match precisely). There is only one data point per strike in the synthetic dataset. Also plotted are the best fit option prices for the Black–Scholes, Bachelier, IGT and MIGT models. (Note that best fit corresponds to minimizing the RMSE of the model *prices* when compared with the data).

Recall from table 4 that the errors in the fit of the Black–Scholes and Bachelier models are roughly constant

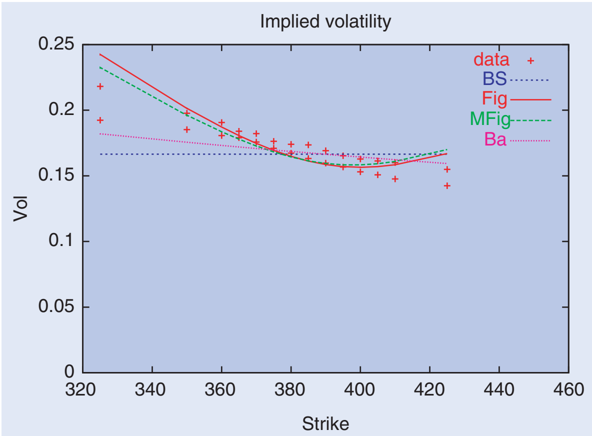

Figure 3. Implied volatilities of puts and calls on 1991/06/03 with maturity 6 months later. Also shown are the best-fit implied volatilities from the best fit Black–Scholes, Bachelier, IG and MIG models.

over strike†. From figures 2 and 3 we see that the fit of the implied volatility is much worse away from the money than at-the-money. However, in the tails the impact on the price of an error in implied volatility is much reduced. In fact, this effect dominates and although the errors in implied volatility are largest in the tails, the pricing errors are smallest there.

From table 4 we see that for the models IG and MIG the errors increase with strike‡. The explanation can be found in figures 2 and 3. The typical set of data is convex and negatively skewed. As a result, the symmetric but convex implied volatilities from IG and MIG fit better for low strikes than for high strikes. Again, the reported results are for pricing errors rather than in terms of implied volatilities, so that volatility errors at-the-money result in the highest pricing errors. This explains why we find the dependence we do (errors increasing with strike).

Figures 2 and 3 also help to explain why in terms of the RMSE expressed in tables 2, 3 and 4, the Bachelier or Brownian motion model gives such an excellent fit to the data when compared against the Black–Scholes model. As is well known, the Bachelier model produces implied skews which are negative, and from figures 2 and 3 we can see that they match the skew of the index data extremely well.

Now consider the results in table 3. Overall we might expect to see similar results to the Model O results in table 2, and again we find that for most of the data the Black–Scholes and Bachelier models outperform the other two models. However, MIG does provide a better fit than IG. The results are most striking when we take the smallest strike with a given maturity and attempt to predict the option price using the observed value for the next lowest strike. This is the case when we expect the no-arbitrage problems of IG to have most effect. We find that MIG significantly outperforms IG in this case, and is the best model of all for puts, so that it seems that for the smallest strikes the model IG is starting to perform badly.

8. Conclusion

One interpretation of the conclusions of Figlewski's (2002) paper might be that, if the role of an option pricing model is to derive option prices for illiquid options from the market prices of liquidly traded options, then it is not necessary to use the Black–Scholes formula and any other function with the right shape will perform roughly as well. Further, in this context it is possible to assess the contribution of the Black–Scholes model by comparing its performance against other candidate passive pricing functions which satisfy the no-static-arbitrage conditions. In his paper Figlewski suggests an appropriate function and finds that sometimes it performs better than the Black–Scholes formula, and sometimes worse, but the fit to data is typically comparable.

Unfortunately, Figlewski's example is not consistent with no-arbitrage. Indeed it prices at up to \$2 a put option with zero strike, which must be worthless. As an alternative, we introduce a modification to Figlewski's function which does not suffer from this deficiency. We find that this modified function fits S&P 500 options data roughly as well as the implied G model and generally they both outperform or underperform the Black Scholes pricing function for the same tests. We do find one test where MIG outperforms IG. This is for extremely low strike options where we expect the arbitrage problems of the Figlewski example to have an effect. We also expect both IG and our modification to perform better on currency options data which tends to be characterized by more symmetric implied volatility smiles.

The good news for the implied G model and modified implied G function is that they fit the data roughly as well as Black–Scholes and that they are much simpler in that they only involve square roots rather than cumulative normal distributions. The bad news for IG and MIG comes when one introduces maturity into the models. It is easy to include maturity, but once this is done then

†The exception is that the RMSE for out-of-the-money calls is very small. The reason for this is that these options have a very small price (much less than a dollar), so that even a large pricing error in relative terms translates to a small error in absolute terms. This effect also explains why the performance of IG and MIG improves markedly for far out-of-the-money calls.

‡Figlewski (2002) obtains the opposite relationship, in that he finds that for both puts and calls errors decrease with strike. The fact that our conclusions are contradictory was one of the main motivations for our attempt to explain this relationship. The fact that we are able to give a plausible explanation in terms of stylized facts such as skew and smile gives us confidence that our numerical results are the correct ones, and that the entries in Figlewski's Exhibit V have indeed been mislabelled.

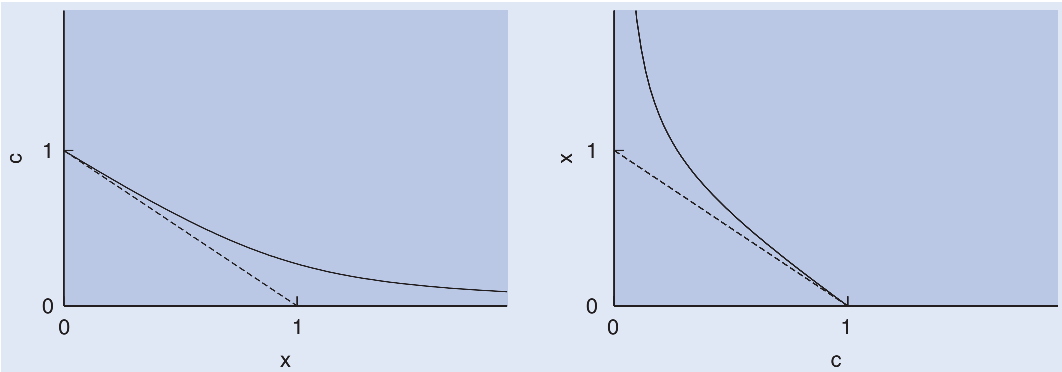

Figure 4. Strictly positive convex function $c = f(x)$ (left graph) and its inverse $x = f^{-1}(c)$ (right graph).

there is only one set of dynamics which is consistent with the option pricing function (at least under a diffusion assumption). Once maturity has been included, then following the analysis of Dupire (1993, 1994) to each 'informationally passive' call pricing function there corresponds an 'active' model and vice versa. Hence it no longer makes sense to distinguish between passive and active models. There is no informationally passive benchmark for option pricing incorporating maturity.

When we apply the Dupire (1993, 1994) analysis we find that the implied dynamics of the implied G and modified implied G models are much more complicated than the Samuelson–Black–Scholes model. Even if one does not explicitly allow for different maturities in the IG model, then the model has to be consistent with some maturity-extended version, and then the dynamics must be complex. In this sense, the simplicity of the Samuelson model makes the Black–Scholes pricing formula the one requiring minimal assumptions. This simplicity will be especially powerful when it comes to pricing exotic options.

Nonetheless, time-extended implied G models do give a surprisingly good fit to data, especially when one considers the fact that they have purely been chosen as simple functions and not optimized in any way. The reason for this good fit is that the models are consistent with an implied volatility smile which reflects the smile in the options data. However, there is another candidate benchmark model which outperforms both the Black–Scholes model and the implied G models. This model is the Bachelier (or arithmetic Brownian motion) model, which can be made to respect positivity of prices by making the underlying process absorbing at zero. The pricing function which arises from this model outperforms the other models in terms of the tests we consider. The main reason for this is that the Bachelier model gives implied volatility skews which match those found in the data.

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Appendix

This appendix motivates the choice of option pricing functions $IG^G_{t,S_t}(T,X)$ and $MIG^g_{t,S_t}(T,X)$ in (3) and (4). We want to find a call price function $C(X)$ which satisfies the static no-arbitrage properties (i)–(iii) (for fixed time t, index level S and maturity T) when considered as a function of discounted strike $X\mathrm{e}^{-r(T-t)}$. Consider the problem in terms of the dimensionless quantities $c = C/S$ and $x = X\mathrm{e}^{-r(T-t)}/S$.

We require $c = f(x)$ to be a strictly positive convex function with $f(0) = 1$, such that $f(x)$ decreases to zero as x increases. Further, from comparisons with the intrinsic value of the option, we have $f(x) \geq 1 - x$. The left graph of figure 4 illustrates the shape of the function.

It turns out that rather than attempting to write down suitable functions $c = f(x)$ it is easier to look for a function $x = f^{-1}(c)$ where f^{-1} is a decreasing, convex function satisfying $f^{-1}(c) \geq 1 - c$. This is graphed in the right part of figure 4.

Rather than working with $f^{-1}(c)$ it is more natural to consider $\tilde{f}(c) = f^{-1}(c) - 1 + c$. Then $\tilde{f}$ must be a convex decreasing function with $\tilde{f}(0) = \infty$ and $\tilde{f}(1) = 0$. If we ignore this last condition, then there is a simple parametric family of solutions given by

$$\tilde{f}(c) = \frac{\tilde{G}}{c}. \qquad \text{(A1)}$$

Then $x = f^{-1}(c) = -c + 1 + \tilde{f}(c)$ so that

$$1 - x = c - \frac{\tilde{G}}{c}.$$

This simplifies to

$$c = \sqrt{\tilde{G} + \frac{(1-x)^2}{4}} + \frac{1-x}{2}.$$

Writing $G = \tilde{G} S_t^2$ we find

$$C(X) = \sqrt{G + \frac{(S_t - X\mathrm{e}^{-r(T-t)})^2}{4}} + \frac{S_t - X\mathrm{e}^{-r(T-t)}}{2}$$

and recover (3), which is Figlewski's (2002) implied G example.

The problem with this pricing function is that it does not satisfy $C(0) = S$ or equivalently $\tilde{f}(1) = 0$. One easy way to modify (A1) so that $\tilde{f}(1) = 0$ is to set

$$\tilde{f}(c) = \frac{\tilde{G}(1-c)}{c}.$$

Then

$$1 - x = c - \frac{\tilde{G}(1-c)}{c}$$

and

$$c = \sqrt{\tilde{G} + \frac{(1 - \tilde{G} - x)^2}{4}} + \frac{1 - \tilde{G} - x}{2}.$$

In terms of the original quantities we have

$$C(X) = \sqrt{gS_t + \frac{(S_t - X\mathrm{e}^{-r(T-t)} - g)^2}{4}} + \frac{S_t - X\mathrm{e}^{-r(T-t)} - g}{2},$$

where $g = \tilde{G} S_t$, which gives us the modified implied G example.