Static-Arbitrage optimal subreplicating strategies for Basket Options

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Abstract

In this paper we investigate the possible values of basket options.

Instead of postulating a model and pricing the basket option using that model, we consider the set of all models which are consistent with the observed prices of vanilla options of all strikes. In the case of basket options on two components we find, within

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this class, the model for which the price of the basket option is smallest. This price, as discovered by Rapuch and Roncalli, is associated to the lower Fréchet copula. We complement their result in this paper by describing an optimal subreplicating strategy. This strategy is associated with an explicit portfolio which consists of being long and short a series of calls with strikes chosen as the zeros of an auxiliary function.

Keywords and Phrases: Basket Options; Anti-monotonicity; Sub-replication; Arbitrage-free bounds; Copula.

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1. **Introduction**

Options on a basket of stocks are fundamental instruments in world financial markets. Examples thereof are exchange traded instruments such as equity index options, usually written on at least 15 stocks, and currency basket options, written on two or more assets. Currency baskets are customized products which are traded over the counter.

An index I on n underlying stocks $S_i, i = 1, \cdots, n$ is usually defined as a basket on the n stocks with fixed weights w_i, so that $I = \sum_{i=1}^{n} w_i S_i$. A European call option on the index, struck at K, with maturity T has a payoff $(I - K)^+$, and arbitrage pricing theory gives the value of this option at time zero as

$$E_\mu \left[e^{-rT} \left(\sum_{i=1}^{n} w_i S_i - K \right)^+ d\mu(S_1, \ldots, S_n) \right],$$

where μ is a risk neutral measure associated with the joint distribution of the underlying prices of S_i at time T. This price is uniquely determined in a complete market for in such markets μ is known unequivocally. In practice, however, markets are incomplete and a myriad of possible risk neutral measures can be used to calculate the option price. The most standard setting in finance, the Black-Scholes setting, assumes the assets are driven by correlated exponential Brownian motions so that the distribution under the risk-neutral measure at time T, assuming no dividends are paid between 0 and T, is

$$S_i(T) = S_i(0) \exp\left(\sigma_i W_t + (r - \frac{\sigma_i^2}{2})t \right) \quad i = 1, \cdots, n$$

where σ_i are constants as are the correlations $\rho_{ij}, i, j = 1, \cdots n$ between the driving Brownian motions.

However, even for options written on one asset, the standard Black-Scholes model is not consistent with the so-called smile effect in option prices, and a substantial amount of research over the last decade has been devoted to pricing and hedging assuming the underlying evolves according to alternative stochastic processes. Among the most popular are level-dependent models, in which the volatility is allowed to depend on spot and time, stochastic volatility models, uncertain volatility models, and jump-diffusion or pure jump processes. Such models can be used to account for

the smile effect in the observed market values of vanilla calls and puts. The plethora of alternative models available for pricing and hedging leave practitioners with a wide spectrum of models at their disposal, but little information about which, if any, is the correct model to use.

In such an environment a complementary approach, useful both for risk management purposes and to provide a sanity check for the prices and hedges obtained from parametric models, is to derive *distribution free* no-arbitrage prices and hedges. This second approach is less ambitious in scope in the sense that it does not aim to derive a unique fair price, but more robust in the sense that it is not dependent on the efficacy of an underlying model. The aim is to provide bounds on the possible price of the basket option which are consistent with no-arbitrage *given* the market prices of vanilla puts and calls. In essence, rather than using a single model, we consider the class of all models which are consistent with the observed call prices, and rather than quoting a single option price we give the range of prices which arise under models from this class.

In this paper this philosophy will be applied to basket options in the setting of a one-period static arbitrage model and we will focus on the case of lower bounds for baskets written on two assets. We will also assume that prices of call options on the two underlying stocks with a continuum of strikes are known. In reality only a discrete number of strikes for each maturity are traded. The case of only a discrete number of strikes is not a straightforward extension of the continuum of strikes case.

We consider only lower bounds in this paper. The case of upper bounds was solved, in the general n-asset case, for both the continuum of strikes case and for the discrete set of strikes case, in our previous paper [18]. (The upper bound problem is also considered in the continuous strike case by Dhaene et al [12], Goovaerts et al [17] and Kaas et al [21] using the theory of stochastic ordering.) It may seem surprising that one cannot treat both upper and lower bounds by the same method. It turns out however that deriving distribution-free lower bounds is far more complex then deriving distribution-free upper bounds. At the root of this difficulty is the fact that optimal upper bounds turn out to be associated with superreplicating strategies for which the hedger takes *long* positions in *all* the underlying components and a zero position in cash and it then only remains to determine, among the strikes trading on

the individual options, which ones are associated with the cheapest possible superreplicating portfolio[2]. In [18] we proved that the optimal superreplicating strategy involves the selection of only two strikes per asset and we gave a simple and computationally efficient way to determine these strikes. In the case of lower bounds it turns out that it is not in general sufficient, as illustrated in this paper in two asset case, to consider subreplicating strategies involving one or two strikes per asset and the optimal strategy involves both long and short positions in calls as well as a cash component. The optimal subreplicating strategy may involve many strikes. These strikes are the zeros of a certain function uniquely determined by the call price functions as functions of strike.

Some insight into the added complexity of optimal lower bounds may be useful and is gained by a review of earlier results in this direction. Let us recall the first one derived in the case of one underlying asset by Bertsimas and Popescu in [4]:

Given prices $q_i = q(K_i) = E[(X - K_i)^+], i = 1, \cdots, n$ of call options with strikes $0 \leq K_1 \leq K_2 \leq \cdots K_n$ on a stock S, the range of all possible valid prices for a call option with strike price K where $K \in (K_j, K_{j+1})$ for some $j = 0, \cdots, n$ is $[q^-(K), q^+(K)]$ where

$$q^-(K) = \max\left(q_j \frac{K - K_{j-1}}{K - K_{j-1}} + q_{j-1}\frac{K_j - K}{K_j - K_{j-1}}, q_{j+1}\frac{K_{j+2} - K}{K_{j+2} - K_{j+1}} + q_{j+2}\frac{K - K_{j+1}}{K_{j+2} - K_{j+1}}\right)$$

$$q^+(K) = q_j \frac{K_{j+1} - K}{K_{j+1} - K_j} + q_{j+1}\frac{K - K_j}{K_{j+1} - K_j}$$

Here $K_0 = 0$, $q_n = q_{n+1} = q_{n+2}$ and $K_{n+2} > K_{n+1} > K_n$, although the precise values of these extra strikes does not matter. The situation is summarized in Figure 1.

Laurence and Wang [23] established a lower bound for basket options in the 2-asset case, under the assumption that there is only one traded asset and that, in addition to an option on each asset, the forward prices are prescribed. The optimal hedging strategy associated to the lower bound depends in a complicated way on the input forward and option prices and involves in some cases, both long and short positions in options and long and short positions in cash. Let $F_i = \frac{S_0^i - c_i}{K_i}$ and let

[2]Even for a small index, such as DJX, with 30 assets in the index and 8 to 13 options traded on each component asset, this entails choosing among the order of 10^{30} possible combinations, if choosing only one strike per asset is optimal and many more if it turned out to be more efficient to "diversify" and select more than one strike per asset.

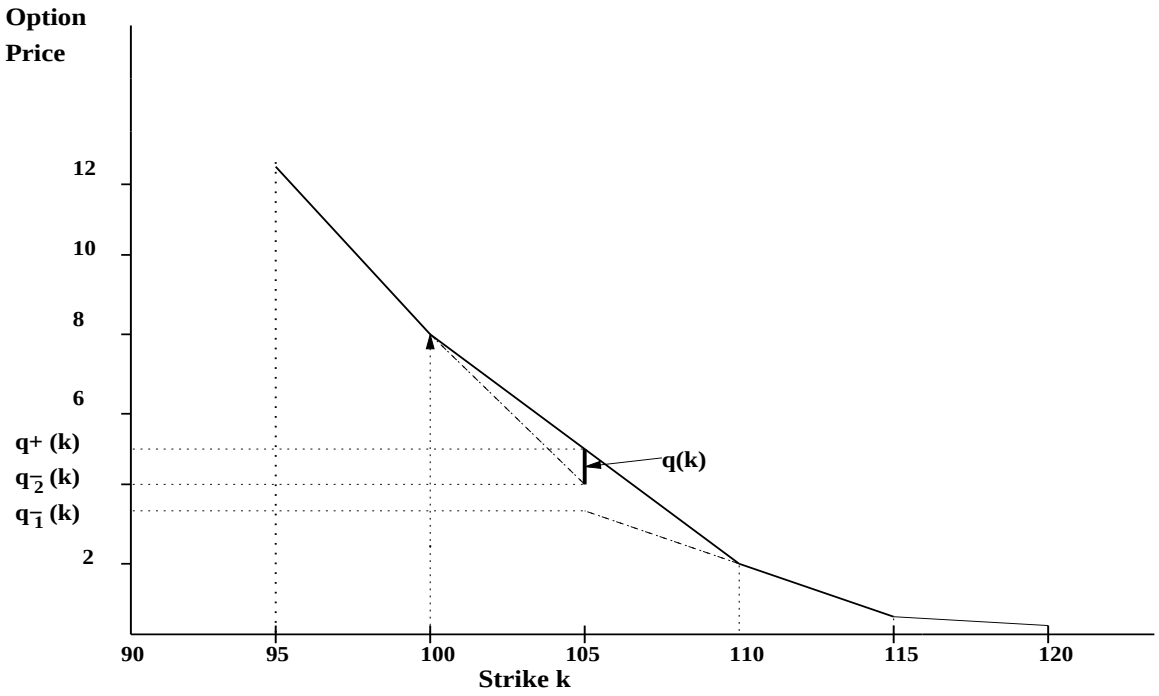

Figure 1: The figure illustrates the range of prices for options whose strikes lie between 100 and 110. The particular case of strike 105 is emphasized. q^+ stands for the upper bound and the larger of q_1^- and q_2^- stands for the lower bound.

$D = K - w_1K_1 - w_2K_2$. If $K > \max(w_1K_1, w_2K_2)$ then the Laurence-Wang lower bound is given by[3]

- For $D \leq 0$,

$$\max\left\{A_1 + w_2K_2F^+, A_2 + w_1K_1F^+, A_1 + A_2 + KF^+, 0\right\}, \tag{1}$$

- For $D \geq 0$,

$$\max\left\{A_1 + (K - w_1K_1)F^+, A_2 + (K - w_2K_2)F^+, A_1 + A_2 + KF^+, 0\right\}, \tag{2}$$

where

$$A_i = w_ic_i - (\frac{K - w_iK_i}{K_i})(S_0^i - c_i) \quad \text{for} \quad i = 1, 2, \tag{3}$$

$$F = F_1 + F_2 - 1, \tag{4}$$

$$F^+ = \max\{F, 0\}. \tag{5}$$

[3]Laurence and Wang allow a non zero short rate, but here and throughout this paper we will take $r = 0$ for simplicity.

A simpler lower bound exists for $K \leq \max(w_1 K_1, w_2 K_2)$, see [24].

There are several open problems concerning the lower bound. Firstly, it is not known, even when there are only two components in the basket, how to extend the above bound to the case when there are several strikes traded on each component. Secondly, the best lower bound is not known even when there is only one option traded on each asset, but $n > 2$. (It should be noted however than in an interesting paper Aspremont and El Ghaoui [2] find optimal bounds in closed form when an option on each component is traded, but forwards on the component assets are not.)

The results described above illustrate how much more complex the situation is in the case of lower bounds than for upper bounds. In this paper we are interested in lower bounds (when $n = 2$) and we assume knowledge of a continuum of strikes on each of two assets. Breeden and Litzenberger's result [7] then implies that knowledge of the full marginals is easily deduced from the call prices (but we have no information on the joint distribution of assets). Indeed Breeden and Litzenberger showed how to deduce the distribution of an asset from the first derivative of the option price with respect to the strike. Applying this to each of our stocks in turn, we then recover knowledge of the full marginal of the stock from knowledge of the (assumed) continuum of strike prices.

Now the problem of determining optimal joint distributions subject to the constraint of prescribed marginals has a long history in mathematics. A comprehensive and nearly up to date reference is the book by Joe [19]. A tool that dates back to the work by Fréchet [16] that has been used to attack such problems is to use copulas. Indeed the celebrated Sklar theorem tells us that for any joint probability distribution F with margins $F_1, \cdots, F_n$ there exists an n-copula C such that for all $x \in \mathbb{R}^n$ we have

$$F(x_1, \ldots, x_n) = C\left(F_1(x_1), \ldots, F_n(x_n)\right)$$

Thus, in seeking optimal joint distributions subject to prescribed marginals, the marginals have in effect been "factored out" and the remaining problem is to determine the optimal copula. It was discovered by Dhaene and Goovaerts [11] that the optimal copula for the basket options in the case $n = 2$ is associated with the lower Fréchet bound. This result has been generalized to more general payoff functions of contingent claims on two assets by Rapuch and Roncalli [27]. To establish their

result Rapuch and Roncalli also rely on a result of Muller and Scarsini concerning the behaviour of convex functions under the concordance order [26]. Since their approaches rely in a central way on the lower Fréchet copula, their results are, as is ours, restricted to the two-asset case. Indeed it is well known [19] that in general the lower Fréchet bound does not correspond to a copula for $n \geq 3$. We complement that result in two ways: the first and main contribution is to provide an optimal subreplicating strategy using what we refer to as STP portfolios. This striking class of portfolios is expressed as a series of long and short positions in call options of different strikes and a long position in cash. An alternative more compact description in terms of long positions in calls and short positions in puts is also given. A second contribution of this paper is that we obtain, as a corollary, a *direct proof of the optimal lower bound* that is *independent* of the Fréchet bounds and of the Muller-Scarsini ordering result. Since, as mentioned above, the Fréchet lower bound does not extend to n dimensions for $n \geq 3$, this opens up a roadmap to an n-dimensional generalization of the present results.

We conclude this section with a literature review. Most of the work on basket options focuses on the Black-Scholes setting. Due the high dimension of these options (most indices involve at least 17 assets) , their analytical and numerical valuation is very challenging even in the Black-Scholes setting. Indeed, although an analytical formula for the call option does exist, see for example the text by Kwok, [22], the integral is difficult to evaluate in practice and one must resort to Monte-Carlo methods, to recursive methods, see Ware and Avelassani [29], or to methods using characteristic functions, see Ju [20], moment matching, see Brigo et al [5] and Dufresne [13]. Also progress in handling diffusion based models with non constant volatility close to expiration has been made in Avellaneda et al [3]. Given the difficulty of calculating exact prices several authors, including Deelstra et al [9] and Kaas et al [21], have looked for upper and lower bounds. In both of the papers a clever conditioning argument is used to give a lower bound on the price of the basket option. However this method assumes that the underlyings follow correlated exponential Brownian motions, so that the lower bound is a model-based lower bound in the Black-Scholes setting, and not a model-independent lower bound such as we propose. Corresponding bounds for

Asian options and in the context of actuarial science are also considered in Dhaene et al [10].

2. **Problem formulation**

We assume that a market exists in which calls and puts of maturity T are traded on the two component assets of the basket. We assume that options with a continuum of strikes are traded on each component asset: we may also think of the underlying assets themselves as call options with zero strike. Our goal is to price a basket option on the two assets which is traded with the same maturity T.

We address two problems whose solutions turn out to be dual to each other in an appropriate variational formulation that will be described below: i) the problem of finding the infimum of all basket option prices when the joint distribution is constrained by the (perfect) knowledge of the marginal distributions provided by the individual call options, ii) The problem of finding the optimal subreplicating strategy consisting of calls, puts and cash. The cost of the optimal subreplicating strategy corresponds to the highest bid price of an investor who is offered the opportunity to be long the basket option, but is not prepared to accept any risk. It is the highest price she is willing to pay, since if she buys the basket at this price and constructs a static hedge by a portfolio of calls, puts and cash according to the optimal portfolio, she is sure the payoff of the basket will be higher than the obligations arising from her hedging portfolio in all states of the world. Moreover the portfolio (of calls, puts and cash) is the most expensive one she can sell whilst still guaranteeing that the basket superreplicates her portfolio.

A simple but important observation is that put-call parity ensures that any put is equivalent to a call and cash. Therefore, in choosing optimal subreplicating strategies and or in minimizing the basket option price subject to the constraint of a continuum of call and put options of all strikes, we may reduce the problem to one in which only calls and cash are the subreplicating instruments. Thus the problem will be formulated in this setting [4].

[4] It will however turn out, as we will see in section § 1.4 that the optimal subreplicating portfolio takes a particularly simple form when we use calls on one asset and puts on the other.

For expository reasons we begin by describing the primal and dual variational problems in a somewhat informal and intuitively appealing form. Their rigorous formulation is deferred to the next section. Consider the following constrained minimization problem

$$\inf_{\mu} \int_{\mathbb{R}^2_+} (x+y-K)^+ \mu(dx,dy)$$

where μ ranges over the space of all risk neutral distributions on $\mathbb{R}^2_+$, subject to the constraints on the marginal distributions

$$\int_{\mathbb{R}_+} (x-k_1)^+ \mu_X(dx) = C_X(k_1),$$
$$\int_{\mathbb{R}_+} (y-k_2)^+ d\mu_Y(dy) = C_Y(k_2),$$
$$\int_{\mathbb{R}^2_+} \mu(dx,dy) = 1.$$

Here μ_X and μ_Y are the marginal distributions.

Throughout this paper we have taken *unit* coefficients (weights) in the basket and zero interest rates and assume that the assets pay no dividends. If the weights are different from one and interest rates are constant we may reduce to the present case by a simple scaling argument (see [24]) .

The dual problem for finding optimal subreplicating strategies is given by

$$\sup_{\lambda,\nu_1,\nu_2} \int_{\mathbb{R}_+} C_X(k_1)\nu_1(dk_1) + \int_{\mathbb{R}_+} C_Y(k_2)\nu_2(dk_2) + \lambda \tag{6}$$

subject to the constraints

$$(x+y-K)^+ - \int_{\mathbb{R}_+} (x-k_1)^+ \nu_1(dk_1) - \int_{\mathbb{R}_+} (y-k_2)^+ \nu_2(dk_2) - \lambda \geq 0, \tag{7}$$
$$\forall x \geq 0 \quad y \geq 0.$$

2.1. **A review of Dhaene and Goovaerts' result.**

Given knowledge of the call functions $C_X(k_1), C_Y(k_2)$, Breeden and Litzenberger's result implies that $\frac{d}{dk_1}C_X = F_X(k_1) - 1$, $\frac{d}{dk_2}C_Y = F_Y(k_2) - 1$ where F_X and F_Y are the distribution functions of X and Y respectively. As mentioned in the introduction, by Sklar's theorem any joint distribution $F(x,y) = P(X \leq x, Y \leq y)$ can be represented as $\mathcal{C}(F_X(x), F_Y(y))$ for some copula $\mathcal{C}$.

Recall the lower and upper Fréchet copula bounds: for any copula $\mathcal{C}(u_1, u_2)$ on $[0,1]^2$ we have

$$\mathcal{C}^-(u_1, u_2) \leq \mathcal{C}(u_1, u_2) \leq \mathcal{C}^+(u_1, u_2),$$

where

$$\mathcal{C}^+ = \min(u_1, u_2),$$
$$\mathcal{C}^- = \max(u_1 + u_2 - 1, 0).$$

Therefore, for any distribution function F with prescribed marginals F_X, F_Y, we have

$$\mathcal{C}^-(F_X(x), F_Y(y)) \leq F(x,y) \leq \mathcal{C}^+(F_X(x), F_Y(y)).$$

From this Dhaene and Goovaerts [11] (Theorem 3, p.206) deduce the following:

Proposition 1. *The price of a call option on a basket $C_{\mathcal{B}}$ on X and Y whose marginal distributions are F_X and F_Y satisfies the bounds*

$$C_{\mathcal{B}}^- \leq C_{\mathcal{B}} \leq C_{\mathcal{B}}^+$$

where $C_{\mathcal{B}}^+$ and $C_{\mathcal{B}}^-$ correspond to the upper and lower Fréchet bounds respectively, that is, the joint distributions of (X,Y) are given respectively by $\mathcal{C}^+(F_X, F_Y)$ and $\mathcal{C}^-(F_X, F_Y)$.

The result has been generalized to *supermodular* payoff by Rapuch and Roncalli [27] from a result of Muller and Scarsini [26].

When the distribution functions F_X and F_Y are continuous, we have

$$\mathcal{C} = \mathcal{C}^- \Leftrightarrow Y = F_Y^{-1}(1 - F_X(X)),$$
$$\mathcal{C} = \mathcal{C}^+ \Leftrightarrow Y = F_Y^{-1}(F_X(X)).$$

For the lower bound note this means that

$$C_{\mathcal{B}}^- = \int_{\mathbb{R}_+} [x + F_Y^{-1}(1 - F_X(x)) - K]^+ dF_X(x). \tag{8}$$

2.2. **A precise formulation of primal and dual problems.**

In this section we state the primal and dual problems in an appropriate infinite dimensional setting. Our approach is adapted from that in Anderson and Nash [1]. For the reader's convenience we summarize the backdrop of these results in an Appendix, see Section 6. Let $\mathcal{M}$ denote the linear space of all finite signed measures on $\mathbb{R}_+^2$ which decay at least linearly at infinity in the sense that, for $\mu \in \mathcal{M}$, $\mu(B_r^c(\xi)) \sim o(1/r)$

as r goes to infinity for every $\xi \in \mathbb{R}^2_+$, where $B^c_r(\xi) = \{\eta \in \mathbb{R}^2_+ : |\eta - \xi| > r\}$. Let Γ be the linear space generated by the functions $p_1(x, y; k_1) := (x - k_1)^+$, $p_2(x, y; k_2) := (y - k_2)^+$ and $p_B(x, y; K) := (x + y - K)^+$ defined on $\mathbb{R}^2_+$. Define the pairing $\langle , \rangle$ between $\mathcal{M}$ and Γ by integration, i.e.,

$$\langle \mu, f \rangle = \int_{\mathbb{R}^2_+} f(x, y)\mu(dxdy)$$

Denote by $\mathcal{M}_+$ the convex cone in $\mathcal{M}$ of all finite positive measures on $\mathbb{R}^2_+$. We consider the following constrained minimization problem

$$\inf_{\mu \in \mathcal{M}_+} \langle \mu, p_B(\cdot; K) \rangle \tag{9}$$

subject to the constraints on the marginal distributions

$$\langle \mu, p_1(\cdot; k_1) \rangle = C_X(k_1), \tag{10}$$

$$\langle \mu, p_2(\cdot; k_2) \rangle = C_Y(k_2), \tag{11}$$

$$\langle \mu, 1 \rangle = 1, \tag{12}$$

where C_X and C_Y are given call price functions which are necessarily nonnegative, decreasing and convex. Let $\mathcal{H}$ be the set of all nonnegative, decreasing and convex functions defined on $\mathbb{R}_+$, $\mathcal{S}$ be the set of all finite signed measures over $\mathbb{R}_+$ and the pairing $\langle , \rangle$ between $\mathcal{H}$ and $\mathcal{S}$ is given by integration. Consequently, the dual problem of primal problem (9-12) is the following constrained maximization problem

$$\sup_{\nu_1, \nu_2, \lambda} \langle C_X, \nu_1 \rangle + \langle C_Y, \nu_2 \rangle + \lambda \tag{13}$$

subject to the constraints

$$\langle \mu, p_B - \langle p_1, \nu_1 \rangle - \langle p_2, \nu_2 \rangle - \lambda \rangle \geq 0, \quad \forall \mu \in \mathcal{M}_+, \tag{14}$$

$$\nu_1, \nu_2 \in \mathcal{S}, \quad \lambda \in \mathbb{R}. \tag{15}$$

As a matter of fact, (14) can be further realized as

$$p_B - \langle p_1, \nu_1 \rangle - \langle p_2, \nu_2 \rangle - \lambda \geq 0, \quad \forall x \geq 0 \quad y \geq 0. \tag{16}$$

Hence we have the dual problem in the form described as (6-7). Here we remark that (6) is an expression for the most expensive value among subreplicating portfolios and (7) is the condition for subreplication.

The complementary slackness condition (38) in the Appendix, written out in our setting therefore reads

$$\langle \mu, p_B(\cdot;K)\rangle = \langle \mu, \langle p_1(\cdot;k_1),\nu_1\rangle + \langle p_2(\cdot;k_2),\nu_2\rangle + \lambda\rangle$$

Representing the inner product in integral form and exchanging the order of integration on the right hand side yields

$$\begin{aligned} \int p_B(\cdot;K)d\mu &= \int \left(\int p_1(\cdot;k_1)d\mu\right)\nu_1(dk_1) + \int\left(\int p_2(\cdot;k_2)d\mu\right)\nu_2(dk_2) + \lambda \\ &= \int C_X(k_1)\nu_1(dk_1) + \int C_Y(k_2)\nu_2(dk_2) + \lambda. \end{aligned} \tag{17}$$

Therefore, in order to prove optimality by applying the complementary slackness condition, we need to find feasible measures μ, ν_1, ν_2 and a real number λ such that the equality (17) holds.

2.3. **A family of optimal subreplicating portfolios.**

In this section we introduce a family of subreplicating portfolios we call STPs[5]. An intuitive derivation of the STPs from lower Fréchet copula is postponed until Section 3. Let C'^{+} denote the right derivative of the call price function, and C'^{-} the left derivative. Define the auxiliary function

$$\phi(x) := {C'_X}^{+}(x) + {C'_Y}^{-}(K-x) + 1. \tag{18}$$

By construction ϕ, which is only defined on $[0,K]$, is right-continuous. We also set $\phi(K) = {C'_X}^{+}(K)$. Clearly ϕ is the difference of two increasing functions and it's total variation is bounded by the constant 2. Define

$$A = \{x : \phi(x) > 0\}. \tag{19}$$

Then A is a countable union of disjoint intervals $A = \cup_j A_j$. For the rest of this section we make the assumption: A is a union of a finite number n of intervals. Indeed the case that A consists of infinitely many intervals is degenerate. A discussion on this assumption is given in Section 3.2.

Suppose that the intervals $(A_j)_{1\le j\le n}$ are placed in their natural order, and that the boundary of A_j is given by the points $\{K_1^{2j-1}, K_1^{2j}\}$. Define $K_2^j = K - K_1^{2n-j+1}$ and

[5]"STP" is short for "sheep-track portfolios" since the graph of such a portfolio is reminiscent of such tracks on British hillsides. A picture thereof is contained in Figure 2 below.

consider the dual variables defined by

$$\bar{\nu}_1(dk_1) = \delta_0(k_1)dk_1 + \sum_{i=1}^{2n}(-1)^i\delta_{K_1^i}(k_1)dk_1, \tag{20}$$

$$\bar{\nu}_2(dk_2) = \delta_0(k_2)dk_2 + \sum_{i=1}^{2n}(-1)^i\delta_{K_2^i}(k_2)dk_2, \tag{21}$$

$$\bar{\lambda} = \sum_{i=1}^{n}(K_1^{2i} - K_1^{2i-1}) - K = \sum_{i=1}^{n}(K_2^{2i} - K_2^{2i-1}) - K. \tag{22}$$

These dual variables are shown to be feasible in the next section.

Remark 2. In the special case that ϕ is C^1 in $[0, K]$, the determination of the strikes K_1^i's in the dual variables given above reduces to find the zeros $\{x : \phi(x) = 0\}$ of ϕ. For the finiteness of the zeros for C^1 functions, we add the following observation that shows that the finiteness will hold except in certain (unavoidable) degenerate cases. Let $I \subset \mathbb{R}$ be a bounded open interval, $h \in C^1(\overline{I})$ and p be a regular value of h, i.e., $h'(x) \neq 0$ for every $x \in h^{-1}(p)$. Then $h^{-1}(p)$ is finite. In our case this means that $\phi(x)$ has only finite number of zeros in $(0, K)$ provided that whenever x is such that $C_X'(x) = -C_Y'(K - x) - 1$, then $C_X''(x) \neq C_Y''(K - x)$, i.e., the densities of X and Y are different at such points.

Given the dual measures as in (20) and (21) the payoff of the the associated portfolio can be expressed by integrating $(x - k_i)$ against $\nu_i(dk_i)$. Partition $\mathbb{R}_+^2$ into $(2n + 1)^2$ pieces denoted by $R_{i,j}$using the definitions

$$\mathbb{R}_+^2 = \bigcup_{i,j=1}^{2n+1} R_{i,j}, \text{ where } R_{i,j} = \{(x, y) \in \mathbb{R}_+^2 : K_1^{i-1} \leq x < K_1^i, {K_2}^{j-1} \leq y < {K_2}^j\}.$$

Here we have used the convention that $K_1^0 = {K_2}^0 = 0$ and $K_1^{2n+1} = {K_2}^{2n+1} = \infty$. The associated portfolio may be expressed in each region $R_{i,j}$ as

$$\left(x + \sum_{a=1}^{i}(-1)^a(x - K_1^a)\right) + \left(y + \sum_{b=1}^{j}(-1)^b(y - K_2^b)\right) + \bar{\lambda}.$$

or in the following more compact form, which applies simultaneously across all regions

$$f_1(x) + f_2(y) + \bar{\lambda}, \tag{23}$$

where the functions f_i are defined by

$$f_i(z) = z^+ + \sum_{a=1}^{n} \{(z - K_i^{2a})^+ - (z - K_i^{2a-1})^+\}. \tag{24}$$

Note that each of $f_1(x)$ and $f_2(y)$ has an immediate interpretation in terms of a hedging strategy involving call options. Moreover, a few lines of algebra show that

$$f_1(K - z) + f_2(z) + \bar{\lambda} = 0,$$

so that (23) simplifies to

$$f_1(x) - f_1(K - y). \tag{25}$$

Thus the hedging portfolio for basket call may be expressed as a portfolio of calls held long and short on x and a portfolio of puts held long and short on y. The advantage of this representation is that it does not involve cash.

The functions f_i will be referred to as *STP functions*. The measures (20) and (21) are in one to one correspondence with the STP portfolio. We will refer to them as the *STP measures*. An illustration of *STP* function f is given in Figure 2.

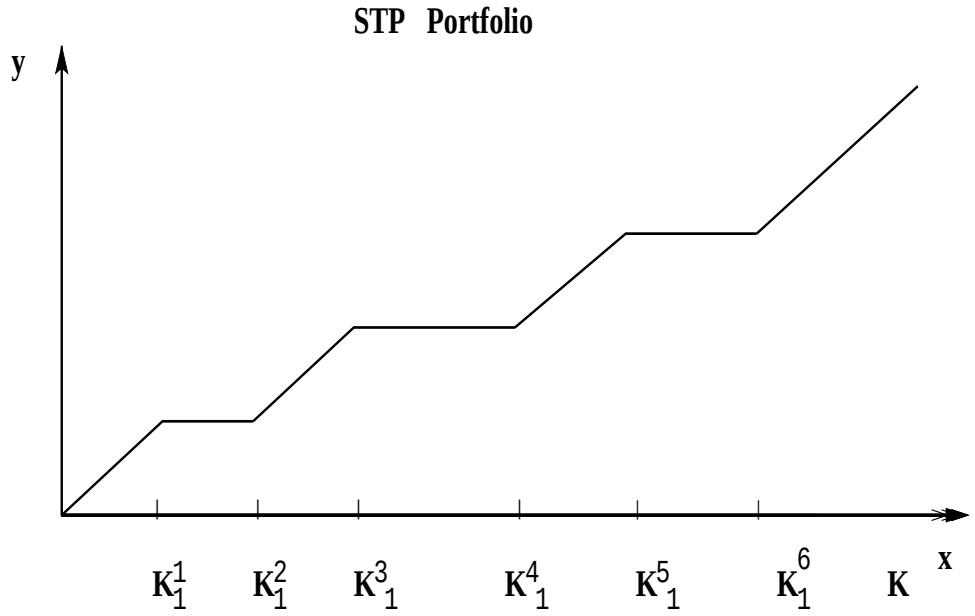

Figure 2: The figure illustrates an STP portfolio which, starting at zero, is piecewise linear with alternating slopes one and zero. The points of transition define K_1^i for odd and even i.

2.4. **Feasibility of dual variables.**

Feasibility is equivalent to establishing the following subreplication property

$$f_1(x) - f_1(K-y) \leq (x+y-K)^+. \tag{26}$$

Since f_1 is piecewise linear with alternately slopes 0 and 1, by the mean value theorem we clearly have

$$f_1(z+u) - f_1(z) \leq u^+ \tag{27}$$

so we have (26). Feasibility is assured for our hedging portfolios provided they satisfy (27).

2.5. **A geometric construction of optimal bivariate processes using STP.**

Notice that equality is achieved in (26) provided that one of the following 3 cases holds:

$$x = K - y \tag{28}$$

$$x > K - y \text{ and } f_1(x) = f_1(K-y) + x + y - K, \text{ i.e., } x, K-y \in (K_1^{2j}, K_1^{2j+1}) \tag{29}$$

$$x < K - y \text{ and } f_1(x) = f_1(K-y), \text{ i.e., } x, K-y \in (K_1^{2j-1}, K_1^{2j}) \tag{30}$$

as is illustrated by Figure 3 below.

Suppose that X and Y are continuous random variables with strictly positive densities on $[0, K]$. In order to construct a bivariate process (X, Y) achieving equality in the inequality (26) it suffices to choose a countermonotic process $Y = G(X)$ with G a non-increasing function in such a way that the support of (X, Y) is concentrated on places where the conditions given in (28), (29) and (30) above hold. Since in this case F_X and F_Y^{-1} are continuous functions this is achieved by the choice $G(X) := F_Y^{-1}(1 - F_X(X))$. Note that the set $(G(x) < K - x)$ is precisely the set $\phi(x) > 0$, so that the points where G crosses the line $x + y = K$ are exactly the boundary points of the set A.

More generally, if $U \sim U[0,1]$ and we define $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(1-U)$ then X and Y have the desired marginals, and moreover the joint law of (X, Y) is such that all the mass is placed at co-ordinates where equality holds in (26).

This is illustrated graphically in Figure 3.

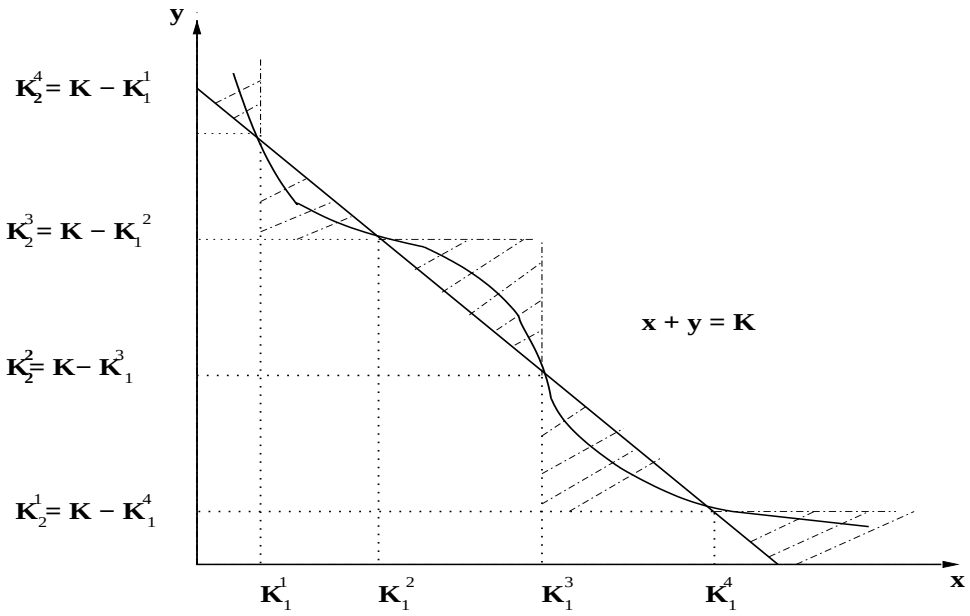

Figure 3: Joint distribution is optimal if supported in the shaded region.

2.6. **Optimality.**

Having established that there is equality in the relations $\mathbb{E}[(X+Y-K)^+] = \mathbb{E}[f_1(X) - f_1(K-Y)]$ when (X,Y) is an anti-monotonic process with support chosen as above, we can easily deduce the following theorem:

Theorem 1. Suppose the call functions $C_X(x)$ and $C_Y(y)$ are such that the set A defined in (19) is a union of finitely many intervals. Let $0 \leq K_0^1 \leq \cdots K_1^{2n} \leq K$ be the endpoints of these intervals, and let f_i be defined as at (24).

Let U be a random variable with a standard uniform distribution. Define $\bar{X} = F_X^{-1}(U)$ and $\bar{Y} = F_Y^{-1}(1-U)$. Then

- The joint distribution function $\bar{\mu}$ associated to the bivariate process $(\bar{X}, \bar{Y})$ is a minimizer for the primal problem for the call on a two asset basket with strike K;
- the portfolio defined via $f_1(x) + f_2(y) + \bar{\lambda}$ is feasible and the optimal measures $\bar{\nu}_1, \bar{\nu}_2$ given by the formulas (20) and (21), together with $\bar{\lambda}$ form a triple that is optimal for the dual problem (6), (7) .

Proof. In essence, all that remains to be proved is that the dual variables we have constructed are optimal not only among measures of the form given in (20) and (21), but also amongst all measures in the original dual problem. The following chain of inequalities makes this clear: since $(\bar{X}, \bar{Y})$ is feasible for the primal problem and $(\bar{\lambda}, \bar{\nu_1}, \bar{\nu_2})$ is feasible for the dual problem, we have

$$\begin{aligned}
\langle C_X, \bar{\nu}_1\rangle + \langle C_Y, \bar{\nu}_2\rangle + \bar{\lambda} &= \sup_{(\nu_1,\nu_2,\lambda)} \langle C_X, \nu_1\rangle + \langle C_Y, \nu_2\rangle + \lambda \\
&\leq \sup_{\nu_1,\nu_2,\lambda \text{ verifies (14),(15)}} \langle C_X, \nu_1\rangle + \langle C_Y, \nu_2\rangle + \lambda \\
&\leq \inf_{\mu\in\mathcal{M}_+,\ \mu \text{ verifies (10),(11),(12)}} \int_{\mathbb{R}^2_+} (x+y-K)^+ \mu(dx,dy) \\
&\leq \int_{\mathbb{R}^2_+} (x+y-K)^+ \bar{\mu}(dx,dy)
\end{aligned}$$

The first and the last elements in the above chain are equal and hence the other inequalities are equalities. Hence $\bar{\mu}$ and $(\bar{\lambda}, \bar{\nu}_1, \bar{\nu}_2)$ respectively are also optimal for the original primal and dual problem and the values of the primal and dual functionals evaluated on these functions are equal. □

2.7. **Optimal subreplicating portfolio for basket put.**

Using put-call parity for the basket we also have lower bound for the price $P_{\mathcal{B}}$ of basket put and the corresponding subreplicating portfolio. Note that the price $P_{\mathcal{B}}$ is given by $P_{\mathcal{B}} = C_{\mathcal{B}} - (x+y) + K$ (recall we have reduced to the case of zero interest rates). Thus one immediately obtains the lower bound for $P_{\mathcal{B}}$ by subtracting the lower bound for the price of basket call by a forward price. Moreover, since the stocks x, y may be thought of as options with strike zero, we may use this relation to determine the STP type portfolio that subreplicates $P_{\mathcal{B}}$. In other words we have that

> The hedging portfolio for the basket put option is equal to the hedging portfolio for a basket call option combined with a portfolio which is short both assets and long K units of cash.

Using the explicit form (23) for the hedging portfolio on the basket and (22) we therefore get that the hedging portfolio for the put of the form

$$f_1(x) - f_1(K-y) - (K-y-x) = \tilde{f}(K-y) - \tilde{f}(x) \tag{31}$$

where $\tilde{f}(z) = z - f_1(z)$, or equivalently,

$$\tilde{f}(x) = \sum_{a=1}^{n} \{(x - K_1^{2a-1})^+ - (x - K_1^{2a})^+\}. \tag{32}$$

As for (25), the hedging portfolio for the basket put is expressed as a portfolio of puts held long and short on y and a portfolio of calls held long and short on x. Also the advantage of the representation (31) is that it does not involve cash.

3. **A derivation of optimal dual from optimal copula**

We complement the treatment in the previous section by outlining how the STP portfolio can be derived from the lower Fréchet copula. Indeed it is this approach that first led us to the form of these portfolios. The second part of the section consists of a discussion on the finiteness assumption on the set A defined by (19).

3.1. **Optimal dual measures.** Consider the following joint distribution of X and Y given by

$$F(x,y) = \max\{F_X(x) + F_Y(y) - 1, 0\} = \left(C_X'^{+}(x) + C_Y'^{+}(y) + 1\right)^+.$$

This joint distribution of X and Y can also be characterized as: X is distributed as $F_X^{-1}(U)$ and Y is distributed as $F_Y^{-1}(1-U)$, where U is a random variable uniformly distributed in $(0,1)$ and F_X^{-1}, F_Y^{-1} are the generalized inverse of F_X and F_Y respectively defined by

$$F_X^{-1}(u) = \inf\{x : F_X(x) > u\},$$
$$F_Y^{-1}(u) = \inf\{y : F_Y(y) > u\}.$$

Now the joint distribution associated with the lower Fréchet copula is feasible for the primal problem and yields the primal value

$$\begin{aligned}
\mathbb{E}[(X+Y-K)^+] &= \int_0^1 (F_X^{-1}(u) + F_Y^{-1}(1-u) - K)^+ du \\
&= \int_E (F_X^{-1}(u) + F_Y^{-1}(1-u) - K) du \\
&= \int_E (F_X^{-1}(u) - K) du + \int_E F_Y^{-1}(1-u) du,
\end{aligned}$$

where $E = \{u \in (0,1) : F_X^{-1}(u) + F_Y^{-1}(1-u) - K > 0\}$. Computing these two integrals we arrive at the formula[6]

$$(33)\mathbb{E}[(X+Y-K)^+] =$$
$$C_X(0) + C_Y(0) - K + \sum_{x \in \partial(A^\circ)} (-1)^{\sigma(x)} C_X(x) + \sum_{y \in \partial(A^\circ)} (-1)^{\sigma(y)+1} C_Y(K-y) + |A|.$$

at least under the assumption that A is a finite union of intervals. Here σ is defined as

$$\sigma(x) = \begin{cases} 1 & \text{if } x \text{ is a left endpoint in } \partial(A^\circ); \\ -1 & \text{if } x \text{ is a right endpoint in } \partial(A^\circ), \end{cases}$$

If we now, for an intuitive derivation, *assume* strong duality we have

$$\begin{aligned} &\int_{\mathbb{R}_+} C_X(k_1)\nu_1(dk_1) + \int_{\mathbb{R}_+} C_Y(k_2)\nu_2(dk_2) + \lambda \\ &= C_X(0) + \sum_{x \in \partial(A^\circ)} (-1)^{\sigma(x)} C_X(x) + C_Y(0) + \sum_{y \in \partial(A^\circ)} (-1)^{\sigma(y)+1} C_Y(K-y) \\ &\quad + |\{0 < x < K : \phi(x) > 0\}| - K. \end{aligned}$$

Hence the optimal dual variables are (by comparing the coefficients)

$$\begin{aligned} \nu_1(dk_1) &= \delta_0(k_1)dk_1 - \sum_{x \in \partial(A^\circ)} (-1)^{\sigma(x)} \delta_x(k_1)dk_1 \\ \nu_2(dk_2) &= \delta_0(k_2)dk_2 + \sum_{y \in \partial(A^\circ)} (-1)^{\sigma(y)} \delta_{K-y}(k_2)dk_2 \\ \lambda &= |\{0 < x < K : \phi(x) > 0\}| - K, \end{aligned}$$

and these are the dual variables (20), (21) and (22) that we described in §2.3.

We can reverse the last stage of the above reasoning to give a second proof of strong duality. I.e., if we define dual variables via (20), (21) and (22), where the K_1^i's are associated with the endpoints of the level set of $\{x : \phi(x) > 0\}$, then the above derivation shows that the complementary slackness condition (17) is satisfied and hence strong duality holds.

3.2. **Finiteness assumption on the set A.** In this section we show that although it is perfectly possible for A to consist of infinitely many intervals, this is a degenerate case, in a sense of "measure zero".

[6]Details of derivation are provided in Section 7.

Lemma 3. *Let E be a measurable subset of an bounded open interval I. Then its characteristic function χ_E is of bounded variation if and only if E consists of finitely many intervals.*

Proof. The intuition is that since the characteristic function χ_E is either 0 or 1, the total variation is the number of times it changes from 0 to 1 and from 1 to 0. Thus we are left with a simple counting argument. We refer to Volpert and Hudjaev [28] for background material on functions of bounded variation. □

The following lemma is a one dimensional version of a theorem in Evans and Gariepy [14]. We refer to their book (see Theorem 1, page 185) for details.

Lemma 4. *Let f be a function of bounded variation defined on an open interval I. Denote by E_t the level set $\{x \in I : f(x) > t\}$ for f. Then, for almost every $t \in \mathbb{R}$, the characteristic function χ_{E_t} of E_t is of bounded variation.*

Hence, by Lemma 4 (since ϕ is of bounded variation), for almost every t the characteristic function of set $\Phi_t := \{\phi > t\}$ is of bounded variation and therefore a finite union of intervals by Lemma 3. Thus the case that $A = \Phi_0$ fails to be finite union of intervals is degenerate.

4. Extending beyond the finite union case.

In this section we show that even in the degenerate case, i.e., χ_A is not of bounded variation, the joint distribution of X and Y constructed in Section 2.6, i.e, $X \sim F_X^{-1}(U)$ and $Y \sim F_Y^{-1}(1-U)$ where U is a random variable uniformly distributed in $[0,1]$, is still primal optimal. However, on the other hand, in this case the dual variables given by (20), and (21) involve infinite sums and therefore are no longer finite signed measures. Clearly in practice it is not realistic to consider portfolios which involve going long and/or short an infinite number of calls. Instead we show that, for any $\epsilon > 0$, there exists an explicit ϵ-optimal subreplicating portfolio in the sense that will be clarified in the rest of the section.

Let Φ_t denote the super level set $\{x \in (0,K) : \phi(x) > t\}$. By Lemma 4, the characteristic function χ_{Φ_t} of Φ_t is of bounded variation for almost every t. Suppose now that we are in the case that χ_A (recall that $\Phi_0 = A$) is not of bounded variation. Then, given any $\epsilon > 0$, there exists a positive $t_\epsilon < \epsilon/K$ with $\chi_{\Phi_{t_\epsilon}}$ of bounded variation (hence Φ_{t_ϵ} is a finite union of intervals by Lemma 3). We shall denote Φ_{t_ϵ} by Φ_ϵ

hereafter for notational convenience. We can then form a portfolio $(\nu_1^\epsilon, \nu_2^\epsilon, \lambda^\epsilon)$ as the one in (20), (21) and (22) by replacing the set A by Φ_ϵ. We shall refer to such portfolio determined by Φ_ϵ as an ϵ-optimal subreplicating portfolio for the reason which will be clear in the following calculation. Recall that in this case $\lambda^\epsilon = |\Phi_\epsilon| - K$. The price of such portfolio satisfies

$$\begin{aligned}
& \int C_X(k_1)d\nu_1^\epsilon(k_1) + \int C_Y(k_2)d\nu_2^\epsilon(k_2) + \lambda^\epsilon \\
= \; & C_X(0) + C_Y(0) + \int_0^K \chi_{\Phi_\epsilon}(x)dC_X(x) - \int_0^K \chi_{\Phi_\epsilon}(y)dC_Y(K-y) + |\Phi_\epsilon| - K \\
= \; & C_X(0) + C_Y(0) + \int_{\Phi_\epsilon} F_X(x)dx - \int_{\Phi_\epsilon} F_Y(K-y)d(K-y) + \int_{\Phi_\epsilon} d(K-y) - K \\
= \; & \int_{\Phi_\epsilon^c} (x-K)dF_X(x) + \int_{\{K-y\in\Phi_\epsilon\}^c} ydF_Y(y) - t_\epsilon|\Phi_\epsilon| \\
= \; & \int_{E_\epsilon} (F_X^{-1}(u) + F_Y^{-1}(1-u+t_\epsilon) - K)du - t_\epsilon|\Phi_\epsilon| \\
\geq \; & \int_E (F_X^{-1}(u) + F_Y^{-1}(1-u) - K)du - \epsilon \\
= \; & \mathbb{E}[(X+Y-K)^+] - \epsilon
\end{aligned}$$

where $E_\epsilon := \{u \in (0,1) : F_X^{-1}(u) + F_Y^{-1}(1-u+t_\epsilon) - K > 0\}$. Here we note that in the inequality we have used that $E := \{u \in (0,1) : F_X^{-1}(u) + F_Y^{-1}(1-u) - K > 0\}$ is contained in E_ϵ (since F_Y^{-1} is nondecreasing and $t_\epsilon > 0$), that the integrand is positive in E_ϵ and that $F_Y^{-1}(1-u+t_\epsilon) \geq F_Y^{-1}(1-u)$ (again since F_Y^{-1} is nondecreasing and $t_\epsilon > 0$). Therefore, the price of the portfolio $(\nu_1^\epsilon, \nu_2^\epsilon, \lambda^\epsilon)$ constructed from the super level set Φ_ϵ is higher than, within an ϵ error, the primal value $\mathbb{E}[(X+Y-K)^+)]$. This is explains the term "ϵ-optimality".

Combing the above ϵ-optimal inequality with the weak duality between primal and dual, we obtain

$$\mathbb{E}[(X+Y-K)^+] - \epsilon \leq \langle C_X, d\nu_1^\epsilon\rangle + \langle C_Y, d\nu_2^\epsilon\rangle + \lambda^\epsilon \leq \mathbb{E}[(X+Y-K)^+]$$

for any $\epsilon > 0$. In this sense, we say that the primal value $\mathbb{E}[(X+Y-K)^+]$ with joint distribution constructed in Section 2.6 is optimal.

5. Numerical Results

In this section we illustrate the hedging strategy in the context of the Black-Scholes model. We determine the optimal hedging strategy, by determining numerically the zeroes of ϕ (see (18) in Section 2.3).

Let us therefore assume that

$$S_{1t} = S_0^1 \exp(\sigma_1 W_{1t} - \frac{1}{2}\sigma_1^2 t), \tag{34}$$

$$S_{2t} = S_0^2 \exp(\sigma_2 W_{2t} - \frac{1}{2}\sigma_2^2 t), \tag{35}$$

where $\sigma_i, i = 1, 2$, are constants and where W_{1t}, W_{2t} are standard Brownian motions. In order to simulate the anti-monotonic case corresponding to the optimal lower bound, we assume the following relation between the driving Brownian motions

$$W_{2t} = -W_{1t}.$$

As is well known, the distribution functions $F^i_{S_T}(x)$ for stock i, $i = 1, 2$, at maturity T are given by $1 - N(d_2^{(i)})$, where N is the distribution function of the standard normal random variable and $d_2^{(i)}$, which appears in the Black-Scholes formula, is given (when $r = 0$) by the expression

$$d_2^{(i)}(x) = \frac{\log(\frac{S_0^i}{x}) - \frac{\sigma_i^2}{2}T}{\sigma_i\sqrt{T}}.$$

Thus the function ϕ whose zeroes we seek to determine is given in our setting by

$$\phi(x) = -N(d_2^{(1)}(x)) - N(d_2^{(2)}(K - x)) + 1.$$

In the paper, we have assumed for simplicity that the weights $w_i, i = 1, 2$, in the portfolio are equal to 1. Adapting the results in this paper to positive weights that are different from 1 involves only trivial adjustments. In Table 1 below we show results when both weights are equal to .5. We have assumed geometric Brownian motions for both stocks. We took $T = .5$, $S_0^1 = S_0^2 = 100$ and $\sigma_1 = .355, \sigma_2 = .2$ and $w_1 = w_2 = .5$. In the first column the basket strike K_B is shown. In the second column is the Monte Carlo price of the basket in the case where stock S_1 and S_2 are anti-monotonic with the prescribed marginals. In the third column is the price of the optimal hedging portfolio. In the next two columns is part of the optimal hedging portfolio that concerns stock S_1 and involves a short w_1 and a long w_1 position in a

K_B	MC Value	Hedging Portfolio's Value	Long w_1 Call with Strike K_1^1	Short w_1 Call with Strike K_1^2
81.5	18.52	18.50	absent	absent
84	16.03	16.00	absent	absent
86.5	13.55	13.50	absent	absent
89	11.02	11.00	absent	absent
91.5	8.5	8.50	absent	absent
94	5.97	6.00	absent	absent
96.5	3.98	3.99	51.24	89.40
99	2.73	2.69	44.47	101.61
100	2.28	2.29	42.50	105.76
102.5	1.54	1.54	38.52	115.19
105	1.02	1.03	35.41	123.73
107.5	0.69	0.69	32.83	131.73
110	0.45	0.46	30.65	139.30
112.5	0.32	0.31	28.78	146.59
115	0.21	0.21	27.12	153.64
117.5	0.14	0.14	25.64	160.48

Table 1: The hedging portfolio and optimal lower bound for primal and dual problem

call with the indicated strikes K_1^1 and K_1^2. To this we must always add a long w_1 position in the underlying stock S_1 and a complementary position in three w_2 puts on S_2: a short w_2 position on a put with strike $2K_B - K_1^1$, a long w_2 position in a put with strike $2K_B - K_1^2$ and a short w_2 position in a put with strike $2K_B$. We are assuming, as we do throughout this paper, that a continuum of strikes are traded on each asset. For the parameters considered, the function ϕ has either two zeroes or none. When there are none (indicated by "absent" in fourth and fifth column), the optimal hedging portfolio consists only of a long position in stock 1 and a short position in a put on S_2, struck at the basket strike.

6. **Appendix I** - Infinite Dimensional Linear Programming

In this section we quote some results for linear programming in infinite dimensional space. Please refer to Anderson and Nash [1] for details.

Let (X, X') and (Y, Y') be two dual pairs of linear spaces and denote by $\langle , \rangle$ for both of their pairings. A linear programming problem is a constrained optimization

problem of the form

$$\begin{array}{ll} \text{minimize} & \langle x, c\rangle \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \tag{36}$$

where b and c are given elements of Y and X' respectively, A is a continuous linear map from X to Y. The dual problem to (36) is given as

$$\begin{array}{ll} \text{maximize} & \langle b, y'\rangle \\ \text{subject to} & -A^*y' + c \in P^* \\ & y' \in Y' \end{array} \tag{37}$$

where P^* is the dual cone of P defined by

$$P^* = \{x' \in X' : \langle x, x'\rangle \geq 0 \text{ for all } x \in P\},$$

and A^* is the adjoint of A defined by $\langle Ax, y'\rangle = \langle x, A^*y'\rangle$ for all $x \in X$ and $y' \in Y'$.

The following two theorems are essential in our following analysis. Recall that a program is called *consistent* if it has a feasible solution and the *value* of a consistent program (36) is defined as the infimum over feasible x of $\langle x, c\rangle$.

Theorem 5. *(Weak duality) If both (36) and (37) are both consistent, then the value of (36) is greater than or equal to the value of (37) and both values are finite.*

Theorem 6. *(Complementary slackness) If x is primal feasible and y' is dual feasible and*

$$\langle x, c - A^*y'\rangle = 0, \tag{38}$$

then x is primal optimal and y' is dual optimal.

7. Appendix II

In this section we provide the details of the derivation for the equality (33) in Section 3.1. Recall that

$$\mathbb{E}[(X + Y - K)^+] = \int_E (F_X^{-1}(u) - K)du + \int_E F_Y^{-1}(1-u)du,$$

where $E = \{u \in (0,1) : F_X^{-1}(u) + F_Y^{-1}(1-u) - K > 0\}$. Now we compute these 2 integrals out as follows.

(1) By applying a change of variables formula (by making $u = F_X(x)$) to the first integral and notice that

$$\begin{aligned} & F_X^{-1}(E) = \{x : x + F_Y^{-1}(1 - F_X(x)) - K > 0\} \\ = & \{x \in [0, K] : F_X(x) + F_Y(K - x) - 1 \leq 0\} \bigcup (K, \infty) \\ = & \{x \in [0, K] : C'_X(x) + C'_Y(K - x) + 1 \leq 0\} \bigcup (K, \infty) \end{aligned}$$

we have

$$\begin{aligned} & \int_E [F_X^{-1}(u) - K]du = \int_{F_X^{-1}(E)} [x - K]dF_X(x) \\ = & \int_0^\infty [x - K]dF_X(x) - \int_A [x - K]dF_X(x) \\ = & C_X(0) - K - (x - K)F_X(x)|_{\partial A} + \int_A F_X(x)dx \\ = & C_X(0) - K - (x - K)F_X(x)|_{\partial A} + \int_A C'_X(x)dx + |A| \end{aligned}$$

where $A = \{x \in (0, K) : C'_X(x) + C'_Y(K - x) + 1 > 0\}$ and $|A|$ is the Lebesgue measure of A.

(2) Similarly for the second integral, we make the change of variable $1 - u = F_Y(y)$ and notice that

$$\begin{aligned} & \{y : 1 - F_Y(y) \in E\} = \{y : F_X^{-1}(1 - F_Y(y)) + y - K > 0\} \\ = & \{y \in [0, K] : F_X(K - y) + F_Y(y) - 1 \leq 0\} \bigcup (K, \infty) \\ = & \{y \in [0, K] : C'_X(K - y) + C'_Y(y) + 1 \leq 0\} \bigcup (K, \infty) \end{aligned}$$

then we have

$$\begin{aligned} & \int_E F_Y^{-1}(1 - u)du = -\int_{\{y:1-F_Y^{-1}(y)\in E\}} ydF_Y(y) \\ = & C_Y(0) - yF_Y(y)|_{\partial B} + \int_B C'_Y(y)dy + |B| \end{aligned}$$

where $B = \{y \in (0, K) : C'_X(K - y) + C'_Y(y) + 1 > 0\}$ and $|B|$ is the Lebesgue measure of B.

Now we have

$$\begin{aligned}\mathbb{E}[(X+Y-K)^+] &= C_X(0)+C_Y(0)-K-(x-K)F_X(x)|_{\partial A}-yF_Y(y)|_{\partial B}\\ &\quad+\int_A C'_X(x)dx+\int_B C'_Y(y)dy+|A|+|B|.\end{aligned}$$

By applying the symmetry $x\in A\Longleftrightarrow K-x\in B$ we obtain

$$-(K-x)F_X(x)|_{\partial A}+yF_Y(y)|_{\partial B}=yF_X(K-y)|_{\partial B}+yF_Y(y)|_{\partial B}=y|_{\partial B}=|B|,$$

here in the second equality we have used the relation $F_X(K-y)+F_Y(y)=1$ on ∂B. Hence, under the assumption that A is a finite union of intervals, we obtain

$$\begin{aligned}&\mathbb{E}[(X+Y-K)^+]=C_X(0)+C_Y(0)-K+\int_A C'_X(x)dx+\int_B C'_Y(y)dy+|A|\\ =\;& C_X(0)+C_Y(0)-K+\int_0^K\chi_A(x)C'_X(x)dx+\int_0^K\chi_B(y)C'_Y(y)dy+|A|\\ =\;& C_X(0)+C_Y(0)-K+\sum_{x\in\partial(A^\circ)}(-1)^{\sigma(x)}C_X(x)+\sum_{y\in\partial(A^\circ)}(-1)^{\sigma(y)+1}C_Y(K-y)+|A|,\end{aligned}$$

where in the last equality we have again used the symmetry $x\in A\Longleftrightarrow K-x\in B$. A° denotes the interior of A and σ is defined as

$$\sigma(x)=\begin{cases}1 & \text{if } x \text{ is a left endpoint in } \partial(A^\circ);\\ -1 & \text{if } x \text{ is a right endpoint in } \partial(A^\circ).\end{cases}$$

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