Static-Arbitrage Upper Bounds for the Prices of Basket Options

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Abstract

In this paper we investigate the possible values of basket options.

Instead of postulating a model and pricing the basket option using that model, we consider the set of all models which are consistent with the observed prices of vanilla options, and, within this class, find the model for which the price of the basket option is largest.

This price is an upper bound on the prices of the basket option which are consistent with no-arbitrage. In the absence of additional assumptions it is the lowest upper bound on the price of the basket option. Associated with the bound is a simple super-replicating strategy involving trading in the individual calls.

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1. Introduction

In this article we consider the problem of pricing a basket option. The standard approach to derivative pricing is to assume a model for the traded assets, a popular one being that the assets are driven by correlated exponential Brownian motions. Then, in a complete market, by standard no-arbitrage theory, the price of the option is the discounted expected payoff of the option under the risk-neutral measure. This price is the unique fair price for the option, but only if the market is frictionless and the model is a true representation of the real world.

For options written on one asset an important direction of research, which has gained in momentum over the last decade, is to consider pricing and hedging with respect to alternative stochastic processes. Among the most popular are level-dependent models, in which the volatility is allowed to depend on spot and time, stochastic volatility models, uncertain volatility models, and jump-diffusion or pure jump processes. Such models can be used to account for the smile effect in the observed market values of vanilla calls and puts. The plethora of alternative models available for pricing and hedging leave practitioners with a wide spectrum of models at their disposal, but little information about which, if any, is the correct model to use. On the other hand, the practitioner may be reassured by the fact that, as Figlewski (2002) has argued, any reasonable model (ie one satisfying certain no-arbitrage conditions and having a calibration parameter) will give accurate prices when appropriately calibrated.

In an environment in which there is model uncertainty, a complementary approach, useful both for risk management purposes and to provide a sanity check for the prices and hedges obtained from parametric models, is to derive distribution free no-arbitrage prices and hedges. This second approach is less ambitious in scope in the sense that it does not aim to derive a unique fair price, but more robust in the sense that it is not dependent on the efficacy of an underlying model. The aim is to provide bounds on the possible price of an exotic option which are consistent with no-arbitrage *given* the market prices of vanilla puts and calls. In essence, rather than using a single model we consider the class of all models which are consistent with the observed call prices, and rather than quoting a single option price we give the range of prices which arise under models from this class. In this paper this philosophy will be applied to basket options in the setting of a one-period static arbitrage model. We will focus on the case of upper bounds and leave the case of lower bounds to future research.

In the first part of our paper, we assume a continuum of strikes and suppose that instead of hedging with just the underlying stocks and cash, investors are allowed to hedge a basket call option with calls on the constituent assets with the same maturity and *all available strikes*. On the one hand this gives the investor a greater range of hedging instruments, but on the other, since we do not postulate any model for the underlying, dynamic hedges based on stochastic calculus and delta-hedging can no longer be expected to work. The upper bound on the price of a basket option is equivalent to the cheapest portfolio of calls and underlying assets which will super-replicate the asset. Conversely, there is a model-independent lower bound on the price which is the price of the most expensive portfolio of calls whose payoff is dominated by the payoff of the basket.

Breeden and Litzenberger (1978) introduced this second approach in the context of European options on a single asset. Consider an option whose payoff depends on the asset value at maturity. Then this payoff function can be written as a weighted sum of call option payoffs with different strikes. As a direct consequence, the price of the option is given by the weighted sum of the prices of the calls options in the portfolio. The Breeden and Litzenberger result both gives a unique fair price for the new option, and suggests a static hedging strategy using calls.

Now consider the case with many underlyings. In the case of a continuum of strikes, Breeden and Litzenberger's result implies that the full marginals are known (but we have no information on the joint distribution of assets). Then, using the theory of copulas, it is possible to show that there is an upper bound on the price of the option which is equal to the Fréchet bound. This Fréchet bound corresponds to the case where the co-dependence structure is chosen so that the underlying assets are co-monotonic. For a basket with two assets this is the approach taken by Rapuch and Roncalli (2001) and Cherubini and Luciano (2001) and has the advantage that it applies simultaneously to a wide class of exotic options, including the basket option. Building on ideas for Asian options of Dhaene et al (2000), see also the overview by Dhaene et al (2002), Deelstra et al (2004) use a related idea, namely that of convex order, to derive a similar result (that the largest possible price occurs when the component assets are co-monotonic) for baskets with an arbitrary number of components. We rederive the result using a direct proof via Lagrangian methods. The advantage of the new proof is that it requires no sophisticated mathematics (such as copulas or Fréchet bounds), it highlights the role of static super-replicating strategies and, most important, it provides an explicit characterization of the cheapest super-replicating strategy.

The cheapest super-replication strategy turns out to be remarkably simple. Indeed, although in principle we consider hedging strategies which consist of arbitrary portfolios of calls on each component asset in the basket, the cheapest super-replicating portfolio requires investment in a simplified call portfolio which includes exactly one strike for each underlying. This may be viewed as surprising as the potential extra diversification available in using a multitude of strikes turns out not to be useful. (The full marginal information is used in determining the optimal strike, but once this strike has been determined, the other calls become irrelevant.)

The second case of interest is when there are only a finite number of calls traded on each underlying. This is a more realistic market situation. If we only know call prices for a few strikes then we only know partial information about the marginals. However, we show how to deduce the upper bound in this case from the result on the full marginals. In this case there exists an optimizing portfolio which consists of options on no more than two strikes per asset, and involves at most $N + 1$ separate options. The simplicity of this super-replicating strategy is one of its attractions.

The result in the case of finitely many strikes on each underlying is new, even in the case of two assets. It also has the advantage that the proof involves finding the cheapest super-replicating strategy for the basket option, and hence has an immediate interpretation as a hedging strategy. This strategy only involves the traded stocks and call options.

Since knowledge of a finite number of strikes only gives a partial information about the marginals, it may seem surprising that there is such a simple characterisation of the least upper bound. The characterization is based on an interpolation technique which, in essence, 'fills in' the missing values of the call price functions, and hence, via Breeden and Litzenberger, completes the partial information about the marginal to 'full' information. The reason that this works is a simple but key observation: the largest convex function passing through m given points is the *linearly interpolated* function.

This paper was motivated by an attempt to extend the results of Laurence and Wang (2004, 2005) who consider the case in which trading is allowed on one single name call option per stock, stocks and cash. In particular it is assumed that there is only one traded strike for each underlying asset. An equivalent upper bound in this case, albeit in a more complicated form, was obtained by Aspremont and El Ghaoui (2003). In all of these papers the authors convert the problem into a dual problem. The fact that there is only one call per underlying means that there are only $2N + 1$ linear constraints, so that the number of dual variables stays small. However as we shall see, the best way to solve the problem with many strikes is to specialize the full marginal result, rather than to 'build upward' from a small number of strikes.

As a practical illustration of our methods, we consider two examples. The first example is artificial, but is designed to illustrate the pitfalls which face an investor who makes an incorrect choice of model. In this example, model mis-specification leads to pricing errors. The second example is based on real data on a DJX contract. In this case, any model-based price depends on the parameter choices and moreover, the true model can never be determined. In contrast, our upper bound depends only on market observables, in the form of traded call prices. Moreover, this bound is associated with a simple and explicit hedge.

We briefly review the literature of basket options. Given the multidimensional nature of these options, their analytical and numerical valuation is very challenging even in the Black-Scholes setting. Indeed, although an analytical formula for the call option does exist, see for example the text by Kwok (1998), the integral is difficult to evaluate in practice and one must resort to Monte-Carlo methods, to recursive methods, see Ware and Avelassani (2001), to methods using characteristic functions, see Ju (2002) or moment matching, see Dufresne (2002) and Brigo et al (2004). Some progress has been made in handling diffusion-based models with non-constant volatility close to expiration in Avellaneda et al (2003), but the resulting prices can be very sensitive to the choices of correlation and volatility.

There is one paper which could be added to the above list concerning basket option pricing in a Black-Scholes environment, and which, as we mentioned earlier, employs related ideas to ours. In their Theorem 1, Deelstra et al (2004) give a similar result to our Theorem 3.1. They then specialize to the Black-Scholes model and show how this bound may be refined when the underlying component assets follow exponential Brownian motions. Our focus is different in that we are interested in model-independent bounds.

Last we note that the problem of finding model independent bounds can also be posed for other exotic options. For example, Hobson (1998) considers model independent bounds on the price of a lookback option, and Brown et al (2001) and Hodges and Neuberger (1998) consider barrier options.

2. Preliminaries

Let C denote the time-0 call price function for options on an underlying with non-negative price process X. These options are supposed to have maturity M. Thus $C(k)$ is the market price of an option with payoff $(X_M - k)^+$ at time M. No arbitrage considerations imply that $C(0) = X_0$ (otherwise a simple buy and hold strategy creates arbitrage), C is decreasing and convex.

If the market prices are given by discounted expectations under a risk-neutral measure (for example if the market is complete, although we do not assume this) then we may write

$$C(k) = \beta \mathbb{E}_\mu \left[(X_M - k)^+ \right]$$

where β is a discount factor and μ is the risk-neutral measure. In this case, as observed by Breeden and Litzenberger (1978), knowing call prices for all strikes is equivalent to knowing the full distribution of the stock price at time M under the pricing measure. If C' is continuous then

$$\beta \mathbb{P} \left[X_M \geq k \right] = -C'(k).$$

More generally the convex function C has left and right derivatives and

$$-\beta \mathbb{P} \left(X_M \geq k \right) = C'(k-) \leq C'(k+) = -\beta \mathbb{P} \left(X_M > k \right).$$

We let k_∞ be the supremum of the support of the distribution of X_M, so that $k_\infty = \sup\{k : C(k) > 0\}$ and denote the distribution function of X_M by G, so that

$$G(k) = \mathbb{P}(X_M \leq k) = 1 - \mathbb{P}(X_M > k) = 1 + (1/\beta) C'(k+).$$

Define the (left-continuous) inverse function G^{-1} by $G^{-1}(u) = \inf\{x \geq 0 : G(x) \geq u\}$.

Now suppose that instead of knowing the call price function for all strikes we only know prices for a finite number of traded strikes. Let these traded strikes be given by $(k_j)_{1\leq j\leq J}$ where $0 = k_0 < k_1 < \cdots < k_J$. We may think of the underlying security as an option with zero strike so that it makes sense to include k_0 with associated call price $C(k_0) = X_0$ in the set of traded strikes. Again, static no-arbitrage implies that $C = \{C(k_j)\}_{0\leq j\leq J}$ must be a decreasing convex function.

There are many call price functions (defined for all strikes) which are consistent with the observed call prices at the traded strikes. Hence knowledge of the traded call prices does not mean that we know the full distribution of X_M. Instead each call price provides a linear constraint on the set of measures consistent with call prices.

For the basket option we suppose there are N underlying assets in the basket labelled $(X^{(i)})_{1\leq i\leq N}$. Maturity-$M$ calls on the asset with index i have market price $C^{(i)}$ and we are interested in pricing a maturity-M basket option with positive weights w_i and strike K. The payoff of this option is

$$P_K(X_M) := \left(\sum_{i=1}^{N} w_i X_M^{(i)} - K\right)^+$$

where $X_M = (X_M^{(1)}, \cdots, X_M^{(N)})$. We denote by $G^{(i)}$ the cumulative distribution function for $X_M^{(i)}$ which is implied by the market call prices $C^{(i)}$, and define $(G^{(i)})^{-1}$ and $k_\infty^{(i)}$ relative to this distribution.

In this article we will be interested in the upper bound on a basket call option. However, as pointed out by Laurence and Wang (2005) and Deelstra et al (2004), given the equality

$$\left(K - \sum_i w_i X_M^{(i)}\right)^+ = \left(\sum_i w_i X_M^{(i)} - K\right)^+ - \left(K - \sum_i w_i X_M^{(i)}\right),$$

there is a put-call parity result for basket options, and a static no-arbitrage upper bound on the price of a basket call can be translated directly into an upper bound on the corresponding basket put option.

3. The full marginal case: a continuum of strikes.

3.1. **The upper bound.** The aim is to find an upper bound on the price $B(K)$ of a basket option with strike K and maturity M, given the prices of maturity-M call options on the individual stocks with all possible strikes. The bound will be model independent, and will be based on the prices of the calls and a simple super-replicating strategy consisting of the purchase of a calls. The relation of this optimization problem to a more general one in which both cash and more complicated portfolios of options are used in the super-replicating strategy, will be discussed in Section 5.

For any vector $\lambda = (\lambda_1, \dots \lambda_N)$ with $\lambda_i \geq 0$ and $\sum_{i=1}^N \lambda_i = 1$ we have

$$\left(\sum_i w_i X_M^{(i)} - K\right)^+ \leq \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i K}{w_i}\right)^+$$

Hence the payoff of the basket option is bounded above by a weighted sum of calls on the individual assets. Further if $B(K)$ is the price of the basket option under any model which is consistent with the call price data, then a simple application of no-arbitrage gives

$$B(K) \leq \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$$

The λ_i are arbitrary and so

$$B(K) \leq \inf_{\lambda_i \geq 0, \sum \lambda_i = 1} \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right).$$

Note that in deriving this upper bound we do not need to assume that call prices are given by discounted expectations under some model, but simply that they are traded prices. Since we are minimising a bounded continuous function over a compact domain the infimum is attained. Let λ_i^* be a minimising choice, then we can write the upper bound as

$$\overline{B}(K) = \sum_i w_i C^{(i)} \left(\lambda_i^* K / w_i\right). \tag{1}$$

Implicit in this inequality is the simple super-replicating strategy which consists of buying w_i calls with strike $\lambda_i^* K / w_i$ on underlying $X^{(i)}$.

3.2. **Characterizing the optimal strikes.** We wish to find the infimum of $\sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right)$ over choices λ_i satisfying $\lambda_i \geq 0, \sum \lambda_i = 1$. Define the Lagrangian

$$L(\lambda, \phi) = \sum_i w_i C^{(i)} \left(\lambda_i K / w_i\right) + \phi \left(\sum_i \lambda_i - 1\right).$$

Recall the definition of $k_\infty^{(i)} = \sup\{k : C^{(i)}(k) > 0\}$ as the smallest strike for which the call price on underlying $X^{(i)}$ is zero.

To establish the value of the optimal solution to this optimization problem we must confront the problem that the convex functions $C^{(i)}(\cdot)$ may have a discontinuous first derivative. If the convex function C has discontinuous first derivative at x, then the sub-differential $\bar{\partial} C(\cdot)$ is an interval consisting of the slopes of all tangents to C at x. We shall show that there exist ϕ^* and $\lambda_i^* = \lambda_i(\phi^*)$ with $\phi^* \in \bar{\partial} C^{(i)}(\lambda_i^* K / w_i)$, $\lambda_i^* \geq 0$ and $\sum \lambda_i^* = 1$, so that

$$\inf_{\lambda : \lambda \geq 0} L(\lambda, \phi^*) = L(\lambda^*, \phi^*) = \sum_{i=1}^N w_i C^{(i)} \left(\lambda_i^* K / w_i\right).$$

Case 1: $k_\infty^{(i)} = \infty$ for all i.
Consider first the 'nice' case in which for each i, $C^{(i)}(k)$ is positive and

$\partial C^{(i)}/\partial k$ is a continuous, strictly increasing (and necessarily negative) function. In this case an unconstrained optimization of $L(\lambda, \phi)$ over λ_i yields optimal values $\lambda_i(\phi)$ such that

$$K \left.\frac{\partial C^{(i)}}{\partial k}\right|_{(\lambda_i(\phi)K/w_i)} = -\phi. \tag{2}$$

In particular the right-hand-side of (2) is independent of i. Further the call price is related to the marginal distribution via $\partial C^{(i)}/\partial k = -\beta \mathbb{P}(X_M^{(i)} > k)$, and we find that the optimal choices $\lambda_i(\phi)$ satisfy

$$\mathbb{P}\left(X_M^{(i)} > \frac{\lambda_i K}{w_i}\right) = -\frac{1}{\beta} \left.\frac{\partial C^{(i)}}{\partial k}\right|_{(\lambda_i(\phi)K/w_i)} = \frac{\phi}{\beta K},$$

or equivalently,

$$\lambda_i(\phi) = \frac{w_i}{K}(G^{(i)})^{-1}\left(1 - \frac{\phi}{\beta K}\right). \tag{3}$$

We can now choose ϕ^* as the solution to $H(\phi) = 0$ where $H(\phi) = \sum_i \lambda_i(\phi) - 1$. Then $\lambda_i^* = \lambda_i(\phi^*)$ is the optimal solution to the constrained problem. Our assumption that $\partial C^{(i)}/\partial k$ is negative for all strikes ensures that $\phi^* > 0$.

To see that a solution to $H(\phi) = 0$ exists note that under our hypotheses on $C^{(i)}$, $H(\phi)$ is continuous and strictly decreasing. Hence, to show that 0 is in the range of $H(\phi)$ it suffices, by the intermediate value theorem to show that $\inf H(\phi) < 0 < \sup H(\phi)$. For $\phi > \beta K$ we clearly have $H(\phi) = -1$. Also

$$\lim_{\phi \to 0} \Lambda(\phi) = \sum w_i k_\infty^{(i)}/K,$$

and the right hand side is equal to $+\infty$ under our assumptions. Note, since we will need it later, that provided that $k_\infty^{(i)}, i = 1, \dots N$ are such that $\sum w_i k_\infty^{(i)}/K > 1$ the same conclusion $\sup H(\phi) > 0$ still holds.

It follows from (3) that

$$\mathbb{P}\left(X_M^{(i)} \le \frac{\lambda_i^* K}{w_i}\right) = G^{(i)}\left(\frac{\lambda_i^* K}{w_i}\right) = 1 - \frac{\phi^*}{\beta K}. \tag{4}$$

Now consider the 'not nice' case in which $\partial C^{(i)}/\partial k$ may fail to be continuous ($G^{(i)}$ has jumps) or fail to be strictly increasing ($(G^{(i)})^{-1}$ has jumps).

For fixed ϕ, the minimization over λ_i still yields an optimum value satisfying

$$K \left|\frac{\partial C^{(i)}}{\partial k}\right|_{(\lambda_i(\phi)K/w_i)-} \ge \phi \ge K \left|\frac{\partial C^{(i)}}{\partial k}\right|_{(\lambda_i(\phi)K/w_i)+}. \tag{5}$$

It follows that $\lambda_i(\phi)$ must satisfy

$$\mathbb{P}\left(X_M^{(i)} < \frac{\lambda_i^* K}{w_i}\right) \le 1 - \frac{\phi}{\beta K} \le \mathbb{P}\left(X_M^{(i)} \le \frac{\lambda_i^* K}{w_i}\right). \tag{6}$$

Let $\lambda_i^-(\phi)$ and $\lambda_i^+(\phi)$ be given by

$$(\lambda_i^-(\phi), \lambda_i^+(\phi)) = \left(\frac{w_i}{K}(G^{(i)})^{-1}\left(1-\frac{\phi}{\beta K}\right), \frac{w_i}{K}(G^{(i)})^{-1}\left(\left(1-\frac{\phi}{\beta K}\right)+\right)\right).$$

Then, still for fixed ϕ, any value of λ_i in the range $[\lambda_i^-(\phi), \lambda_i^+(\phi)]$ leads to an optimal value of L.

Our goal is now to find a value ϕ^* which ensures that the constraint $\sum_i \lambda_i(\phi) = 1$ is satisfied, and hence leads to a solution of the constrained optimization problem.

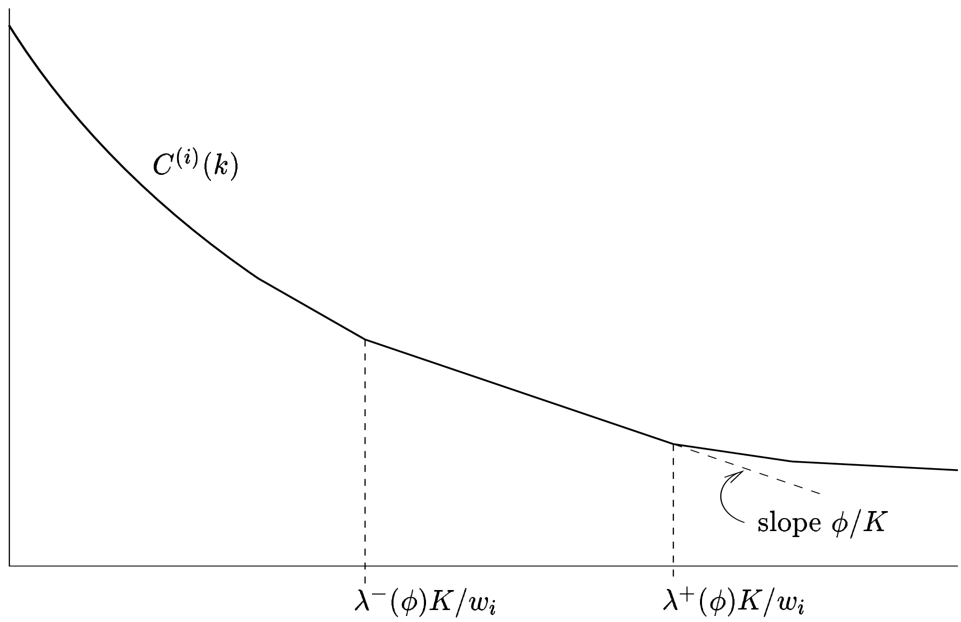

FIGURE 1. When the tangent to $C^{(i)}$ with gradient $-\phi/K$ intersects with $C^{(i)}$ over an interval, there is a range of solutions to the defining equation (5) for $\lambda_i(\phi)$.

Observe that $\lambda_i^\pm$ is decreasing in ϕ. Let $H^-(\phi) = \sum_i \lambda_i^-(\phi) - 1$, let $H^+(\phi) = \sum_i \lambda_i^+(\phi) - 1$, and let $\phi^* = \inf\{\phi : H^-(\phi) \le 0\}$. Then, $H^-(\phi^*) \le 0 \le H^+(\phi^*)$ and there exists a vector $(\lambda_i^*)_{1\le i\le N}$ with $\lambda_i^-(\phi^*) \le \lambda_i^* \le \lambda_i^+(\phi^*)$ for all i, such that $\sum_i \lambda_i^* = 1$. For these values λ_i^* we have the following extension of (4),

$$G^{(i)}\left(\frac{\lambda_i^* K}{w_i}-\right) \le 1 - \frac{\phi^*}{\beta K} \le G^{(i)}\left(\frac{\lambda_i^* K}{w_i}\right). \tag{7}$$

The model-independent bound is given by (1).

Case 2: $k_\infty^{(i)} < \infty$ for some i, but $\sum_i w_i k_\infty^{(i)} > K$.

If we try to repeat the analysis of Case 1, then the only potential problem is the fact that we may not have that ϕ^* is positive. But, if $\phi^* = 0$ then $(\lambda_i^* K/w_i) \geq k_\infty^{(i)}$ for all i (from (7)) and

$$K = \sum_i K\lambda_i^* \geq \sum w_i k_\infty^{(i)} > K$$

which is a contradiction. Hence $\phi^* > 0$, (7) defines optimizing values with $(\lambda_i^* K/w_i) \leq k_\infty^{(i)}$ for all indices i and (6) holds. Again, the model-independent bound $\overline{\mathcal{B}}(K)$ is given by (1).

Case 3: $k_\infty^{(i)} < \infty$ for all i and $\sum_i w_i k_\infty^{(i)} \leq K$.

In this case it is essentially impossible for the basket option to mature in-the-money. We deduce a bound directly in this case. We have

$$\begin{aligned}\left(\sum_i w_i X_M^{(i)} - K\right) &= \left(\sum_i w_i X_M^{(i)} - \sum_i w_i k_\infty^{(i)} + \sum_i w_i k_\infty^{(i)} - K\right) \\ &\leq \sum_i w_i \left(X_M^{(i)} - k_\infty^{(i)}\right)\end{aligned}$$

and

$$\left(\sum_i w_i X_M^{(i)} - K\right)^+ \leq \sum_i w_i \left(X_M^{(i)} - k_\infty^{(i)}\right)^+.$$

It follows that

$$B(K) \leq \overline{\mathcal{B}}(K) = \sum_i w_i C^{(i)}\left(k_\infty^{(i)}\right) = 0$$

3.3. **Optimality of the bound.** It remains to show that the upper bound in (1) is a least upper bound. (Note that in Case 3 above, zero is trivially the least upper bound.) This will follow if we can find a model which is consistent with the given call prices (ie marginals) and for which the model price is $B(K) = \overline{\mathcal{B}}(K)$. Under this model the prices of calls are given by the discounted expected payoffs. Equivalently we find a model for which the strategy of buying w_i calls with strike $\lambda_i^* K/w_i$ on underlying $X^{(i)}$ makes zero profit.

One way to construct the random variables $X_M^{(i)}$ is to take a random variable $U \sim U[0,1]$ and define $X_M^{(i)} = (G^{(i)})^{-1}(U)$. Suppose we construct the family of random variables $(X_M^{(i)})_{1 \leq i \leq N}$ in this way (using the same random variable U for each). Then, at least in the 'nice' case we have

$$X_M^{(i)} = (G^{(i)})^{-1}(G^{(i')}(X_M^{(i')}))$$

so that $X_M^{(i)}$ is an increasing function of $X_M^{(i)}$ for each pair i, i'. More generally it remains true that the $X_M^{(i)}$ are co-monotonic in the sense that if $(k_1, \ldots k_N)$ are chosen so that

$$\mathbb{P}(X_M^{(i)} < k_i) \leq \alpha \leq \mathbb{P}(X_M^{(i)} \leq k_i)$$

then we must have

$$(X_M^{(i)} < k_i) \subseteq (X_M^{(i')} \le k_{i'}); \qquad \forall i, i'.$$

In particular if we set $k_i = \lambda_i^* K / w_i$ then we deduce from (6) that

$$\left(X_M^{(i)} < \frac{\lambda_i^* K}{w_i} \right) \subseteq \left(X_M^{(i')} \le \frac{\lambda_{i'}^* K}{w_{i'}} \right); \qquad \forall i, i'. \tag{8}$$

Consider the inequality

$$\left(\sum_i w_i X_M^{(i)} - K \right)^+ \le \sum_i w_i \left(X_M^{(i)} - \frac{\lambda_i^* K}{w_i} \right)^+. \tag{9}$$

If $X_M^{(i)} \ge \lambda_i^* K / w_i$ for all i, then $\sum_i w_i X_M^{(i)} - K = \sum_i w_i (X_M^{(i)} - \lambda_i^* K / w_i) \ge 0$. Similarly if $X_M^{(i)} \le \lambda_i^* K / w_i$ for all i then $\sum_i w_i X_M^{(i)} \le K$. Hence we can only have a strict inequality in (9) if for some pair i, i' we have $X_M^{(i)} > \lambda_i^* K / w_i$ and $X_M^{(i')} < \lambda_{i'}^* K / w_{i'}$. But for our co-monotonic construction these events are mutually exclusive by (8). Hence for this model

$$\begin{aligned} B(K) = \mathbb{E}\left[\left(\sum_i w_i X_M^{(i)} - K \right)^+ \right] &= \mathbb{E}\left[\sum_i w_i (X_M^{(i)} - \lambda_i^* K / w_i)^+ \right] \\ &= \sum_i w_i C^{(i)}(\lambda_i^* K / w_i) = \overline{\mathcal{B}}(K) \end{aligned}$$

and the model price equals the upper bound. We have proved the following theorem:

Theorem 3.1. *Suppose maturity-M calls with all strikes are traded on assets $X^{(i)}$ and we wish to price a maturity-M basket option with payoff $(\sum_i w_i X_M^{(i)} - K)^+$.*

Then, for any model which is consistent with the observed call prices $C^{(i)}$, the fair price $B(K)$ for the option satisfies $B(K) \le \overline{\mathcal{B}}(K)$ where

$$\overline{\mathcal{B}}(K) = \sum_i w_i C^{(i)} \left(\lambda_i^* K / w_i \right)$$

and the multipliers λ_i^* are given by (7). Moreover there is a model which is consistent with the individual call prices for which $\overline{\mathcal{B}}(K)$ is the fair price of the option. Hence $\overline{\mathcal{B}}(K)$ is the smallest model-independent bound on the price of the basket option.

4. The finite market case.

The analysis of the previous chapter assumes firstly, that the full marginals are known, and secondly, that once an optimal decomposition λ_i^* has been identified, the associated super-replicating strategy can be executed. In particular it assumes that calls on $X^{(i)}$ with strike $\lambda_i^* K / w_i$ are traded in the market.

In this section we aim to show that if only finitely many strikes are traded on each asset then we can find a super-replicating strategy for the basket option that only involves investment in these traded calls. This super-replicating strategy gives an upper bound on the basket price. Furthermore, for the optimal choice of super-replicating strategy, we show that there is an upper bound $\overline{\mathcal{B}}_F(K)$ which is a least upper bound for the price of a basket option in the given market. We do this by showing that there is a model consistent with the call price data for which $\overline{\mathcal{B}}_F(K)$ is the unique fair price.

Suppose that the traded calls on asset $X^{(i)}$ have strikes $(k_j^{(i)})_{1\leq j\leq J^{(i)}}$ where $k_j^{(i)} < k_{j+1}^{(i)}$. We consider the asset itself as a call with strike $0 = k_0^{(i)}$. The observed call price data for asset $X^{(i)}$ is given by $C^{(i)}(k_j^{(i)})$ for $0 \leq j \leq J^{(i)}$.

Case 1: $C(k^{(i)}_{J^{(i)}}) = 0$ **for all** i**, and** $\sum_i w_i k^{(i)}_{J^{(i)}} > K$**.**

For $1 \leq i \leq N$, and $0 \leq j \leq J^{(i)}$ define $\overline{C}^{(i)}(k)$ by

$$\begin{aligned} \overline{C}^{(i)}(k_j^{(i)}) &= C^{(i)}(k_j^{(i)}), & 0 \leq j \leq J^{(i)} \\ \overline{C}^{(i)}(k) &= 0 & \text{for } k \geq k^{(i)}_{J^{(i)}}, \end{aligned}$$

and by linear interpolation between neighboring values of $k_j^{(i)}$, see Figure 2. Then $\overline{C}^{(i)}$ is the largest decreasing convex function which agrees with $C^{(i)}$ at the traded strikes. Let $\overline{G}^{(i)}$ be the distribution function associated with this call price function, so that $\overline{G}^{(i)}(x) = \mathbb{P}(X_M^{(i)} \leq x) = 1 + (1/\beta)(\overline{C}^{(i)})'(x+)$.

For $1 \leq i \leq N$, and $0 \leq j \leq J^{(i)}$ define $\Delta_j^{(i)}$ by $\Delta_0^{(i)} = \beta$ and

$$\Delta_j^{(i)} = \frac{C^{(i)}\left(k_{j-1}^{(i)}\right) - C^{(i)}\left(k_j^{(i)}\right)}{k_j^{(i)} - k_{j-1}^{(i)}}.$$

Then the measure associated with $\overline{G}^{(i)}$ is a purely atomic distribution which places mass $(\Delta_j^{(i)} - \Delta_{j+1}^{(i)})/\beta$ at $k_j^{(i)}$, where, by convention, we set $\Delta^{(i)}_{J^{(i)}+1} = 0$. It is easy to check that this is a probability distribution with mean $C^{(i)}(0)/\beta = X_0^{(i)}/\beta$.

If $\Delta_j^{(i)} = \Delta_{j+1}^{(i)}$ then $\overline{C}^{(i)}$ is linear over the interval $(k_{j-1}^{(i)}, k_{j+1}^{(i)})$, and $\overline{G}^{(i)}$ places no mass at $k_j^{(i)}$. (Again, by convention, $k^{(i)}_{J^{(i)}+1} = \infty$.) We may, and shall, remove such $k_j^{(i)}$ from the set of traded strikes. It is clear that this can only increase the price of the cheapest super-replicating strategy, but since we show that the bound is attained for a model which is consistent with $\overline{C}^{(i)}$, and $\overline{C}^{(i)}(k_j^{(i)}) = C^{(i)}(k_j^{(i)})$ at the strikes we remove, it follows that the smallest model-independent upper bound on the price of the basket option option is in fact unchanged.

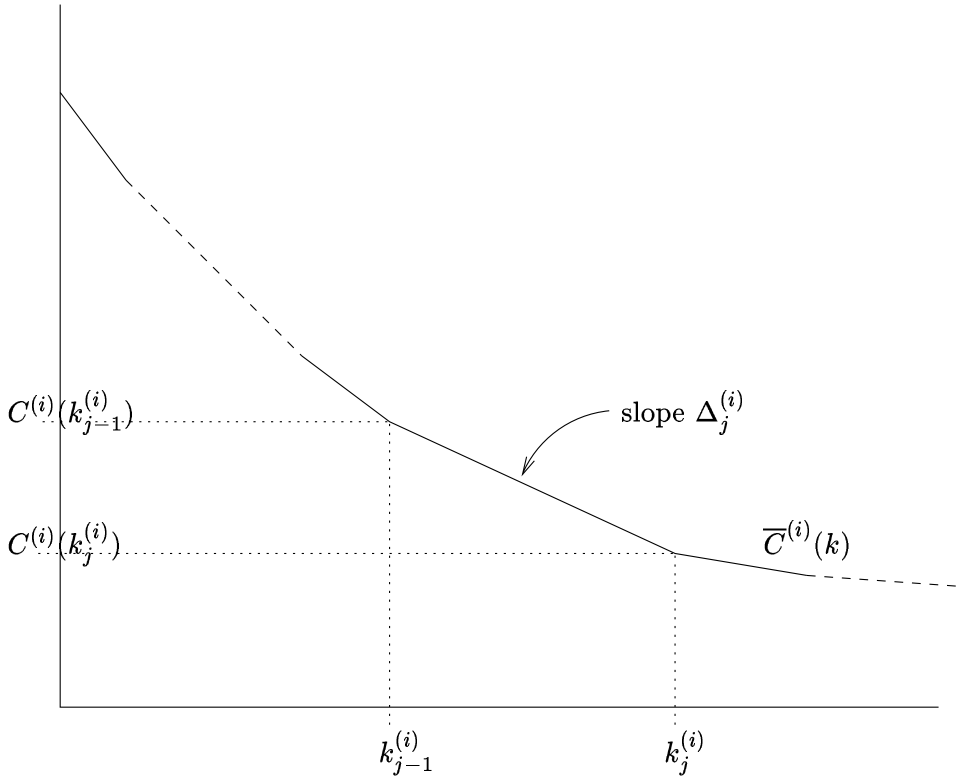

FIGURE 2. The interpolated call price function. $\Delta_j^{(i)}$ gives the modulus of the gradient of $\overline{C}^{(i)}$ over $(k_{j-1}^{(i)}, k_j^{(i)})$.

We now apply the analysis of the previous section to the call price functions $\overline{C}$ and associated distributions $\overline{G}^{(i)}$. Recall the definitions (now in terms of $\overline{\lambda}$ to stress the dependence on $\overline{C}$)

$$\overline{\lambda}_i^{\pm}(\phi) = \frac{w_i}{K}(\overline{G}^{(i)})^{-1}\left(\left(1 - \frac{\phi}{\beta K}\right) \pm\right),$$

and note that, since the associated measure is purely atomic, with atoms at the traded strikes, it follows that $K\overline{\lambda}_i^{\pm}(\phi)/w_i$ lie in the set of traded strikes. Recall also that we can find an optimal Lagrange multiplier ϕ^* and a vector $\overline{\lambda}^*$ such that $\overline{\lambda}_i^-(\phi^*) \leq \overline{\lambda}_i^* \leq \overline{\lambda}_i^+(\phi^*)$ and $\sum_i \overline{\lambda}_i^* = 1$

Let I denote the set of stock indices i such that $K\overline{\lambda}_i^*/w_i$ is a traded strike. (This happens for certain if $\overline{\lambda}_i^-(\phi^*) = \overline{\lambda}_i^+(\phi^*)$ and also if $\overline{\lambda}_i^* \in \{\overline{\lambda}_i^-(\phi^*), \overline{\lambda}_i^+(\phi^*)\}$.) In this case set $\overline{j}(i)$ to be the index j such that $\overline{\lambda}_i^* = w_i k_{\overline{j}(i)}^{(i)}/K$.

If $I = \{1, \ldots, N\}$, then all the strikes in the optimal super-replicating portfolio are traded, and by the results of the previous section we can identify the cheapest super-replicating strategy, and the smallest arbitrage-free upper bound consistent with the call price functions $\overline{C}^{(i)}$. This bound is given by

$$\sum_i w_i \overline{C}^{(i)} \left(k^{(i)}_{\overline{\jmath}(i)} \right).$$

Since $\overline{C}^{(i)}$ agrees with $C^{(i)}$ at the traded strikes we can replace $\overline{C}^{(i)}$ with $C^{(i)}$ in this expression. Further, since there is a model, consistent with $\overline{C}$, for which this price is attained, and since $\overline{C}^{(i)}$ agrees with $C^{(i)}$ at the traded strikes, this bound is the least upper bound consistent with the traded call prices.

Now suppose $I^c = \{1, \dots N\} \setminus I \neq \emptyset$. Then for $i \in I^c$ we have $\overline{\lambda}^*_i \in (\overline{\lambda}^-(\phi^*), \overline{\lambda}^+(\phi^*))$. For all such i define $\overline{\jmath}(i)$ to be the index j such that $\overline{\lambda}^+_i(\phi^*) = w_i k^{(i)}_{\overline{\jmath}(i)}/K$, whence $\overline{\lambda}^-_i(\phi^*) = w_i k^{(i)}_{\overline{\jmath}(i)-1}/K$. Define

$$\theta^*_i = \frac{\overline{\lambda}^*_i - \overline{\lambda}^-_i(\phi^*)}{\overline{\lambda}^+_i(\phi^*) - \overline{\lambda}^-_i(\phi^*)} = \frac{(K\overline{\lambda}^*_i/w_i) - k^{(i)}_{\overline{\jmath}(i)-1}}{k^{(i)}_{\overline{\jmath}(i)} - k^{(i)}_{\overline{\jmath}(i)-1}}. \tag{10}$$

We have

$$\left(\sum_i w_i X^{(i)}_M - K \right)^+ \leq \sum_{i \in I} w_i \left(X^{(i)}_M - \frac{\overline{\lambda}^*_i K}{w_i} \right)^+ + \sum_{i \in I^c} w_i \left(X^{(i)}_M - \frac{\bar{\lambda}^*_i K}{w_i} \right)^+.$$

For $i \in I$ we can replace the relevant term in the sum with $w_i(X^{(i)}_M - k^{(i)}_{\overline{\jmath}(i)})^+$. For $i \in I^c$ we have $\overline{\lambda}^*_i K/w_i = (1-\theta^*_i)k^{(i)}_{\overline{\jmath}(i)-1} + \theta^*_i k^{(i)}_{\overline{\jmath}(i)}$, and hence

$$w_i \left(X^{(i)}_M - \frac{\bar{\lambda}^*_i K}{w_i} \right)^+ \leq (1-\theta^*_i)^+ w_i (X^{(i)}_M - k^{(i)}_{\overline{\jmath}(i)-1}) + \theta^*_i w_i (X^{(i)}_M - k^{(i)}_{\overline{\jmath}(i)})^+ \tag{11}$$

From this we deduce a bound

$$\sum_{i \in I} w_i C^{(i)} \left(k^{(i)}_{\overline{\jmath}(i)} \right) + \sum_{i \in I^c} w_i \left\{ (1-\theta^*_i) C^{(i)} \left(k^{(i)}_{\overline{\jmath}(i)-1} \right) + \theta^*_i C^{(i)} \left(k^{(i)}_{\overline{\jmath}(i)} \right) \right\}, \tag{12}$$

where we have replaced $\overline{C}^{(i)}$ with $C^{(i)}$ since all the strikes are traded. Associated with this bound is a super-replicating strategy consisting of buying w_i calls on $X^{(i)}$ with strike $k^{(i)}_{\overline{\jmath}(i)}$ for each $i \in I$, and, for $i \in I^c$, a combination of $(1-\theta^*_i)w_i$ calls with strike $k^{(i)}_{\overline{\jmath}(i)-1}$ and $\theta^*_i w_i$ calls with strike $k^{(i^*)}_{\overline{\jmath}(i)}$.

Note that there is strict inequality in (11) if and only if $X^{(i)}_M \in (k^{(i)}_{\overline{\jmath}(i)-1}, k^{(i)}_{\overline{\jmath}(i)})$. However the purely atomic distribution $\overline{G}^{(i)}$ places no mass in this interval, and hence when we take expectations on both sides of (11) with respect to $\overline{G}^{(i)}$, we get equality. Hence the upper bound in (12) is the price attained in the co-monotonic model, and it is a least model-independent upper bound.

Case 2: $\sum_i w_i k^{(i)}_{J(i)} > K$ and $C(k^{(i)}_{J(i)}) > 0$ for some i.

Fix $\Delta_\infty > 0$ such that

$$\Delta_\infty < \min_i \Delta^{(i)}_{J(i)},$$

and, for all i such that $C(k^{(i)}_{J(i)}) > 0$ set

$$k^{(i)}_{J(i)+1} = k^{(i)}_{J(i)} + \frac{C(k^{(i)}_{J(i)})}{\Delta_\infty}. \tag{13}$$

Suppose $C(k^{(i)}_{J(i)+1}) = 0$. For the stock $X^{(i)}$ this value of k acts as a synthetic extra strike for which the call price is zero.

By introducing these synthetic strikes we have reduced the problem to the previous case. The bound in that case will still be the least model-independent upper bound, provided that the associated super-replicating strategy does not involve investments in the synthetic calls.

It is sufficient to show that $(\lambda^*_i K/w_i) \leq k^{(i)}_{J(i)}$ for all i for which $C(k^{(i)}_{J(i)}) > 0$. Using the definition of Δ_∞, for all i we have

$$K \left| \frac{\partial \overline{C}^{(i)}}{\partial k} \right|_{k^{(i)}_{J(i)}-} = K\Delta^{(i)}_{J(i)} > K\Delta_\infty.$$

Suppose that for some i' for which $C(k^{(i')}_{J(i')}) > 0$ we have $(\lambda^*_{i'} K/w_{i'}) > k^{(i')}_{J(i')}$. Then $\Delta_\infty = |\partial \overline{C}^{(i')}/\partial k|_{k^{(i')}_{J(i')}+}$ and it follows from (5), evaluated at ϕ^*, that for all i,

$$K \left| \frac{\partial \overline{C}^{(i)}}{\partial k} \right|_{k^{(i)}_{J(i)}-} > K \left| \frac{\partial \overline{C}^{(i')}}{\partial k} \right|_{k^{(i')}_{J(i')}+} \geq K \left| \frac{\partial \overline{C}^{(i')}}{\partial k} \right|_{(\overline{\lambda}^*_{i'} K/w_{i'})-} \geq \phi^*.$$

Hence $(\overline{\lambda}^*_i K/w_i) \geq \overline{\lambda}^-_i(\phi^*) K/w_i) \geq k^{(i)}_{J(i)}$ for all i, but then

$$K \sum_i \overline{\lambda}^*_i \geq \sum_i w_i(\overline{\lambda}^*_i K/w_i) > \sum_i w_i k^{(i)}_{J(i)} \geq K$$

with the strict inequality arising from i' in the sum. This is a contradiction to $\sum_i \overline{\lambda}^*_i = 1$. Hence the optimal strategy does not involve investments in the synthetic calls.

Case 3: $\sum_i w_i k^{(i)}_{J(i)} \leq K$.

This case is the analogue of the third case in the section on the full marginals.

In this case we have the simple bound

$$\left(\sum_i w_i X^{(i)}_M - K \right)^+ \leq \left(\sum_i w_i \left(X^{(i)}_M - k^{(i)}_{J(i)} \right) \right)^+ \leq \sum_i w_i \left(X^{(i)}_M - k^{(i)}_{J(i)} \right)^+$$

It follows that $B(K) \leq \overline{\mathcal{B}}_F(K)$ where

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)} \left(k^{(i)}_{J(i)} \right). \tag{14}$$

If this bound is zero, then clearly it is a least upper bound. Otherwise, it remains to show that $\overline{\mathcal{B}}_F(K)$ is a least upper bound. We do this by finding a sequence of models which are consistent with the traded call prices, and for which the price tends to $\overline{\mathcal{B}}_F(K)$.

If $\overline{\mathcal{B}}_F(K) > 0$ then there is at least one index i for which $C^{(i)}(k^{(i)}_{J(i)}) > 0$. Again we introduce synthetic strikes, as in (13) for which the call prices are taken to be zero. Using the argument in the previous case, it follows that $(\overline{\lambda}^*_i K/w_i) \geq k^{(i)}_{J(i)}$ for all i. The bound in (12) becomes

$$\sum_{i \in I} w_i C^{(i)}\left(k^{(i)}_{J(i)}\right) + \sum_{i \in I^c} w_i (1-\theta^*_i) C^{(i)}\left(k^{(i)}_{J(i)}\right). \tag{15}$$

and there is a co-monotonic model for which this is the fair price. It is not hard to see that the vector λ^*_i is optimal for all values of Δ_∞ (satisfying $0 < \Delta_\infty < \min_i \Delta^i_{J(i)}$) simultaneously. However, if we consider θ^*_i as a function of Δ_∞, then

$$\theta^*_i = \frac{(K\overline{\lambda}^*_i/w_i) - k^{(i)}_{J(i)}}{k^{(i)}_{J(i)+1} - k^{(i)}_{J(i)}} = \frac{(K\overline{\lambda}^*_i/w_i) - k^{(i)}_{J(i)}}{C(k^{(i)}_{J(i)})}\Delta_\infty. \tag{16}$$

This expression tends to zero as $\Delta_\infty \downarrow 0$. As a result we can find a sequence of models, parameterised by Δ_∞, which are consistent with the observed call prices, and for which the basket option price in (15) converges to the bound in (14).

We summarize the information from all the cases in a theorem.

Theorem 4.1. *Suppose maturity-M calls with strikes $(k^{(i)}_j)_{\{1 \leq j \leq J^{(i)}\}}$ are traded on assets $X^{(i)}$ and we wish to price a maturity-M basket option with payoff $\sum_i (w_i X^{(i)}_M - K)^+$.*

Then, for any model which is consistent with the observed call prices $C^{(i)}(k^{(i)}_j)$, the fair price $B(K)$ for the option satisfies $B(K) \leq \overline{\mathcal{B}}_F(K)$ where, for $\sum_i w_i k^{(i)}_{J(i)} > K$,

$$\overline{\mathcal{B}}_F(K) = \sum_{i \in I} w_i C^{(i)}\left(k^{(i)}_{\overline{j}(i)}\right) + \sum_{i \in I^c} w_i \left\{(1-\theta^*_i) C^{(i)}\left(k^{(i)}_{\overline{j}(i)-1}\right) + \theta^*_i C^{(i)}\left(k^{(i)}_{\overline{j}(i)}\right)\right\}$$

where $\overline{j}(i)$ and θ^*_i are as defined in the text and for $\sum_i w_i k^{(i)}_{J(i)} \leq K$

$$\overline{\mathcal{B}}_F(K) = \sum_i w_i C^{(i)}\left(k^{(i)}_{J(i)}\right).$$

Moreover in both cases $\overline{\mathcal{B}}_F(K)$ is the smallest model-independent bound on the price of the basket option, in the sense that we can find models which are consistent with the observed call prices and for which the fair price for the basket option is arbitrarily close to $\overline{\mathcal{B}}_F(K)$.

5. General Remarks

5.1. **The primal and dual problems.** In this section we discuss at a formal level the relationship between the original problem, in the continuum of strikes case, the corresponding dual problem, and the special optimization problem considered in Section 3. Recall that, in principle, for each asset all strikes are available for hedging the basket. Our goal in this section is to formulate the primal and dual problems in this general setting.

Let $\mathcal{M}_+^N$ denote the set of all positive finite measures on $\mathbb{R}_+^N$ and let $\mathcal{M}$ denote the set of all finite signed measures over $\mathbb{R}_+$. Recall that X_M is shorthand for $(X_M^{(1)}, \dots, X_M^{(N)})$.

The primal problem is to find a measure μ in $\mathcal{M}_+^N$ that is optimal for

$$\mathcal{P} = \sup_{\mu \in \mathcal{M}_+^N} \beta \int_{\mathbb{R}_+^N} P_K(X_M)\mu(dX_M), \tag{17}$$

subject to the constraints

$$\int_{\mathbb{R}_+^N} \beta \left(X_M^{(i)} - k^{(i)}\right)^+ \mu(dX_M) = C^{(i)}(k^{(i)}) \quad \text{for } i = 1, \dots, N$$

$$\int_{\mathbb{R}_+^N} \mu(dX_M) = 1.$$

The dual problem is to find

$$\mathcal{D} = \inf_{\nu^{(i)}, \psi} \sum_{i=1}^{N} \int_{\mathbb{R}_+} C^{(i)}(k^{(i)})\nu^{(i)}(dk^{(i)}) + \psi \tag{18}$$

subject to the constraints

$$P_K(X_M) \le \sum_{i=1}^{N} \int_{\mathbb{R}_+} \left(X_M^{(i)} - k^{(i)}\right)^+ \nu^{(i)}(dk^{(i)}) + \psi/\beta, \tag{19}$$

$$\nu^{(i)} \in \mathcal{M}, \text{ for } i = 1, \cdots, N, \quad \psi \in \mathbb{R}.$$

Here ψ represents a cash amount and $\nu^{(i)}$ a portfolio of calls on the underlying $X^{(i)}$.

In Section 3 we restricted our analysis to a restricted class of portfolios with payoffs of the form

$$\sum_{i=1}^{n} w_i (X_M^{(i)} - \lambda_i K / w_i)^+.$$

In particular, we solved (18) subject to (19) and

$$\nu^{(i)} \equiv w_i \delta_{\{k^{(i)} = \lambda_i K / w_i\}}, \qquad \psi = 0 \tag{20}$$

where in (20), $\lambda = (\lambda_1, \dots \lambda_N)$ is any constant vector such that $\lambda_i \ge 0$ and $\sum_{i=1}^N \lambda_i = 1$. If we set $\mathcal{D}'$ to be the infimum in (18) taken over this smaller class of super-replicating portfolios then we have $\mathcal{P} \le \mathcal{D} \le \mathcal{D}'$. However, by exhibiting both an explicit joint probability measure for the primal problem and an explicit super-replicating strategy for the dual problem, and by showing that the corresponding values of the primal and dual objective functionals coincide, we have demonstrated that these values are optimal for the primal and for the dual problems. In other words, we have established strong duality in the present setting, and as a corollary the equivalence between the two dual problems. This is the theoretical underpinning of the construction in Section 3.

5.2. **Implementation of the algorithm.** Recall that we have defined a set of quantities $\{\Delta_j^{(i)}; 1 \le i \le N; 1 \le j \le J^{(i)}\}$. Together with Δ_∞ these quantities correspond to the slopes of the piecewise linear call price functions $\overline{C}^{(i)}$. Note that $\Delta_\infty < \Delta_j^{(i)} \le \beta$, and for fixed i, $\Delta_j^{(i)}$ is decreasing in j. Now place these slopes in decreasing order so that

$$\Delta_{j_1}^{(i_1)} \ge \Delta_{j_2}^{(i_2)} \ge \ldots \ge \Delta_{j_{N'}}^{(i_{N'})}$$

where $N' = \sum_i J^{(i)}$ is the total number of elements $\Delta_j^{(i)}$. When two or more slopes are identical we do not care about the order in which they are placed in the list, except that if $\Delta_j^{(i)} = \Delta_{j+1}^{(i)}$ then we insist that $\Delta_j^{(i)}$ appears before $\Delta_{j+1}^{(i)}$.

The key point is that in searching for the optimal Lagrange multiplier ϕ^* it is sufficient to consider only elements in this list. Moreover, by keeping careful records of the number of times the underlying asset $X^{(i)}$ has had a slope in the list it is possible to determine $\lambda_i^{\pm}(\phi^*)$. These two observations can be used to greatly simplify the optimisation procedure.

6. Numerical Results

In this section we present numerical results which illustrate several theoretical and practical aspects of the upper bound and the associated super-replicating strategy which has been derived in this paper. The results relate to a pair of examples.

The first example is a toy example, which is designed to illustrate the fact that incorrect model choice leads to mispricing, and can lead to violation of the upper bounds. In this example the underlying processes are not exponential Brownian motions and an agent who bases option prices on the log-normal model will inevitably make pricing errors. Instead the underlying processes are based on linear Brownian motions (absorbed at zero). The relative simplicity of the underlying processes means that we can calculate the true price and identify the mispricing.

The second example is based on real data for options on a DJX contract. In this case the true underlying model is unknown. We describe how an agent might attempt to price the basket option in this situation via a Black-Scholes approach, and compare the resulting prices with our alternative upper bound.

6.1. **Model mis-specification and mis-pricing.** Suppose there are two assets in the basket with price processes $X^{(1)}$ and $X^{(2)}$ and equal weights $w_1 = w_2 = 1/2$. Suppose $\beta = 1$, $X_0^{(1)} = X_0^{(2)} = x$ and $M = 1$.

Suppose for $k \geq 0$ the call prices for each asset are given by $C^{(1)}(k) = C^{(2)}(k) = C(k)$ where

$$C(k) = \eta \left[\left(\Phi'(\kappa_-) - \kappa_-(1 - \Phi(\kappa_-)\right) - \left(\Phi'(\kappa_+) - \kappa_+(1 - \Phi(\kappa_+)\right) \right] . \tag{21}$$

Here Φ is the cumulative normal distribution function, Φ' the normal density function and $\kappa_\pm = (k \pm x)/\eta$. The quantity η plays the role of a volatility parameter.

The function $C(k)$ is a decreasing, convex function of k and $C(0) = x$. In fact C is the call price function which arises from a model in which the asset price follows a Brownian motion which is absorbed at 0.

If $\eta = 39.7349$ and $x = 100$ then an at-the-money call option costs 15.8519. This is consistent with a Black-Scholes implied volatility of 40%.

Consider pricing a basket option with strike K. Using the symmetry between $X^{(1)}$ and $X^{(2)}$ it is easily seen that $\lambda_1^* = \lambda_2^* = 1/2$ and that the arbitrage-free model-independent upper bound is $C(K)$, where the function C is as given in (21).

Now consider the pricing problem faced by an agent who uses the Black-Scholes model to price the basket option. Suppose the agent calibrates his model to the at-the-money Black-Scholes implied volatility of 0.4, and uses correlations belonging to the set $\{0.3, 0.7, 1.0\}$. In effect, rather than using the whole call price function, this agent is only using $C(100)$ for each underlying.

Suppose that $K = 110$ (so that the basket call is out-of-the-money). We can price this call using both a model-based approach and our model-free alternative. The results are presented in Table 1. For the model-based comparison we use the results of Deelstra et al [7]. The approach in [7], and especially the lower bound, gives good approximation to the model price which is why we have used it for comparison purposes.

ρ	Monte-Carlo Price	DLV Lower Bound	DLV Upper Bound	Arbitrage-free Upper Bound
0.3	9.12	9.0280	10.2168	11.3513
0.7	10.88	10.8647	11.3373	11.3513
1.0	12.1081	12.1081	12.1081	11.3513

TABLE 1. Prices of basket options with a strike of 110. The numbers in the middle three columns are taken directly from Deelstra et al [7]. Note that these prices are based on using an incorrect model which has been calibrated using at-the-money call prices. In contrast the arbitrage-free upper bound is based on the correct model, as represented by the call prices in (21). Note that this price does not depend on the correlation, or indeed any other model parameters.

The key conclusion from Table 1 is that use of an incorrect model can lead to serious mispricing. When correlation is close to one, so that the co-monotonic model approximates the true covariance structure, naive usage of a pricing scheme based on Black-Scholes can lead to overpricing of the basket option. If a trader believes his model-based price, and offers to sell the option at a price above 11.36 (perhaps because he also believes the correlation between the two assets to be very high) then there is a simple static trading strategy for taking advantage of this mispricing.

6.2. **The DJX contact.**

6.2.1. *The data.* Our analysis is based on a DJX contract, using the prices as traded on May 17^{th}, 2004 on the June 04 contract with expiry June 18^{th}, 2004. The data is given in Table 2. The prices given are mid-market prices.

In fact these market prices are not quite convex. (This lack of convexity is tiny in comparison with the bid-ask spread. Hence there are no realisable arbitrage opportunities in the original data.) As a result the numbers in Table 2 represent 'cleaned' data which has been modified in the few cases where necessary to make call prices on the individual assets convex.

There are 30 stocks within the DJX basket, equally weighted with weights 0.071. The option is traded European style. On each individual stock there are between 4 and 14 traded strikes, with a typical number of strikes being about 8. As well as quoting the strikes and prices of these options on the individual components, Table 2 also shows the implied volatilities of these options assuming that the Black-Scholes model holds.

One of the columns of Table 2 contains the (mid-market) prices of call options on the index with various strikes ranging from 52 to 107. Note that the DJX index itself had a spot value of 99.07.

We consider two approaches which an agent might follow to price the basket options on the index.

6.2.2. *A model based approach.* Suppose the agent assumes that the components of the DJX index each follow exponential Brownian motions. In principle, it is then possible to calculate the model-based price of the index option by calculating the discounted expectation. This calculation is typically quite difficult, but can be greatly simplified by using one of the many fast and accurate approximations which have been suggested in the literature, including, for example, those listed in the Introduction. In fact we use a Monte-Carlo approach. (For each traded strike for the DJX option we simulated 50,000 values. As a result the standard error of each of our Monte-Carlo prices is about 0.04.)

In order to apply any of these approaches it is necessary to specify the parameters of the model, which consist of the individual volatilities and correlations. Choosing such parameters is a non-trivial exercise. Eight traded options on a component asset may have 8 different implied volatilities, and the trader may have to choose between 8^{30} volatility vectors. In practice, traders and analysts have developed rules of thumb to help them choose these parameters, for example when valuing an at-the-money DJX contract

Ticker Symbol													ATM Imp. Vol
AA	0.00	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00				0.43
	28.70	6.25	4.00	2.10	0.88	0.23	0.10	0.08	0.08				
AIG	0.00	60.00	65.00	70.00	75.00	80.00	85.00						0.28
	69.39	9.65	5.25	2.03	0.43	0.10	0.05						
AXP	0.00	42.50	45.00	47.50	50.00	55.00	60.00						0.22
	48.77	6.35	4.10	2.13	0.75	0.08	0.08						
BA	0.00	32.50	35.00	37.50	40.00	42.50	45.00	47.50	50.00				0.24
	42.73	10.20	7.00	5.20	3.00	1.33	0.35	0.10	0.05				
C	30.00	32.50	35.00	37.50	40.00	42.50	45.00	47.50	50.00	55.00	60.00	65.00	0.26
	44.86	14.93	12.44	9.94	7.45	5.10	2.90	1.33	0.38	0.13	0.03	0.03	
CAT	0.00	60.00	65.00	70.00	75.00	80.00	85.00						0.29
	73.54	13.70	8.95	4.80	1.93	0.58	0.18						
DD	0.00	37.50	40.00	42.50	45.00	47.50	50.00						0.25
	41.19	3.95	1.90	0.63	0.15	0.08	0.08						
DIS	0.00	17.50	20.00	22.50	25.00	27.50							0.28
	22.90	5.40	2.95	0.98	0.18	0.05							
GE	0.00	15.00	17.50	20.00	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00	0.23
	29.97	14.99	12.49	9.99	7.50	5.00	2.63	0.83	0.15	0.03	0.03	0.03	
GM	0.00	30.00	32.50	35.00	37.50	40.00	42.50	45.00	47.50	50.00	55.00	60.00	0.35
	43.40	13.50	11.05	8.60	6.25	4.05	2.28	1.00	0.35	0.15	0.03	0.03	
HD	0.00	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00		0.31
	33.47	10.95	8.45	6.00	3.70	1.78	0.55	0.13	0.03	0.03	0.03		
HON	0.00	20.00	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00	0.26
	32.60	12.60	10.10	7.60	5.10	2.78	1.08	0.23	0.05	0.03	0.03	0.03	
HPQ	0.00	15.00	17.50	20.00	22.50	25.00	27.50						0.38
	19.50	4.50	2.25	0.65	0.13	0.03	0.03						
IBM	0.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	0.23
	85.53	25.55	20.55	15.60	10.75	6.20	2.58	0.70	0.15	0.05	0.03	0.03	
INTC	0.00	17.50	20.00	22.50	25.00	27.50	30.00	32.50	35.00				0.33
	26.84	9.39	6.89	4.40	2.25	0.75	0.15	0.03	0.03				
JNJ	0.00	35.00	40.00	45.00	50.00	55.00	60.00	65.00	70.00				0.19
	54.70	19.74	14.75	9.80	4.95	1.10	0.08	0.05	0.03				
JPM	0.00	25.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00	47.50	50.00	0.30
	35.31	10.35	7.90	5.50	3.20	1.40	0.40	0.08	0.08	0.05	0.03	0.03	
KO	0.00	42.50	45.00	47.50	50.00	55.00	60.00						0.17
	49.81	7.35	4.95	2.65	0.95	0.05	0.05						
MCD	0.00	17.50	20.00	22.50	25.00	27.50	30.00	35.00	40.00	45.00			0.29
	25.87	8.39	5.90	3.40	1.38	0.30	0.10	0.03	0.03	0.03			
MMM	0.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00					0.20
	83.22	18.20	13.20	8.30	4.10	1.23	0.20	0.05					
MO	0.00	35.00	37.50	40.00	42.50	45.00	47.50	50.00	55.00	60.00	65.00	70.00	0.24
	49.47	14.50	12.00	9.55	7.10	4.75	2.70	1.15	0.13	0.05	0.03	0.03	
MRK	0.00	37.50	40.00	42.50	45.00	47.50	50.00						0.18
	46.64	9.15	6.65	4.25	2.08	0.65	0.13						
MSFT	0.00	5.00	7.50	10.00	12.50	15.00	17.50	20.00	22.50	25.00	27.50	30.00	0.26
	25.54	20.55	18.05	15.55	13.10	10.60	8.10	5.60	3.15	1.10	0.18	0.03	
PFE	0.00	17.50	20.00	22.50	25.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	0.24
	35.50	18.00	15.50	13.00	10.50	8.00	5.60	3.20	1.28	0.30	0.10	0.05	
PG	0.00	80.00	85.00	90.00	95.00	100.00	105.00	110.00	115.00	120.00	125.00		0.19
	106.28	26.25	21.30	16.35	11.50	6.95	3.15	0.95	0.20	0.08	0.08		
SBC	0.00	15.00	17.50	20.00	22.50	25.00	27.50	30.00	32.50	35.00	40.00		0.24
	24.30	9.30	6.80	4.30	1.98	0.40	0.05	0.03	0.03	0.03	0.03		
UTX	0.00	75.00	80.00	85.00	90.00	95.00							0.22
	82.34	7.45	3.55	1.15	0.28	0.08							
VZ	0.00	32.50	35.00	37.50	40.00								0.24
	35.78	3.50	1.45	0.35	0.08								
WMT	0.00	40.00	42.50	45.00	47.50	50.00	55.00	60.00	65.00	70.00	75.00		0.23
	54.70	14.70	12.20	9.70	7.30	5.00	1.38	0.13	0.03	0.03	0.03		
XOM	0.00	32.50	35.00	37.50	40.00	42.50	45.00	47.50	50.00				0.21
	43.05	10.60	8.10	5.65	3.30	1.38	0.35	0.05	0.03				

TABLE 2. CBOE data from May 17th, 2004 on the June 2004 contract which expires June 18. There are 30 stocks in the basket with equal weights 0.071. The first column records the ticker symbol for each of the 30 assets in the basket. The second column records the stock price. The entry of 0 for each stock refers to the fact the the stock can be considered to be a call with strike zero. The next 11 columns are call price data. For each stock, and for each traded call on that stock, the table lists the strike and associated price. (For some stocks more than 11 calls are traded, these have been omitted for reasons of space.) The final column lists the Black-Scholes at-the-money implied volatility.

they might decide to use the at-the-money implied volatilities for each component asset, but these rules of thumb put the investor at severe risk of mis-specification of the parameters.

DJX Strikes	DJX Prices	UB	MC ATM BS Price rho=0	MC ATM BS Price rho=.5	MC ATM BS Price rho=.75	MC ATM BS Price rho=.9	MC ATM BS Price rho=.99
52	47.10	47.09	47.14	47.14	47.15	47.16	47.18
56	43.10	43.10	43.16	43.18	43.18	43.19	43.17
60	39.10	39.11	39.16	39.18	39.13	39.13	39.14
64	35.10	35.11	35.16	35.16	35.16	35.20	35.17
68	31.10	31.12	31.17	31.17	31.22	31.17	31.16
70	29.10	29.13	29.18	29.19	29.18	29.17	29.11
72	27.10	27.14	27.19	27.22	27.18	27.13	27.18
76	23.10	23.15	23.18	23.16	23.18	23.15	23.19
80	19.10	19.18	19.20	19.18	19.15	19.19	19.22
84	15.20	15.24	15.21	15.24	15.23	15.18	15.23
88	11.30	11.42	11.20	11.26	11.25	11.25	11.36
90	9.40	9.61	9.21	9.28	9.35	9.41	9.44
92	7.50	7.90	7.21	7.34	7.53	7.67	7.73
94	5.80	6.32	5.22	5.58	5.83	6.01	6.08
95	4.95	5.57	4.22	4.79	5.06	5.26	5.34
96	4.15	4.85	3.22	4.01	4.35	4.54	4.66
97	3.35	4.19	2.24	3.28	3.69	3.92	4.01
98	2.73	3.58	1.35	2.70	3.12	3.34	3.44
99	2.13	3.02	0.67	2.16	2.58	2.75	2.96
100	1.60	2.53	0.25	1.69	2.10	2.33	2.43
102	0.78	1.73	0.01	0.99	1.37	1.55	1.71
103	0.50	1.42	0.00	0.71	1.05	1.26	1.36
104	0.33	1.16	0.00	0.52	0.82	1.02	1.13
105	0.15	0.95	0.00	0.36	0.63	0.79	0.89
106	0.15	0.75	0.00	0.25	0.48	0.60	0.70
107	0.15	0.59	0.00	0.16	0.35	0.48	0.53

TABLE 3. Prices on DJX option contracts on May 17^{th}, 2004, on the June 04 contract. In order the columns represent: the strike, the trade price, the upper bound calculated via our algorithm, and Monte-Carlo prices generated using a Black-Scholes model with volatilities chosen to match at-the-money volatilities of traded calls, and correlations $\rho = 0.0$, 0.5, 0.75, 0.9 and 0.99. Each Monte-Carlo price was generated using 50,000 samples. The standard errors are of the order of 0.04.

We consider an agent who uses the Black-Scholes exponential Brownian motion model with volatilities chosen to match the price of an at-the-money call on each individual component stock. In effect therefore the agent has replaced the full set of call price information with the information content of the at-the-money calls. In Table 3 we give the (Monte-Carlo) price of basket options based on this model for the range of traded strikes, and for a range of correlations. For simplicity, we assume that the correlation between each pair of assets are identical, (again this might be a traders rule of thumb).

6.2.3. *The model-independent bound.* As an alternative to a model-based approach the agent could use the information on the calls on the individual assets to calculate the upper bounds given in Theorem 4.1. This can be done with a single calculation which took 20 seconds to run on a Pentium 3 with 3.06 GH and 2 MG Ram. The results for the model-independent upper bound are also reported in Table 3. In addition to specifying a price, this approach determines a cheapest super-replicating option, consisting of a portfolio of calls on the individual assets. For each traded strike on the basket option this portfolio of calls is described in Table 4.

In fact, Table 4 only gives the results for the first 10 underlyings in the basket (on grounds of space). In general, for a given strike for the basket, the cheapest super-replicating portfolio involves buying 0.071 calls on each component with strike as listed in the corresponding row of the table. There are a few exceptions to this rule, where the optimal super-replicating portfolio may involve buying a positive combination (with total weight 0.071) of calls involving a pair of strikes. For example, for the DJX option with strike 99, the optimal portfolio involves buying calls on DD with strikes 40 and 42.5.

DJX Strikes	DJX Call Prices	AA	AIG	AXP	BA	C	CAT	DD	DIS	GE	GM
52	47.1	0	0	0	37.5	0	0	0	17.5	25	0
56	43.1	0	0	0/42.5	37.5	0	0	0	17.5	25	0
60	39.1	22.5	0	42.5	37.5	0/37.5	0	0	17.5	25	0
64	35.1	22.5	0	42.5	37.5	37.5	0/60	0	17.5	25	0
68	31.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25	0/30
70	29.1	22.5	0	42.5	37.5	37.5	60	0	17.5	25	0/30
72	27.1	22.5	0/60	42.5	37.5	37.5	60	0	17.5	25	30
76	23.1	22.5	60	42.5	37.5	37.5	60	0	17.5	25	30
80	19.1	22.5	60	42.5	37.5	37.5	60	0/37.5	17.5	25	30
84	15.2	22.5	60	42.5	37.5	37.5	60	37.5	20	25	35
88	11.3	22.5	60	42.5	37.5	40	65	37.5	20	27.5	37.5
90	9.4	25	60	45	37.5/40	40	65	37.5	20	27.5	37.5
92	7.5	25	65	45	40	42.5	70	37.5/40	20	27.5	40
94	5.8	25	65	47.5	40	42.5	70	40	22.5	27.5	40
95	4.95	27.5	65	47.5	40	42.5	70	40	22.5	27.5	40
96	4.15	27.5	65	47.5	40	42.5	70	40	22.5	30	42.5
97	3.35	27.5	70	47.5	42.5	42.5	70	40	22.5	30	42.5
98	2.725	27.5	70	47.5	42.5	45	70	40	22.5	30	42.5
99	2.125	27.5	70	50	42.5	45	75	40/42.5	22.5	30	42.5
100	1.6	30	70	50	42.5	45	75	42.5	22.5	30	45
102	0.775	30	70	50	45	47.5	75	42.5	22.5	30	45
103	0.5	30	75	50	45	47.5	75/80	42.5	25	30/32.5	45
104	0.325	32.5	75	50	45	47.5	80	42.5	25	32.5	47.5
105	0.15	32.5	75	50	45	47.5	80	42.5	25	32.5	47.5
106	0.15	32.5	75	50	45	47.5	80	45	25	32.5	47.5
107	0.15	32.5	75	50	45	47.5	80	45	25	32.5	47.5

TABLE 4. The super-replicating portfolio. For each strike on the DJX, and for each component of the basket, we list the relevant strike to hold in the cheapest super-replicacting portfolio. A strike of 0 corresponds to holding the asset. For space reasons we only give the strikes for the first 10 components. In most cases there is a single strike listed. In others the optimal portfolio involves a combination of two strikes. Note that the optimal strike to hold on each component asset increases as the strike on the DJX increases.

6.2.4. *Discussion.* It is clear simply from the set of traded strikes (the spot price is 99.07, but the set of strikes ranges from 52 to 107) that the market takes an asymmetric view about the likely future direction of price movements. Recall that the basket options are very short maturity (of about one month). The market is much more worried about crashes and protection against market falls than about market gains. The same effect can also be seen in the prices of traded calls. If we compare calls with strike 94 and 104 (chosen to be roughly equally distant from the spot, both in absolute terms and on a log scale), then allowing for the intrinsic value, the former has price 0.73 whereas the latter has price 0.33.

Now consider the prices that arise under the Black-Scholes model for varying values of the correlation. These prices are increasing in correlation. When $\rho = 0$ the option value excluding intrinsic value is almost zero outside the range (97-102). This is the central limit theorem coming in to play. Clearly the market does not act as if the component assets in the basket are independent.

As correlation rises from 0 to 1, prices increase until the model price exceeds the market price. Note however, that if we compare the option values for strikes of 94 and 104 as before, then for each value of correlation in the Black-Scholes model we find they are roughly equal, — this is inevitable from the symmetries of the exponential Brownian model viewed on a log scale — and it is not possible to capture the market asymmetries. Note that the best-fit correlation for model prices to market prices is to take $\rho = 0.5$ for near and out of the money options, but $\rho = 0.75$ for far-in-the-money options.

Now consider the prices that arise from our upper bound. In general these exceed the market price, but for strikes well below the spot value they are remarkably close to the traded price. Further, the upper bound does display some of the same biases as the market prices. Relative to the Black-Scholes model there is some tendency for in-the-money calls to be priced more highly than out-of-the-money calls.

The key features of our bound are that it is robust, it is model independent, and it is associated with a simple super-replicating strategy. Instead of using the bound to price options, it may be more appropriate to use it as a signal to indicate when options are being mispriced. Reassuringly, on that basis none of the DJX options is mispriced to the extent that it is inconsistent with no-arbitrage, although some of the prices calculated using the Black-Scholes model with high correlations come quite close.

7. Conclusion

In the Black-Scholes model, or indeed any complete model, there is an expression for the fair price of a basket option. However, this price is sometimes difficult to calculate and often very sensitive to correlations between the assets within the basket. Furthermore, the traded prices of vanilla options may be inconsistent with the model, leading to the danger that model-based prices result in arbitrage opportunities.

Instead of assuming a particular model, and pricing basket options in the framework of that model, in this article we derived model-independent upper bounds for the basket option price.

The first case we treated was that of an idealized market in which options with a continuum of strikes are traded on each stock. In this case the pricing bounds correspond in a direct fashion to super-replicating strategies involving long positions in calls on the individual assets. An optimal portfolio of calls, or equivalently a cheapest super-replicating strategy, is obtained by purchasing w_i calls on underlying $X^{(i)}$, (recall that w_i is the weight of $X^{(i)}$ in the basket), and by choosing the strikes of these calls appropriately. Alternatively we can consider the class of models which are consistent with the traded call prices, and search for the model within this class under which the basket option is most expensive. It turns out that the highest price is obtained from a model in which the individual assets are co-monotonic.

A more realistic situation is when only finitely many strikes are traded on each underlying. In this setting we have provided a simple constructive algorithm, from which it is possible to derive a model-independent, no-arbitrage upper bound. Further, we have presented an explicit, model-independent super-replicating strategy. The key features of this strategy are that it is the cheapest super-replicating strategy, and it is extremely simple to execute.

Although many different strikes may be traded on each option, the optimal strategy involves positions in at most two options per stock (both held long). Surprisingly our results show that even when options of other strikes are available, it is not optimal to use them to hedge the basket.

In cases where correlation is very high, the upper bound is tight. However, in other cases the bound can be much greater than the true price. For this reason we do not suggest that the bound should replace other pricing methods. Instead, it should provide a check on the sensible choices of model parameters. Further, in markets where the basket option is traded, it can provide a signal to highlight mispricing within the market.

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