Bounds for the Utility-Indifference Prices of Non-traded Assets in Incomplete Markets

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Abstract

We consider a special class of financial models with both traded and non-traded assets and show that the utility indifference (bid) price of a contingent claim on a non-traded asset is bounded above by the expectation under the minimal martingale measure. This bound also represents the marginal bid price for the claim.

The key conclusion is that the bound and the marginal bid price are independent of both the utility function and initial wealth of the agent. Thus all utility maximising agents charge the same marginal price for the claim. This conclusion is in some sense the opposite conclusion to that of Hubalek and Schachermayer (2001), who show that any price is consistent with some equivalent martingale measure.

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1 Introduction

The key insights of the Black-Scholes option pricing methodology are firstly, that in a complete market it is possible to replicate a contingent claim, and secondly, that the initial fortune which is required to finance the replicating strategy is the fair price of the option. In particular, the price of a contingent claim is determined unambiguously by the principles of no-arbitrage, and is independent of the risk preferences of agents.

The conclusion that there are unique option prices in the Black-Scholes model is lost as soon incompleteness is introduced into the model. This can happen in many ways, for example following the introduction of transaction costs, or if the assumptions of the model do not allow the agent to follow the replicating strategy. In these cases any non-attainable contingent claim carries risk, and any pricing rule makes implicit or explicit assumptions about utilities and preferences.

The typical problem we have in mind (see Hubalek and Schachermayer (2001), Davis (1998), Henderson and Hobson (2002a, 2002b)) is as follows. There are two risky assets, one of which is traded, but the second is not. Although the price processes for the assets may be driven by correlated Brownian motions, the coefficients of the dynamics for the traded asset do not depend on the untraded asset. An agent is due to receive a claim which is contingent on the non-traded asset. How much is that random claim worth? This is the situation in real options, see Dixit and Pindyck (1994). An illustration from Hubalek and Schachermayer (2001) is when the two assets are different brands of crude oil, only one of which is liquidly traded.

This problem is an example of the problem of pricing a claim in an incomplete market, and is similar to those considered in Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), Duffie and Richardson (1991) and many others. In common with Duffie and Richardson (1991) and Davis (1998) we model our agents as maximisers of expected utility. An alternative approach is to select a martingale measure (for example the minimal martingale measure) and to use that for pricing.

The utility maximisation problem is a basic problem in finance and was first studied in a continuous time model by Merton (1969). A powerful approach to this problem is the dual variational method, see, for example, Karatzas et al (1991), Kramkov and Schachermayer (1999) and Schachermayer (2001). These papers provide a complete solution of the optimal investment problem in an incomplete market. The paper by Karatzas et al (1991), provides the foundations for both the notation and style of arguments in this paper.

In order to address the question of the pricing of contingent claims in an incomplete market, Hodges and Neuberger (1989) introduced the notion of the utility indifference price. The utility indifference bid price is the amount the agent is prepared to pay which leaves him indifferent between paying for, and receiving the claim, and not paying for, and not receiving the claim. In order to derive this price we need to solve the utility maximisation problem both without the claim (see the references in the previous paragraph) and in the case with a random endowment. Cvitanic at al (2001) have made some steps towards a solution of the random endowment problem in a general setting. For the exponential utility function, Delbaen et al (2002) characterise the solution to the pricing problem and determine the associated dual problem.

The goal of this paper is to compare the utility indifference price across different choices of utility function. Such comparisons are very rare in the literature, although there have been some studies which investigate the impact of changing the risk aversion within a parametric family of utility functions, see Sircar and Zariphopoulou (2005) (stochastic volatility models and exponential utility), Henderson et al (2005) (stochastic volatility models and power-law utilities) and Bouchard et al (2001) (transactions costs and exponential utility). In general, the bid price offered by an agent must depend on her choice of utility function, and there is a wide range of prices which can be realised as the utility indifference price. However, in our specific non-traded asset setting, we show that there is a simple, non-trivial upper bound on the bid price for the option which is independent of the choice of utility. This bound is the price of the claim under the minimal martingale measure. Further, this bound represents the marginal price, or equivalently the unit price she would be prepared to pay for an infinitesimal quantity of the option.

The rest of this paper is structured as follows. In the next section we describe the model and the main concepts in an abstract setting. We state the results, both purely in terms of probability, and in terms of their financial interpretation. In Section 3 we prove the main theorems. The key observation is that the bounds we derive are independent of the choice of utility function. There is a set of analogous results for ask prices which we give in Section 4. In Section 5 we show how the non-traded assets model fits into this framework and Section 6 describes the results for certain common parametric families of utility functions for which explicit calculations are sometimes possible. In Section 7 we consider a fundamentally different model, which is a special case of a stochastic volatility model, and which also fits into the general framework of Section 2. Section 8 concludes.

2 The main results and the associated financial model

We suppose that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, together with a convex function $U : \mathbb{R} \to [-\infty, \infty)$, which is strictly increasing, strictly concave and continuously differentiable on its domain, with derivative tending to zero at infinity.

Assumption 2.1. (a) Suppose that the probability space supports a non-negative $\mathcal{G}$-measurable random

variable ζ satisfying $0 < \mathbb{E}[\zeta] < \infty$.
(b) Define $\mathcal{A}_{\mathcal{G}}(x) = \{X \in m\mathcal{G} : \mathbb{E}[\zeta X] \leq x\}$, where $m\mathcal{G}$ is the set of $\mathcal{G}$-measurable functions, and let $\mathcal{A}_{\mathcal{F}}$ be defined similarly, but with respect to the σ-algebra $\mathcal{F}$. Suppose we are given an increasing family $\{\mathcal{A}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{A}(x)$ is the set of admissible random variables for a given constraint level x, with $\mathcal{A}_{\mathcal{G}}(x) \subseteq \mathcal{A}(x) \subseteq \mathcal{A}_{\mathcal{F}}(x)$, and with the property that

$$c\mathcal{A}(x') + (1-c)\mathcal{A}(x'') \subseteq \mathcal{A}(cx' + (1-c)x'') \qquad \forall c \in (0,1).$$

Let H be an element of $m\mathcal{F}^+$, the set of non-negative $\mathcal{F}$-measurable random variables. We consider an optimal control problem involving U and H. Set $V(x) = \sup_{X\in\mathcal{A}(x)} \mathbb{E}[U(X)]$. To avoid trivialities we assume that $V(x) < \infty$, for some, and then all x. Define

$$\mathcal{V}(x,k) = \sup_{X\in\mathcal{A}(x)} E[U(X+kH)], \tag{1}$$

so that $\mathcal{V}(x,0) = V(x)$, and

$$p(k) = \inf\{q : \mathcal{V}(x-q,k) \geq V(x)\}. \tag{2}$$

We now make a technical assumption, see also Karatzas et al (1991, Equation 6.2),

Assumption 2.2. Suppose that for all $w > 0$ we have $\mathbb{E}[|\zeta(U')^{-1}(w\zeta)|] < \infty$.

Then, the main results of this paper are that, provided $V(x-) > -\infty$,

Theorem A. $p(k) \leq k\mathbb{E}[H\zeta]$, and

Theorem B. $D_+p|_{k=0} = \mathbb{E}[H\zeta]$, where D_+ denotes the right derivative.

In particular, both the bound for $p(k)$ and the derivative of $p(k)$ near zero depend on the random variable ζ, but not on the function U, or the constraint level x.

Let us now try to motivate these results by explaining why they are important and relevant in the theory of mathematical finance.

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a stochastic basis for a financial market, where $\mathbb{P}$ is the real world probability measure for an agent. The σ-algebra $\mathcal{F}$ represents all possible events in this model. We suppose that there is also a sub-σ-algebra $\mathcal{G}$ which corresponds to the events associated with a complete market which is embedded within the larger financial model. The random variable ζ plays the role of the (unique) state-price density in the complete market model, and one of many state-price densities in the larger model.

The analysis of this paper is based on this rather special assumption that there exists the complete market model contained within the larger financial market. We do not claim that this assumption is appropriate in a general financial model, but rather that it is appropriate in certain contexts, and that then some strong conclusions about the utility indifference pricing of derivatives follow.

We focus on a single agent in this model who is assumed to have a concave utility function U. By tradition the agent is female. She is assumed to be a maximiser of expected utility of wealth. We assume that the agent begins with initial wealth x and that the set of candidate or admissible target wealths for the agent is the set $\mathcal{A}(x)$. It is natural to assume that $\mathcal{A}(x)$ includes all terminal wealths which can be generated in the complete sub-market (hence the assumption $\mathcal{A}(x) \supseteq \mathcal{A}_{\mathcal{G}}(x)$), and conversely that every admissible wealth must satisfy a budget constraint relative to each state-price density in the incomplete model. Let $\mathcal{Z}$ denote the set of state-price densities in the incomplete model. Then we might define

$$\mathcal{A}(x) = \tilde{\mathcal{A}}_{\mathcal{F}}(x) = \{X \in m\mathcal{F} : \sup_{\xi \in \mathcal{Z}} \mathbb{E}[\xi X] \leq x\},$$

and more generally other restrictions on trading strategies may be imposed such that $\mathcal{A}(x) \subset \tilde{\mathcal{A}}_{\mathcal{F}}(x)$. In either case we have $\mathcal{A}(x) \subseteq \mathcal{A}_{\mathcal{F}}(x) = \{X \in m\mathcal{F} : \mathbb{E}[\zeta X] \leq x\}$.

Karatzas et al (1991) restrict the set of admissible random variables further to include only those elements for which $\mathbb{E}[U(X)^-] < \infty$. However, as we argue in Remark 3.6, this restriction is not necessary, since our assumptions guarantee that $\mathbb{E}[U(X)^+] < \infty$ for all $x \in \mathcal{A}(x)$, and hence $\mathbb{E}[U(X)]$ is always well defined, even if it may equal minus infinity.

The recent literature (Schachermayer (2001), Strasser (2004)) also contains a discussion of the appropriate definition of admissibility, with special reference to utility functions defined on the real line. In a dynamic setting the budget constraint is usually augmented by a further condition which ensures that the discounted gains from trade is a supermartingale. However in our special setting it turns out that $\mathcal{A}_{\mathcal{G}}(x) \subseteq \mathcal{A}(x) \subseteq \mathcal{A}_{\mathcal{F}}(x)$ is both a sufficient and appropriate definition of admissibility, not least because we do not want to declare inadmissible to the agent operating in the full market any strategies which would normally be declared admissible in the complete sub-market corresponding to $\mathcal{G}$.

Our aim in this paper is to consider the problem where the agent is to receive $k > 0$ units of a random non-negative payout H. We take the claim H as fixed throughout. The agent's value function, now a function of initial wealth and endowment k, is given by (1).

We want to decide how much the agent is prepared to pay for the claim H. The utility indifference (bid) price (Hodges and Neuberger (1989)) is the amount of money the agent could pay now which would leave her indifferent between paying and receiving the random claim H, and not paying, and not receiving the claim. Stated mathematically, if there is a unique q with $\mathcal{V}(x - q, k) = V(x)$ then $q = p(k)$ and we say that $p(k)$ is the *utility indifference price.* More generally, it may be that this quantity is not well defined so we define the *bid price* $p(k)$ for k units of the claim to be as in (2). We can also define the *marginal bid price* for the agent to be $D_{+}p|_{k=0}$ assuming this derivative exists. This definition is related to the definition of the *fair* price of a derivative given in Davis (1998). Davis defines the fair price to be $Dp|_{k=0}$, provided that $D_{+}p|_{k=0} = D_{-}p|_{k=0}$.

The existence of the marginal price in a general incomplete market is the subject of a recent paper by Hugonnier et al (2005). For a fixed utility defined on $\mathbb{R}^{+}$, these authors are interested in the conditions under which a marginal price exists and is unique (although it will depend on the choice of utility). Their definition of marginal price corresponds to the fair price of Davis. Loosely stated the result is that the marginal price is well defined for all bounded contingent claims provided that the solution to the dual problem defines an equivalent local martingale measure. This later condition plays a similar role to Assumption 2.2.

The main results of this paper can be translated into the following statements. Under Assumptions 2.1 and 2.2,

Theorem Aa. *If $h = \mathbb{E}[H\zeta]$ then kh is an upper bound on the bid price for k units of the claim H,* and

Theorem Ba. *The marginal bid price for the non-negative claim H is given by h.*

Note that h is independent of both the wealth of the agent and her particular utility function. It is also independent of the set $\mathcal{Z}$ of state-price densities.

We can also show that kh is a lower bound on the ask price for k units of the claim, where the ask price is defined in the natural fashion. If H is bounded then h is also the marginal ask price.

The results of this section have been described in a general setting, subject to Assumption 2.1 on the existence of a $\mathcal{G}$-measurable state-price-density, and the definition of admissible strategies. We now describe the type of situation where this assumption is satisfied. The key example we have in mind is a model of non-traded assets. Suppose there are two risky assets given by correlated (constant parameter) exponential Brownian motions. Suppose that only one of these assets is traded and consider the problem of trying to price an option on the second asset. The financial sub-market consisting of the traded asset alone is a standard Black-Scholes model, is complete, and has a unique state-price density. Assumption 2.1(a) is satisfied in this example, and under some natural assumptions on the set of admissible strategies Assumption 2.1(b) also holds. We return to this example in Section 5.

3 Proofs

The aim of this section is to prove Theorems A and B under Assumptions 2.1 and 2.2. We begin by stating some easy properties of the value function which follow immediately from the properties of $\mathcal{A}$ and the fact that U is concave.

Lemma 3.1. (i) $\mathcal{V}(x,k)$ *is increasing in the first argument. If* $-\infty < \mathcal{V}(x,k) < U(\infty) \leq \infty$ *then* $\mathcal{V}$ *is strictly increasing.*
(ii) $\mathcal{V}$ *is concave in the* (x,k) *plane.*

Let $z_* = \inf\{z : U(z) > -\infty\}$ and let $y_* = D_+U(z_*)$. For most commonly used utility functions $y_* = \infty$. Let the inverse to the derivative of U be denoted by I. The assumptions on U ensure that I is a well-defined, continuous, strictly decreasing function on $(0, y_*)$. Let $I(y) = I(y_*)$ for $y \geq y_*$ if necessary.

Let $\tilde{x} = x/\mathbb{E}[\zeta]$. Note that $X = \tilde{x}$ is an admissible element of $\mathcal{A}(x)$.

Lemma 3.2. $U(\tilde{x}) > -\infty$ *if and only if* $V(x) > -\infty$.

Proof. Clearly, $V(x) \geq U(\tilde{x})$. Conversely, if $X \in \mathcal{A}(x)$ then $\mathbb{P}(X \leq \tilde{x}) > 0$, and if $U(\tilde{x}) = -\infty$ then $\mathbb{E}[U(X)] = -\infty$ and hence $V(x) = -\infty$. □

The next result, adapted from Karatzas et al (1991), describes the form of the optimal random variable $\hat{X}_T$. Note that if $V(x-) > -\infty$ then $U(\tilde{x}-) > -\infty$.

Lemma 3.3. *Suppose that* $V(x-) > -\infty$. *Then the optimal admissible element* $\hat{X} \equiv \hat{X}^x$ *is given by* $\hat{X}^x \equiv I(\chi\zeta)$, *where* χ *is a Lagrange multiplier chosen to satisfy* $x \equiv \mathbb{E}[\zeta I(\chi\zeta)]$.

Proof. For $w > 0$ define $\gamma(w) = \mathbb{E}[\zeta I(w\zeta)]$. Then $\gamma(w)$ is a continuous, decreasing function, which is well defined by Assumption 2.2. On $\{w : \gamma(w) > z_*\mathbb{E}[\zeta]\}$ the function γ is strictly decreasing, and we can define an inverse Γ defined on $(z_*\mathbb{E}[\zeta], \infty)$. It is simple to show that $\gamma(\infty) = \mathbb{E}[\zeta]I(\infty)$ so that if $V(x-) > -\infty$ or equivalently $U(\tilde{x}-) > -\infty$ then $\Gamma(x) < \infty$. If we define $\hat{X} \equiv I(\Gamma(x)\zeta)$ then $\hat{X} \in m\mathcal{G}$ and satisfies $\mathbb{E}[\zeta\hat{X}] = \gamma(\Gamma(x)) = x$, so that $\hat{X} \in \mathcal{A}_{\mathcal{G}}(x) \subseteq \mathcal{A}(x)$.

It remains to show that $\hat{X}$ is optimal. Observe that for any a and b, $U(b) \geq U(a) + (b-a)U'(b)$. For any $X \in \mathcal{A}(x)$

$$U(\hat{X}) \geq U(X) + (\hat{X} - X)U'(\hat{X}) = U(X) + (\hat{X} - X)\Gamma(x)\zeta \tag{3}$$

and this last term has non-negative expectation under $\mathbb{P}$. Hence $\mathbb{E}[U(\hat{X})] \geq \mathbb{E}[U(X)]$. □

Corollary 3.4. *For* $z > 0$, $\hat{X}^{x+z} \geq \hat{X}^x$, *almost surely.*

Remark 3.5. Assumption 2.2 is used to show that the functions γ and Γ are well defined and hence that $\hat{X}$ is optimal. This assumption can be weakened, see for example Kramkov and Schachermayer (1999), where the notion of 'reasonable asymptotic utility' is introduced and used to show that $\hat{X} = I(\chi\zeta)$ is still optimal.

Remark 3.6. In Karatzas et al (1991) the domain of admissible strategies is restricted to the class for which $\mathbb{E}[U(X)^-] < \infty$. In fact this is not necessary. If $X \in \mathcal{A}(x)$, then by definition $\mathbb{E}[(X\zeta)^+] < \infty$, and from (3), $U(X) \leq U(\hat{X}) + (X - \hat{X})\Gamma(x)\zeta$. It follows that

$$U(X)^+ \leq U(\hat{X})^+ + \Gamma(x)\zeta X^+ + \Gamma(x)\zeta\hat{X}^-$$

and we conclude that $\mathbb{E}[U(X)^+]$ is necessarily finite for all $X \in \mathcal{A}(x)$, and $\mathbb{E}[U(X)]$ is well defined.

Now we can prove the first main result.

Theorem 3.7. *Suppose* $-\infty < V(x-)$. *The quantity* $p(k)$ *is bounded above by* kh, *where* $h = \mathbb{E}[\zeta H]$.

Proof. If $h = \infty$ then there is nothing to prove.

Otherwise consider x such that $V(x-) > -\infty$. Let $\hat{X}^x$ denote the optimal solution to the control problem when $k = 0$. Suppose $z > kh$ and consider the optimal control problem (1) but with admissible random variables constrained to lie in $\mathcal{A}(x - z)$. Then, using $U(b) \leq U(a) + (b - a)U'(a)$,

$$\begin{aligned}
\mathbb{E}[U(X^{x-z} + kH)] - \mathbb{E}[U(\hat{X}^x)] &\leq \mathbb{E}[(X^{x-z} + kH - \hat{X}^x)U'(\hat{X}^x)] \\
&= \mathbb{E}[(X^{x-z} + kH - \hat{X}^x)\Gamma(x)\zeta] \\
&\leq \Gamma(x)(x - z + k\mathbb{E}[\zeta H] - x) \quad < \quad 0
\end{aligned}$$

Optimising over admissible random variables we find $\mathcal{V}(x - z, k) < V(x)$. In particular $p(k) \leq z$. Since $z > kh$ is arbitrary, it follows that $p(k) \leq kh$. □

Now we wish to investigate the way in which $p(k)$ depends on k. Clearly, if $V(x-) > -\infty$ and if $h < \infty$ then $p(k)$ is finite. More generally, it is possible to show that provided $V(x-) > -\infty$, then $p(k)$ exits in $[0, \infty)$ as long as $\mathcal{V}(x, k) < U(\infty)$. For the rest of this section we will be interested in $p(k)$ when k is small.

Lemma 3.8. *For* $k > 0$, $p(k)$ *is non-decreasing and* $p(k)/k$ *is non-increasing. In particular, if* $p(k)$ *exists in* $(0, \infty)$ *for any* k, *then* $0 < p(k) < \infty$ *for all* k. *Furthermore* $\lim_{k\downarrow 0+} p(k)/k = (D_+p)|_{k=0}$ *exists in* $(0, \infty]$.

Proof. The first statement is obvious. For the statement about $p(k)/k$ fix $0 < k < k'$, and assume $p(k) < \infty$ and $p(k') > 0$, else there is nothing to prove. Choose $z > p(k)$ and $u < p(k')$. Then, by the concavity of $\mathcal{V}$

$$\mathcal{V}(x - z, k) < \mathcal{V}(x, 0) \leq \frac{k}{k'}\mathcal{V}(x - u, k') + \frac{(k' - k)}{k'}\mathcal{V}(x, 0) \leq \mathcal{V}(x - ku/k', k).$$

Hence $z > ku/k'$, and since z and u are arbitrary it follows that $p(k)/k \geq p(k')/k'$. □

The next result shows that if k is sufficiently small then $p(k)$ is well defined.

Lemma 3.9. *If $-\infty < V(x-) \leq V(x) < U(\infty) \leq \infty$, and $h = \mathbb{E}[H\zeta] < \infty$ then for sufficiently small k, $\mathcal{V}(x - p(k), k) = V(x)$.*

Proof. First note that for any $X^x \in \mathcal{A}(x)$,

$$\mathbb{E}[U(X^x + kH)] \leq \mathbb{E}[U(\hat{X}^x) + (kH + X^x - \hat{X}^x)U'(\hat{X}^x)] = V(x) + \Gamma(x)hk$$

Hence $\mathcal{V}(x, k) < U(\infty)$ for sufficiently small k.

Again for sufficiently small k, $V(x - 2kh) > -\infty$ and hence

$$-\infty < V(x - 2kh) < \mathcal{V}(x - 2kh, k) < \mathcal{V}(x - kh, k) \leq V(x) < \mathcal{V}(x, k) < U(\infty).$$

By Lemma 3.1 $\mathcal{V}(z, k)$ is concave and strictly increasing in its first argument for $z \in [x - 2kh, x]$ so that $\mathcal{V}(x - q, k) = V(x)$ has a unique solution in the interval $[x - kh, x]$. □

By Lemma 3.8 we know that D_+p exists at $k = 0$, and by Theorem 3.7 we know that $(D_+p)|_{k=0}$ is bounded above by h, and that this bound is independent of the concave function U and the level x. We want to show that $(D_+p)|_{k=0} = h$.

Theorem 3.10. *Suppose $-\infty < V(x-) \leq V(x) < U(\infty)$. Then*

$$(D_+p)|_{k=0} = h = \mathbb{E}[\zeta H] \tag{4}$$

Proof. By assumption U' is continuous, which implies in turn that both I and γ are strictly decreasing, at least over suitable ranges, and hence Γ is continuous at x.

Suppose first that H is bounded. Consider the optimal control problem for fixed k and $X \in \mathcal{A}(x - kz)$. By concavity of U, $U(b) \leq U(a) + (b - a)U'(a)$ and for any admissible element

$$\begin{aligned} U(X^{x-kz} + kH) - U(\hat{X}^x) &\leq (X^{x-kz} + kH - \hat{X}^x)U'(\hat{X}^x) \\ &= (X^{x-kz} + kH - \hat{X}^x)\Gamma(x)\zeta \end{aligned}$$

Taking expectations, optimising over admissible random variables and dividing by k we find

$$\frac{\mathcal{V}(x - kz, k) - V(x)}{k} \leq \Gamma(x)\{\mathbb{E}[\zeta H] - z\}. \tag{5}$$

Conversely $U(b) \geq U(a) + (b - a)U'(b)$ and $\hat{X}^{x-kz}$ is admissible, so

$$\begin{aligned} \mathcal{V}(x - kz, k) - V(x) &\geq \mathbb{E}[U(\hat{X}^{x-kz} + kH)] - \mathbb{E}[U(\hat{X}^x)] \\ &\geq \mathbb{E}[(\hat{X}^{x-kz} + kH - \hat{X}^x)U'(\hat{X}^{x-kz} + kH)] \\ &= \mathbb{E}[(\hat{X}^{x-kz} - \hat{X}^x)U'(\hat{X}^{x-kz} + kH)] + k\mathbb{E}[HU'(\hat{X}^{x-kz} + kH)]. \end{aligned} \tag{6}$$

Now U' is a decreasing function so that $U'(\hat{X}^{x-kz} + kH) \leq U'(\hat{X}^{x-kz}) = \Gamma(x-kz)\zeta$. Note that $\Gamma(x-kz) < \infty$ for sufficiently small k by the assumption that $V(x-) > -\infty$. Further, by Corollary 3.4 $\hat{X}^{x-kz} - \hat{X}^x \leq 0$, so that the first term in (6) is bounded below by $\Gamma(x-kz)\mathbb{E}[\zeta(\hat{X}^{x-kz} - \hat{X}^x)] = -kz\Gamma(x-kz)$. Now consider the term $\mathbb{E}[HU'(\hat{X}^{x-kz} + kH)]$. If H is bounded then by the continuity of Γ and the dominated convergence theorem as $k \downarrow 0$ this converges to $\Gamma(x)\mathbb{E}[\zeta H]$. Collecting together our analysis of the two terms in (6) we deduce that

$$\lim_{k\downarrow 0+} \frac{\mathcal{V}(x-zk,k) - V(x)}{k} \geq \Gamma(x)\{\mathbb{E}[\zeta H] - z\}.$$

Combining this result with (5) we have

$$\lim_{k\downarrow 0+} \frac{\mathcal{V}(x-zk,k) - V(x)}{k} = \Gamma(x)\{\mathbb{E}[\zeta H] - z\}. \tag{7}$$

By Lemma 3.9, for sufficiently small k, $V(x) = \mathcal{V}(x-p(k),k)$ and so

$$\left(-\frac{\partial p(k)}{\partial k}, 1\right) \cdot \nabla\mathcal{V} = 0$$

provided the various quantities exist. By Lemma 3.8, $(D_+p)|_{k=0}$ exists, and so, given (7), we deduce (4).

Now suppose that H is not bounded. Let $H_n = H \wedge n$ and let $p_n(k)$ denote the solution of (2) for k units of the claim H_n. We have

$$D_+p|_{k=0} \geq D_+p_n|_{k=0} = \mathbb{E}[\zeta H_n] \uparrow \mathbb{E}[\zeta H].$$

But $p(k) \leq k\mathbb{E}[\zeta H]$ by Theorem 3.7 and so we deduce (4) for unbounded non-negative claims H.

Remark 3.11. The key result that makes the theorem work is the fact that Γ is a continuous function. In the proof this was achieved by assuming that U had a continuous first derivative. However, this is not a necessary assumption and if the state price density ζ is a continuous random variable then Γ may still be continuous even if the derivative of U has jumps.

4 Ask prices

In the previous section we defined and proved results for the solution of (1) and (2) for positive k. In economic terms these quantities give the bid price for the claim H. Now we consider negative k, which correspond to ask prices.

Firstly observe that we can trivially extend the definition of $\mathcal{V}(x,k)$ given in (1) to include negative k. Given this extension we can define $q(l)$ via $q(l) = \inf\{q : \mathcal{V}(x+q,-l) \geq V(x)\}$. Alternatively we can set $q(l) = -p(-l)$.

Since $\mathcal{V}$ is concave it is easy to deduce that $q(l) \geq p(l)$. Furthermore, the arguments leading to Theorem 3.7 still hold, so that under our main assumptions we have that lh is a lower bound on $q(l)$.

The corresponding results on the $D_+q|_{k=0} \equiv D_-p|_{k=0}$ require additional assumptions on the random variable H. For example, in Lemma 3.9 additional conditions are needed on H to ensure that $q(l)$ is finite. Essentially some condition is needed to ensure that $\mathcal{V}(x,-l) > -\infty$. As we show in Example 6.1, $\mathbb{E}[\zeta H] < \infty$ is not sufficient. However, if H is bounded, then $\lim_{l\downarrow 0+} q(l)/l = D_+q|_{l=0} = \mathbb{E}[\zeta H]$. The proof of this result proceeds almost exactly as in the proof of Theorem 3.10, apart from the fact that it is not possible to extend from bounded claims to unbounded claims.

In the case where the marginal bid and ask prices both exist and agree with each other, then the marginal price is sometimes called the Davis price.

5 The non-traded assets problem

Having defined various concepts and stated the main results we now wish to describe the fundamental example that we have in mind. This example is a model with a non-traded asset, sometimes called a model with basis risk. In its simplest version the financial market consists of a bond with price B growing in deterministic fashion at rate r, and two further assets with prices S and Y described by exponential Brownian motions. The dynamics are given by

$$dB_t = B_t r dt, \qquad dS_t = S_t(\sigma dW + \mu dt), \qquad dY_t = Y_t(\alpha dZ + \beta dt), \tag{8}$$

with $B_0 = b, S_0 = s$, and $Y_0 = y$. The Brownian motions W and Z are correlated with correlation ρ, and we write $dZ = \rho dW + \overline{\rho} dW^{\perp}$ where $\overline{\rho} = \sqrt{1-\rho^2}$ and $W^{\perp}$ is a Brownian motion which is orthogonal to W.

The problem facing our agent is to price a contingent claim on the non-traded asset Y, payable at some fixed time-horizon T, given that she is only able to hedge using the traded asset S and the bond B. We need to describe the state-price densities and admissible trading strategies for the problem. We begin by describing the σ-algebras. Let $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the filtration generated by the pair of Brownian motions W and $W^{\perp}$, and let $\mathcal{F} = \mathcal{F}_T$. Let $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ and $\mathcal{G} = \mathcal{G}_T$ be generated by W alone.

The filtration $\mathbb{G}$ is generated by the asset S. In isolation the asset S and bond B form a Black-Scholes market. If the other perfect market assumptions of the Black-Scholes model are satisfied then this market is complete and there are unique preference independent prices for options on the asset S. Each option can be replicated by a dynamic hedging strategy.

The asset Y plays the role of a non-traded asset. The fundamental problem is to price a contingent claim on Y. If hedging on Y is allowed then the market is again complete and there are unique preference independent prices for all $\mathcal{F}_T$ measurable claims. We are interested instead in the situation where hedging using Y is not possible.

Since Y is not traded there are many martingale measures for this problem. Let $\lambda = (\mu - r)/\sigma$ and let

$$\xi_t^\psi = \exp(-rt)\exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right)\exp\left(\int_0^t \psi_u dW_u^\perp - \frac{1}{2}\int_0^t \psi_u^2 du\right)$$

for some $\mathcal{F}_t$-adapted process ψ such that $\int_0^T \psi_u^2 du < \infty$, and such that the final exponential is a martingale. If we identify ξ^ψ with ξ_T^ψ then $\mathcal{Z} = \{\xi^\psi\}$ is the set of state-price densities. Finally let $\zeta = \xi^0$ (both as processes and random variables on $\mathcal{F}_T$). Then $\zeta \in m\mathcal{G}^+$ and Assumption 2.1(a) is satisfied. The quantity ζ plays the role of the unique state price density in the reduced complete model $(\Omega, \mathcal{G}, \mathbb{P})$. Note that $\zeta_t S_t$ is a $\mathbb{P}$-martingale.

We call ζ the *minimal* state price density. This is because ζ is related to the minimal martingale measure of Föllmer and Schweizer (1991). The minimal martingale measure changes the drifts of the Brownian motions driving the traded assets to make the discounted prices of the traded assets into martingales. It leaves unchanged the drifts of any Brownian motions which are orthogonal to the Brownian motions driving the traded assets. Hence in our setting $W_t + \lambda t$ and $W_t^\perp$ are Brownian motions under the minimal martingale measure. Note that under Assumption 2.1, ζ is also the minimal entropy measure and, more generally, the minimal distance martingale measure in the sense of Goll and Ruschendorff (2001).

Now we consider the space of admissible strategies. In the absence of any contingent claim the agent seeks to maximise the expected utility of wealth at time T. The space $\mathcal{A}_\mathcal{G}(x) = \{X \in m\mathcal{G}_T : \mathbb{E}[X\zeta] \leq x\}$ of attainable wealths satisfying the budget constraint in the reduced model can be rewritten as

$$\mathcal{A}_\mathcal{G}(x) = \{X : X = X_T - C; C \in m\mathcal{G}_T^+, X_t = \zeta_t^{-1}\left(x + \int_0^T \psi_t dW_t\right);\ \zeta_t X_t \text{ a martingale }, \psi_t \in m\mathcal{G}_t\}.$$

This is the space of terminal wealths with can be generated using a dynamic trading strategy involving investments in S alone. (Note that $dW_t = d(\zeta_t S_t)/(\sigma - \lambda)$, so that the integral in the above definition is a discounted gains from trade.) The dynamic strategy is adapted to the filtration generated by S, and chosen such that the discounted gains from trade are a martingale. This rules out doubling strategies. At the final time-point some wealth may be discarded.

It remains to specify the space of admissible wealths $\mathcal{A}(x)$. The concept that we wish to represent is the idea that an agent should not be allowed to trade on Y, but she should be allowed to use information about the current value (and past history of Y) in determining how much to invest in the asset S. This means that the integrand driving the gains from trade need not be adapted to $\mathcal{G}_t$, but rather should be

$\mathcal{F}_t$-adapted. Given the representation of $\mathcal{A}_{\mathcal{G}}(x)$ it is natural to define

$$\mathcal{A}(x) = \{X : X = X_T - C; C \in m\mathcal{F}^+, X_t = \zeta_t^{-1}\left(x + \int_0^T \psi_t dW_t\right);\ \zeta_t X_t \text{ a martingale}, \psi_t \in m\mathcal{F}_t\}.$$

With this representation it follows that Assumption 2.1(b) is satisfied.

Remark 5.1. It is also possible to view the model (8) as a restriction of a complete market model in which both S and Y are traded assets. (The restriction is that the class of strategies is reduced from investments in both risky assets to investments in S alone.) Under market completion (i.e. if trading in Y is allowed) there is a unique state-price density given by $\xi^{\hat{\psi}}$ where $\hat{\psi} = -(\beta - r)/\alpha$ and the unique fair price of a European contingent claim with payoff $H(S_T, Y_T)$ at T is given by

$$\mathbb{E}[H(S_T, Y_T)\xi^{\hat{\psi}}].$$

There are no general relationships comparing the utility indifference price $p(k)$ (defined in the non-traded assets model where Y is not traded) with $k\mathbb{E}[H(S_T, Y_T)\xi^{\hat{\psi}}]$ (the complete market price when Y is traded). For example, if $H = Y_T$ then $\mathbb{E}[Y_T \xi^{\hat{\psi}}] \equiv y$ whereas

$$\lim_{k \downarrow 0} \frac{p(k)}{k} = \mathbb{E}[Y_T e^{-rT} e^{-\lambda W_T - \lambda^2 T/2}] = y e^{-\lambda \alpha \rho T}.$$

Thus the marginal utility indifference price in the non-traded assets model can be larger or smaller than the unit price under market completion, depending on the sign of $\lambda\alpha\rho$. In contrast, the main results of this paper refer to the complete market generated by S alone which exists as a subset of the incomplete model.

Remark 5.2. The definition of admissible wealths we give above is perfectly natural, and ideally suited to providing the proofs of the main results, but is not the only definition of an admissible strategy used in the literature on incomplete markets. Consider for example the notion of an *acceptable strategy* from Delbaen and Schachermayer (1997). In the complete market setting their definition is slightly more restrictive than the one used here, in that the space of random variables C which may be discarded at time T is not necessarily the set of all positive random variables. (The reason for this is that Delbaen and Schachermayer want to be able to consider holding the same strategies long and short.)

Essentially the definition of an acceptable strategy in Delbaen and Schachermayer (1997) is that it should generate a wealth process which dominates as a process a *maximal admissible* strategy where a maximal admissible strategy is a gains from trade process which is a true martingale under *some* equivalent martingale measure. In contrast, the definition we give above is related to the idea that our admissible strategies dominate gains from trade processes which are true martingales under the *minimal* martingale measure. Note that in the complete market setting these two definitions are equivalent.

6 Examples

6.1 Exponential Utility

Consider the family of utility functions with constant absolute risk aversion. These utilities are defined for negative wealth and take the form

$$U(x) = U_\theta(x) = -\frac{1}{\theta}e^{-\theta x}, \qquad x \in \mathbb{R},$$

with θ a positive parameter. For the dynamics given in Section 5 we find that the value function of the agent is given by

$$V(x) = -\frac{1}{\theta}\exp\left(-\theta x e^{rT} - \frac{\lambda^2 T}{2}\right), \qquad x \in \mathbb{R}.$$

For this utility $U'(x) = e^{-\theta x}$ is continuous and tends to zero for large x. The inverse function I is given by $I(y) = -(\ln y)/\theta$. It follows that Assumption 2.2 is satisfied and $\gamma(w) = (-1/\theta)\mathbb{E}[\zeta(\ln w + \ln\zeta)] = -e^{-rT}(\ln w - rT + \lambda^2 T/2)/\theta$. We find $\Gamma(x) = \exp(-\theta x e^{rT} + rT - \lambda^2 T/2)$.

Suppose the claim takes the form $g = g(Y_T)$. Then the utility indifference bid price for k units of the claim is given in Henderson and Hobson (2002b) or Henderson (2002) as

$$p(k) = -\frac{1}{\theta(1-\rho^2)}\ln\mathbb{E}\left[\zeta\exp\left(-k\theta(1-\rho^2)g(Y_T)\right)\right]. \tag{9}$$

Note that the price does not depend on x since wealth factors out of this problem. For g a non-negative claim, and $k > 0$, it follows from Jensen's inequality that $p(k) \leq k\mathbb{E}[\zeta g(Y_T)]$. Further, on differentiation we find

$$D_+ p|_{k=0} = \mathbb{E}[\zeta g(Y_T)].$$

Hence $\mathbb{E}[\zeta g(Y_T)]$, which is independent of the risk aversion parameter θ, is both an upper bound on the bid price for the claim and the marginal bid price.

Now suppose that we are interested in the ask price. A formula for the ask price is obtained by replacing k with $-k$ in the right hand side of (9) and multiplying by -1. However if $g(Y_T) = Y_T$ then the exponential moment is infinite, and the utility indifference ask price is also infinite. Hence, trivially, $\mathbb{E}[\zeta Y_T]$ is a lower bound on the ask price for the claim, but it does not represent the marginal ask price.

6.2 Power Law Utility

For a positive risk aversion parameter R consider the utility function

$$U(x) = U_R(x) = \frac{x^{1-R}}{1-R}, \qquad x \geq 0, \tag{10}$$

with $U(x) = -\infty$ for negative x. The value function for the agent with no endowment is given by

$$V(x) = \frac{x^{1-R} e^{(1-R)rT}}{1-R} \exp\left\{ \frac{(1-R)\lambda^2 T}{2R} \right\},$$

with $V(x) = -\infty$ for $x < 0$.

The inverse to U' is given by $I(y) = y^{-1/R}$ and $\gamma(w) = w^{-1/R}\mathbb{E}[\zeta^{1-(1/R)}] = w^{-1/R}e^{(1-R)rT/R}\exp\{(1-R)\lambda^2/(2R^2)\}$ and for each R, Assumption 2.2 holds.

In general there are no closed form expressions for options prices. However Henderson (2002, Theorem 4.2) gives an expansion for the price in powers of the number of options bought. Suppose the wealth x is positive. If the claim is units of the non-traded asset, then for $k \geq 0$ the price is given as

$$p(k) = k\mathbb{E}[\zeta Y_T] - k^2 e^{-rT} \frac{y^2}{x} \frac{\alpha^2 R(1-\rho^2)}{2\Lambda} e^{2(\beta - \alpha\rho\lambda)/\sigma)T}[e^{\Lambda T} - 1] + O(k^3)$$

where $\Lambda = \alpha^2 - \frac{2\rho\alpha\lambda}{R} + \frac{\lambda^2}{R^2}$. For a more general claim $g = g(Y_T)$ the price to leading order is $p(k) = k\mathbb{E}[\zeta g(Y_T)] + O(k^2)$. In either case the marginal bid price is $\mathbb{E}[\zeta g(Y_T)]$ which is independent of the risk aversion parameter R. As for exponential utility the marginal ask price can be infinite for unbounded claims.

Note that if we take $R = 1$ then we recover the formulæ for logarithmic utility.

6.3 A counterexample for which the bounds are not attained.

Consider an agent with power-law utility function given by (10) with $R < 1$. Suppose that this agent has zero initial wealth and $r = 0$. We consider the bid price of this agent for k units of the claim Y_T where $Y_T = \exp(\alpha Z_T + (\beta - \alpha^2/2)T)$, where for simplicity Z is independent of the Brownian motion driving the traded asset.

In this case $V(0) = 0$. If the agent bids any positive amount p for the claim then there is a positive probability that the agent has negative final wealth, and therefore her utility is $-\infty$. However, even if the agent follows the strategy of investing zero in the traded asset, her value function is

$$\mathbb{E}[U(k\exp(\alpha B_T + (\beta - \alpha^2/2)T))] = \frac{k^{1-R}}{1-R} \exp\left(\beta(1-R)T - \alpha^2 \frac{R(1-R)}{2} T \right) > 0.$$

Hence there is no solution to the equation $\mathcal{V}(-p, k) = 0$. For this problem the bid price for the claim is zero, as is the marginal bid price, which is not equal to $h = \mathbb{E}[\zeta Y_T]$. The utility indifference price is not defined. However in this case $V(0-) = -\infty$ so that the hypotheses of the theorems are not satisfied.

7 'Almost complete' stochastic volatility models

In this section we consider a completely different situation to the main example of Section 5. Here we suppose that the triple (B, S, Y) consists of a bond, a traded asset, and a process which governs the stochastic volatility of that asset.

Consider the model

$$dB_t = B_t r dt, \qquad \frac{dS_t}{S_t} = \sigma(Y_t, t)dW + (r + \lambda_t \sigma(Y_t, t))dt, \qquad dY_t = \alpha(Y_t, t)dZ + \beta(Y_t, t)dt, \quad (11)$$

where, as before, $dZ_t = \rho dW_t + \rho^{\perp} dW_t^{\perp}$. This is a standard stochastic volatility model, see, for example, Hobson (1998) for a review and a list of popular parametric forms for α, β etc. As written, the process Y driving the volatility is an autonomous diffusion, but the results below remain valid even if α, β, ρ and σ are arbitrary adapted functions. The drift of the traded asset is $(r + \lambda_t \sigma(Y_t, t))$ so we have parameterised the dynamics of S in terms of the Sharpe ratio λ_t.

Again S plays the role of a traded asset. This time Y is associated with the volatility of S rather than being a second financial asset, but again Y is not traded. The state-price densities for the model take the form

$$\xi_T^{\psi} = e^{-rT} \exp\left(-\int_0^T \lambda_t dW_t - \frac{1}{2}\int_0^T \lambda_t^2 dt \right) \exp\left(\int_0^T \psi_t dW_t^{\perp} - \frac{1}{2}\int_0^T \psi_t^2 dt \right).$$

The model is incomplete and there are no unique preference-independent option prices. Indeed Frey and Sin (1999) show that in general there are no non-trivial bounds on the prices of call options in this model. In other words the set of prices which are consistent with some risk neutral pricing measure is the interval $(0, S_0)$.

In general, in a stochastic volatility model there is no non-trivial utility-independent bound on the utility indifference bid price of an option. However, suppose that the Sharpe ratio is deterministic, $\lambda_t = \lambda(t)$. (In the mathematical literature, a stochastic volatility model with this property is said to be 'almost complete', see Pham *et al* (1998).) Then we can verify that Assumption 2.1(a) is satisfied, and under appropriate modeling assumptions we can conclude that Theorems Aa and Ba hold. In particular, the marginal utility indifference bid price for a contingent claim H does not depend on the utility function of the agent and is given by $\mathbb{E}[\zeta H]$, where $\zeta = \xi_T^0$.

8 Concluding Remarks

In an incomplete market there are no unique, preference independent option prices. Instead it is necessary either to choose a pricing measure from the family of equivalent martingale measures, or to model the

the preferences directly. Utility indifference option pricing is a consistent pricing scheme which reduces to Black-Scholes option pricing in a complete market. Prices are non-linear, so that in general agents will pay a lower unit price for larger quantities of a financial asset. These larger quantities are associated with higher risk.

The perceived disadvantage of the utility based option pricing approach is that since the price depends on the choice of utility function, it seems unlikely that there are any general pricing principles which hold uniformly across all utility functions, and for all initial wealths. In our non-traded asset model we have shown that this is not the case, and that there is a bound for the utility indifference bid and ask price for the option which is valid for all sufficiently regular utility choices. Furthermore, this bound also acts as the marginal price.

If the bid price for every agent is below the ask price for every agent then under no circumstances would any trading occur. Hence if all agents have zero initial endowment of Y, and all agents have the same physical measure $\mathbb{P}$, then no trading in a claim with payoff $H(Y)$ would occur, even if the agents have different initial wealths or utility functions. Trading only occurs if the agents have initial endowments of non-traded asset, or if they have different expectations of Y.

The results of this paper should be contrasted with the results of Hubalek and Schachermayer (2001). They consider the specific example we introduced in Section 5, and conclude that for a call option on the non-traded asset, every positive price is consistent with some equivalent martingale measure. The reason for this is that although a change of measure leaves the discounted price of a traded asset as a martingale, the drift on the non-traded asset is undetermined: there are martingale measures for which the non-traded asset has arbitrarily large drift (positive or negative). As a consequence the expected payoff of the call option can be made arbitrarily large or small with a judicious choice of pricing measure, and no-arbitrage arguments give only trivial bounds on the price of a contingent claim.

We reach an opposite conclusion: for an agent who seeks to maximise expected utility, the marginal bid price for a claim on the non-traded asset is uniquely specified. One reason why we get this uniqueness is that we assume that the agent has zero initial endowment of the non-traded asset. If we were to relax this assumption then the uniqueness would be lost. Note that there is no role in the no-arbitrage arguments of Hubalek and Schachermayer for the agent's initial endowment of the non-traded asset, and hence they must obtain wider bounds. However, when the assumption of zero initial endowment is appropriate, our conclusion is very strong, and the only candidate for the marginal utility indifference price of the claim is the discounted expected value under the minimal martingale measure.

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