Complete Models with Stochastic Volatility

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Abstract

The paper proposes an original class of models for the continuous time price process of a financial security with non-constant volatility. The idea is to define instantaneous volatility in terms of exponentially-weighted moments of historic log-price. The instantaneous volatility is therefore driven by the same stochastic factors as the price process, so that unlike many other models of non-constant volatility, it is not necessary to introduce additional sources of randomness. Thus the market is complete and there are unique, preference-independent options prices.

We find a partial differential equation for the price of a European Call Option. Smiles and skews are found in the resulting plots of implied volatility.

Keywords: Option pricing, stochastic volatility, complete markets, smiles.

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1 Stochastic Volatility

The work on option pricing of Black and Scholes (1973) represents one of the most striking developments in financial economics. In practice both the pricing and hedging of derivative securities is today governed by Black-Scholes, to the extent that prices are often quoted in terms of the volatility parameters implied by the model.

The Black-Scholes model is based upon the common assumption that the proportional price changes of the asset form a Gaussian process with stationary independent increments. This assumption has been the subject of much attention over the intervening years. An insight of the Black-Scholes model is that the crucial parameter is the volatility of the underlying price process: consequently research has focussed on this parameter. Empirical analysis of stock volatility has shown that it is *not* constant — see Blattberg and Gonedes (1974), Scott (1987) and the references therein. Moreover the prices at which derivatives (and especially call options) are traded are inconsistent with a constant volatility assumption.

For this reason a number of authors have suggested variants of the Black-Scholes model. These alternative theories separate into two broad genres. (Since we are interested in models of changing volatility we exclude models with jumps such as the jump-diffusion model of Merton (1976).) The first approach, as represented by Cox and Ross (1976), Geske (1979), Rubinstein (1983) and recently Bensoussan *et al* (1994) describes the stock price as a diffusion with level dependent volatility. This may either be a modelling assumption or follow from more fundamental properties, for example by relating stock price to the value and the debt of a firm. The second approach, exemplified by Johnson and Shanno (1987), Scott (1987), Hull and White (1987, 1988) and Wiggins (1987) defines the volatility as an autonomous diffusion driven by a second Brownian motion. (The asset price process is driven by the first Brownian motion.) Further details of these two approaches, which we label as level-dependent volatility and stochastic volatility, are given in Sections 2.1 and 2.2 respectively. Hull (1993) and Merton (1990) are also a valuable source of reference.

In this paper we suggest a new class of non-constant volatility models, which can be extended to include the first of the above classes, but also share many characteristics with the second approach. The volatility is non-constant, but it is an endogenous factor in the sense that it is defined in terms of the past behaviour of the stock price. This is done in such a way that the price and volatility form a multi-dimensional Markov process.

We believe that this class of models has twin advantages over existing stochastic volatility models. Firstly, unlike the stochastic volatility models, since no new sources of randomness are introduced, the market remains complete and there are unique preference-independent prices for contingent claims. Secondly, in contrast with the level-dependent volatility models, it is possible to specify a single, simple model within the new class which over time will exhibit smiles and skews of different directions.

A natural effect of the model is to make volatility self-reinforcing. Since volatility is defined in terms of past behaviour of the asset price it will be high precisely when there have been large movements in the recent past. This is designed to reflect real world perceptions of market volatility, particularly if practitioners are to compare historic volatility with implied volatility.

An observed effect in options markets is the presence of 'smiles' and 'skews' in the implied volatilities across strikes. Rubinstein (1985) has demonstrated the presence of this phenomena in options data and noted moreover that the direction of the skew may change over time. Similarly Fung and Hsieh (1991) discovered smiles in the prices of options based on underlying securities in each of the stock, bond and currency markets. Fung and Hsieh also ask whether implied volatility is a better predictor of realised volatility over the lifetime of an option than historic volatility.

A potential explanation for the presence of smiles and skews is the concept of non-constant or stochastic volatility. Hence it is natural to investigate the implied volatility implications of any model of non-constant volatility. Such analysis has been attempted for the Constant Elasticity of Variance model of Cox and Ross by Beckers (1980) and for the stochastic volatility model of Hull and White by Stein and Stein (1991) and Paxson (1994). In this paper we show that the proposed class of volatility models has the desirable feature of explaining smiles and skews.

There are similarities between the proposed class of stochastic volatility models and the $ARCH$ (Engle (1982)) and $GARCH$ (Bollerslev (1986)) models favoured for financial time series modelling by econometricians. These models are formulated in discrete time and postulate a log-price process for the stock which has a conditional variance depending on a set of exogenous and lagged endogenous variables and past residuals. See Duan (1995) for a discussion of $GARCH$ models and option pricing.

The remainder of this paper is structured as follows. Section 2 describes the set of existing models with level-dependent or stochastic volatility and outlines their implications for option pricing. The new model itself is introduced in Section 3. The penultimate section describes the implications of this model for option pricing and demonstrates the existence of skews and smiles via numerical solution of the option pricing partial differential equation. The direction of the skew is seen to be a function of the recent history of the asset price process. The final section provides a summary of the theoretical findings and a comparison with competing models. The task of comparing the predictions made by the new class of volatility models with market experience is left to a subsequent paper. Finally, in an appendix, we discuss some of the technical results that we require to ensure that when we switch to a martingale measure, the new (pricing) measure is equivalent to the original (real world) measure. Our measures arise as the laws of solutions of SDEs and the conditions for equivalence are of interest in their own right.

2 Non-constant Volatility Models

The ubiquitous Black-Scholes pricing formula for stock options assumes that the price $(P_t)_{t\leq T}$ of a stock is the solution to a stochastic differential equation (SDE)

$$dP_t = P_t(\sigma dB_t + \mu dt) \tag{1}$$

where σ is a known and constant volatility parameter and B is a Brownian motion.

Notwithstanding the widespread use of the Black-Scholes formula, concern has been expressed about some of the assumptions necessary for the derivation of the formula. The most criticised of these assumptions are the requirement of a perfect frictionless market (and in particular the absence of transaction costs) and the imposition of a constant volatility. It is exclusively the second of these assumptions that we address here.

The basic financial environment in which our asset trades consists of the stock with price process P_t and a bond which pays a fixed and constant rate of interest r. There is a perfect market with no transaction costs and no restrictions on short selling of stock or bond provided that the net wealth of the trader remains non-negative. In particular a trader may sell stock or bonds that he does not own provided that by the end of the trading period he has repurchased sufficient quantities to cover his obligations.

2.1 Level Dependent Volatility

There is a long history of alternative models for the stock price process in which P_t satisfies the SDE

$$\frac{dP_t}{P_t} = \sigma_P(P_t)dB_t + \mu dt \tag{2}$$

where σ_P is a function of P. For example in their Constant Elasticity of Variance (CEV) model Cox and Ross (1976) take $\sigma_P(x) \equiv \sigma x^{-(1-\alpha)}$. This is a diffusion model for the price process and following Harrison and Pliska (1981) the model is complete with unique, preference-independent option prices. In general these prices may only be known as the solution to a partial differential equation.

Cox and Ross suggest (2) in direct competition to the exponential Brownian motion model, suggesting that the form of the equation may represent leverage effects. Geske (1979), Rubinstein (1985) and Bensoussan *et al* (1994) derive explicit forms for (2) by reference to the value and debt of the firm.

Suppose the aim is to price a European Call option with exercise date T and strike price K. By considering a riskless portfolio consisting of the stock call option, asset and cash, standard arguments show that the value $H = H(P_t, T-t)$ of the call is the solution to

$$0 = rP\frac{\partial H}{\partial P} - rH - \frac{\partial H}{\partial t} + \tfrac{1}{2}(P\sigma_P)^2\frac{\partial^2 H}{\partial P^2} \tag{3}$$

with boundary condition

$$H(P,0) = (P-K)^+.$$

Beckers (1980) investigates the solution to (3) for the Constant Elasticity of Variance model of Cox and Ross. Compared with standard Black-Scholes prices, options prices for the CEV model are higher for in-the-money options and lower for out-of-the-money options. Equivalently implied volatilities decrease as the strike of the option increases.

Geske (1979) suggests modelling stock prices as an option on the value of a firm with debt. If the value V_t of the firm is given by an exponential Brownian motion, (so that it has constant volatility σ_V), then the stock-price P_t is given by the Black-Scholes formula with the strike set equal to the value of the debt D, and the exercise date set equal to the maturity date of the debt T_D:

$$P_t = V_t\Phi\left(\frac{\log(V_t/D) + (r + \frac{1}{2}\sigma_V^2)(T_D - t)}{\sigma_V\sqrt{T_D - t}}\right)$$

$$-De^{-r(T_D-t)}\Phi\left(\frac{\log(V_t/D)+(r-\frac{1}{2}\sigma_V^2)(T_D-t)}{\sigma_V\sqrt{T_D-t}}\right);$$

where $\Phi(.)$ denotes the cumulative normal distribution. In practice we infer V from the observables P, D and T_D. The volatility of the stock price is then

$$\begin{aligned}\sigma_P &\equiv \sigma_V\frac{V_t}{P_t}\frac{\partial P_t}{\partial V}\\ &= \sigma_V\frac{V_t}{P_t}\Phi\left(\frac{\log(V_t/D)+(r+\frac{1}{2}\sigma_V^2)(T_D-t)}{\sigma_V\sqrt{T_D-t}}\right)\\ &> \sigma_V.\end{aligned}$$

The value of a call option in this model is given by the solution of (3) with this new specification of σ_P. If $D=0$ or $T_D=\infty$ then the option price collapses to the usual Black-Scholes result. Otherwise the effect is to modify the Black-Scholes stock option price: if for example the stock price falls then stock volatility rises and the Geske option price rises relative to the Black-Scholes price, although it still falls in absolute terms.

Figure 1 shows the implied volatility curves for the CEV model and the Geske model. The impled volatility is that value for the volatility, which, when substituted into the Black-Scholes formula gives the options price as calculated from the alternative model. For each alternative model it is seen that the implied volatility falls as the strike price rises. Formulæ for the prices of call options are taken from Schroder (1989) and Geske (1979). As Schroder points out $\alpha = 2/3$ is a special case of the CEV model for which the prices of call options can be expressed in terms of normal distributions.

Rubinstein (1985) analysed options price data to see if there was any systematic variation of implied volatility with strike. For one of the periods under consideration he found that implied volatility did indeed decrease as the option moved out-of-the-money. However in a later period the implied volatility increased as the strike increased. The level-dependent volatility models of Cox and Ross and Geske are unable to explain this change over time in the direction of the implied volatility skew.

In a pair of innovative articles Dupire (1993, 1994) incorporates options price data into the specification of a level dependent volatility model. Rather than postulate the dynamics of the asset price process he uses options price data to infer the form of the volatility so that the model is guaranteed to price European call options consistently with the market. In spirit this model is similar to term structure models of interest rates.

Dupire assumes that the asset price is a diffusion (and thus that the volatility of the asset depends on the price level and time alone), and that the price of an option is the

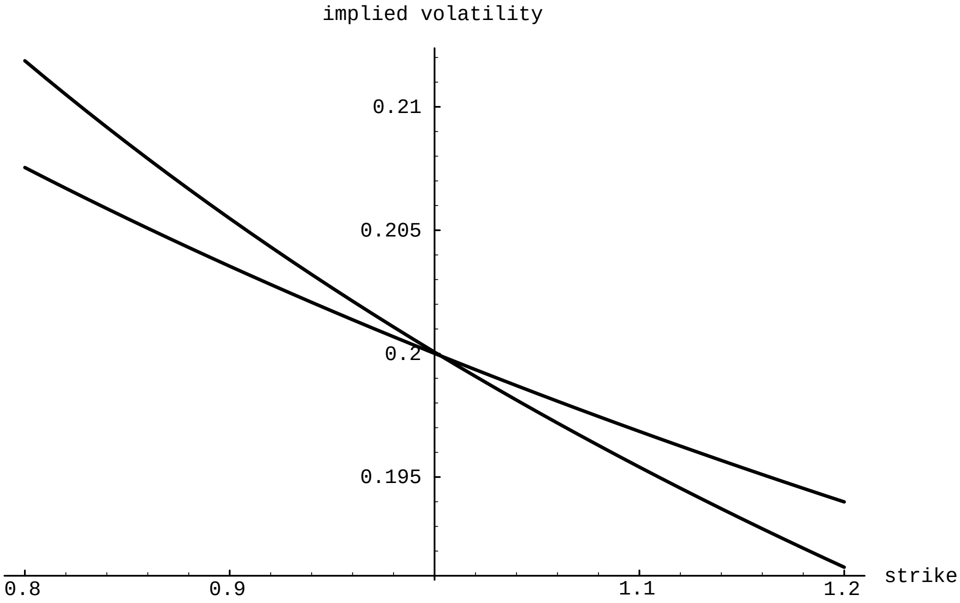

Figure 1: Implied Volatilities for the CEV and Geske models. In each case the option is a European Call with time to exercise $T = 0.25$, and the initial price is unity. For the CEV model (less steep curve) $\sigma = 0.2$ and $\alpha = 2/3$; for the Geske model $T_D = 0.25$, $D = 1$ and the volatility of the value of the firm is 0.1.

discounted expected value of its payoff under some equivalent martingale measure. He shows that if there is a continuum of market prices for European calls of every potential strike, and for every potential exercise date then this uniquely specifies the volatility. However this model cannot *explain* smiles or skews, since they are taken as inputs.

Platen and Schweizer (1995) have developed a further model in this category. They postulate a model in which a non-constant level-dependent volatility arises endogenously. They assume that market traders have portfolios consisting of both the underlying asset and option liabilities which they must dynamically hedge. The effect of the hedging requirements is to affect the asset volatility. As in this paper the authors derive a partial differential equation for an option price which they solve numerically.

2.2 Stochastic Volatility via an SDE

Several authors have proposed models of volatility in which the volatility is defined via a stochastic equation. The following model is due to Hull and White (1987). For related papers see Johnson and Shanno (1987), Scott (1987), Wiggins (1987) and Hull and White (1988). Hofmann *et al* (1993) consider a more general Markovian model which contains the model of Hull and White as a special case.

Let the stock price P_t and the volatility σ_t be defined via the pair of stochastic differential equations

$$dP_t = P_t(\sigma_t dB_t + \mu(P_t, \sigma_t, t)dt) \tag{4}$$

$$dV_t = V_t(\delta(\sigma_t, t)dW_t + \gamma(\sigma_t, t)dt) \tag{5}$$

where $V_t \equiv \sigma_t^2$ and B and W are Brownian motions with covariance $dBdW = \varrho dt$, for some correlation $-1 < \varrho < 1$.

If the volatility is a traded asset then there are two risky securities and the market is complete. Otherwise the introduction of the second Brownian motion W makes this model incomplete. In particular there are no unique prices for stock options.

It is feasible to consider the above model with $|\varrho| = 1$. In this case completeness is regained, but the volatility becomes a complicated function of the history of the price process.

Now consider the option pricing implications of these stochastic volatility models. Our analysis follows Scott (1987). Suppose that the aim is to price a European call option with exercise date T and strike price K. Then by considering a portfolio consisting of the riskless bond, the asset and two call options with different maturities, Scott shows that the option pricing function $H \equiv H(P_t, V_t, T-t)$ must satisfy

$$-H_t + \tfrac{1}{2}vp^2 H_{pp} + \varrho\delta v^{3/2} p H_{pv} + \tfrac{1}{2}\delta^2 v^2 H_{vv} - rH + rpH_p = bH_v,$$

subject to the boundary condition

$$H(P, V, 0) = (P-K)^+.$$

Here $b \equiv b(P_t, V_t, t)$ is independent of the exercise date T.

The function b cannot be deduced from arbitrage considerations alone. Consequently there is no unique option pricing function.

Although there is no unique price for the option some authors have suggested particular choices. When $\varrho = 0$ the choice $b = -\gamma V$ corresponds to setting the market price of risk to be zero (see for example Stein and Stein (1991); Wiggins (1987) builds on the equilibrium approach of Cox, Ingersoll and Ross (1985) to provide some justification). It is also equivalent to pricing options under the minimal martingale measure of Föllmer and Schweizer (1990). Suppport for this idea is to be found in Hofmann *et al* (1993). However other authors propose different criteria for determining b. For a utility based approach see for example Karatzas *et al* (1991) and Duffie and Skiadas (1994).

The implications for options pricing of a stochastic volatility model have been considered by many authors. Stein and Stein (1991), who assume no correlation between the pair of Brownian motions driving the asset price and the volatility, find implied volatility smiles. By allowing a negative correlation between the asset and the volatility Wiggins (1987) shows that implied volatility may be higher for in-the-money options than out-of-the-money options. Both he and Scott (1987) find that, when considered across a range of strikes, options prices from models with stochastic volatility provide a superior fit to market prices when compared with a constant volatility model. Paxson (1994) is able to conclude that, in most but not all cases, stochastic volatility can account for both smiles and skews. On a theoretical note, Renault and Touzi (1992) have shown that under a zero correlation ($\varrho = 0$) assumption a stochastic volatility model must exhibit volatility smiles in the option price.

2.3 *GARCH* Models

The acronym *GARCH* stands for generalised autoregressive conditional heteroskedastic and describes a popular class of discrete time models used to model time series with non-constant volatility. In discrete time the *GARCH*(1,1) model for the log-price process Z_t and conditional variance σ_t is of the form

$$(6) \qquad \begin{aligned} Z_t &= Z_{t-1} + \mu_t + \sigma_t \epsilon_t \\ \sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \alpha \epsilon_{t-1}^2 \sigma_{t-1}^2 \end{aligned}$$

Here ϵ_t is an *i.i.d.* sequence of zero mean unit variance random variables. If μ_t is a function of σ_t then the model is *GARCH*(1,1)-M. More generally *GARCH* models allow for σ_t to be an arbitrary function of past conditional variances and past residuals.

GARCH models generate data with fatter tails than those from a model where σ is constant in (6). This is consistent with many studies on observed stock prices.

Whilst *GARCH* models may capture essential qualitative properties of an asset price process, they are an unsuitable class of models for a prospective option replicator since, save in the simplest binomial cases, exact replication is infeasible in discrete time. In general there is no natural continuous time analogue of the discrete time *GARCH* process. With judicious choice of parameter values Nelson (1990) has demonstrated convergence in the Skorokhod topology to a continuous-time process similar to the model described in Section 2.2; however if this limit process is sampled at equally spaced discrete time points then the resulting process is *not GARCH*. Recently however a new class of *weak ARCH* models has been proposed by Drost and Nijman (1993). These models have the embeddibility property that if a weak ARCH process is sampled at regular intervals then the resulting process is again weak *ARCH*.

Kind *et al* (1991) proved a convergence result for a different stochastic volatility model. In their *GARCH*-type model the quadratic variation of the stock has an interpretation as the 'historic' volatility defined over a finite time window. Unfortunately in the continuous time limit the volatility process is deterministic. In either case the *GARCH* model fails to yield preference independent option pricing in the presence of stochastic volatility. However under assumptions on the utility of the investor Duan (1995) is able to derive a unique price for an option.

3 A Complete Model with Stochastic Volatility

In this section we define a new class of stock-price models. The new feature is the specification of instantaneous volatility in terms of exponentially weighted moments of the historic log-price. This introduces a feedback effect into the volatility process: present shocks in the asset price result in higher future uncertainty.

The first step is to introduce some convenient notation. Define the discounted log-price process Z_t by $Z_t = \log(P_t e^{-rt})$. Define also the offset function of order m, denoted $S_t^{(m)}$, by

$$S_t^{(m)} = \int_0^\infty \lambda e^{-\lambda u}(Z_t - Z_{t-u})^m du \tag{7}$$

The constant λ is a parameter of the model which describes the rate at which past information is discounted. Then, for some value n,

Assumption 3.1 *Z_t solves the SDE*

$$dZ_t = \sigma(S_t^{(1)}, \ldots, S_t^{(n)})dB_t + \mu(S_t^{(1)}, \ldots, S_t^{(n)})dt \tag{8}$$

where $\sigma(.)$ and $\mu(.)$ are Lipschitz functions, and $\sigma(.)$ is strictly positive.

Remark 3.1 More generally it is possible to allow $\sigma(.)$ to be a function of the price level P_t also. In this sense this model can be extended to include the class of level-dependent volatility processes as a special case.

Remark 3.2 The key feature to note at this early stage is that no new Brownian motions (or other sources of uncertainty) have been introduced in the specification of the price process. This will imply that the model yields unique option prices without the need to specify market prices for risk.

Remark 3.3 The model is designed so that movements in the price of an asset may result in changes in the volatility of that asset. If the volatility of stock prices is related to the amount of trading in a stock then this corresponds to the scenario that price changes encourage further interest and activity in the market.

The reason for our definition of the processes $S_t^{(m)}$ is seen in the following lemma.

Lemma 3.1 *$(Z_t, S_t^{(1)}, \ldots, S_t^{(n)})$ forms a Markov process. The offset processes $S^{(m)}$ satisfy the coupled SDEs*

$$dS_t^{(m)} = mS_t^{(m-1)}dZ_t + \frac{m(m-1)}{2}S_t^{(m-2)}d\langle Z\rangle_t - \lambda S_t^{(m)}dt \tag{9}$$

Proof

Since the functions σ and μ are assumed to be Lipschitz, the existence and uniqueness of a non-explosive solution to (8) and (9) is guaranteed (see Rogers and Williams (1987, p132)). Moreover the Markov property also follows (Rogers and Williams (1987, p162)).

The stochastic calculus is easier with the following equivalent definition for $S^{(m)}$:

$$\begin{aligned} e^{\lambda t} S_t^{(m)} &= \int_{-\infty}^{t} \lambda e^{\lambda u} (Z_t - Z_u)^m du \\ &= \sum_{k=0}^{m} \binom{m}{k} (Z_t)^k \int_{-\infty}^{t} \lambda e^{\lambda u} (-Z_u)^{m-k} du. \end{aligned}$$

Then

$$\begin{aligned} &\lambda e^{\lambda t} S_t^{(m)} dt + e^{\lambda t} dS_t^{(m)} \\ &= \sum_{k=0}^{m} \binom{m}{k} \left\{ \int_{-\infty}^{t} \lambda e^{\lambda u} (-Z_u)^{m-k} du \left[k (Z_t)^{k-1} dZ_t + \frac{k(k-1)}{2} (Z_t)^{k-2} d\langle Z \rangle_t \right] \right. \\ &\qquad \left. + \lambda e^{\lambda t} (-Z_t)^{m-k} dt (Z_t)^k \right\} \\ &= m \sum_{k=1}^{m} \binom{m-1}{k-1} \left\{ \int_{-\infty}^{t} \lambda e^{\lambda u} (-Z_u)^{(m-1)-(k-1)} du \right\} (Z_t)^{k-1} dZ_t \\ &\quad + \frac{m(m-1)}{2} \sum_{k=2}^{m} \binom{m-2}{k-2} \left\{ \int_{-\infty}^{t} \lambda e^{\lambda u} (-Z_u)^{m-2-(k-2)} du \right\} (Z_t)^{k-2} d\langle Z \rangle_t \\ &= e^{\lambda t} \left\{ m S_t^{(m-1)} dZ_t + \frac{m(m-1)}{2} S_t^{(m-2)} d\langle Z \rangle_t \right\}. \end{aligned}$$

□

4 Option pricing

4.1 The General Theory

For any model of stock prices a key feature is the price equation for options. The aim of this section is to price a European Contingent Claim with exercise date T and payoff $q(P_T)$. For simplicity we assume that $n = 1$ in (8) so that the volatility depends only on the first order offset $S^{(1)}$. As a result we can simplify notation in this section by writing S as a shorthand for $S^{(1)}$. Note that (9) collapses to $dS_t = dZ_t - \lambda S_t dt$ and that we can combine (8) and (9) to conclude that S is an autonomous diffusion satisfying the stochastic differential equation

$$dS_t = \sigma(S_t) dB_t + (\mu(S_t) - \lambda S_t) dt. \tag{10}$$

Then

$$\log\{P_t e^{-rt}\} \equiv Z_t = Z_0 + (S_t - S_0) + \lambda \int_0^t S_u du.$$

Note also that S_t is adapted to the filtration $\mathcal{F}_t$ of B.

Define $\theta(S) = \frac{1}{2}\sigma(S) + \{\mu(S)/\sigma(S)\}$ and consider the process $\tilde{B}_t \equiv B_t + \int_0^t \theta(S_u)du$. Let $\mathbb{P}$ be Wiener measure for B. Define a new measure $\tilde{\mathbb{P}}$ by specifying that on $\mathcal{F}_t$

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{-\int_0^t \theta(S_u)dB_u - \tfrac{1}{2}\int_0^t \theta(S_u)^2 du\right\}.$$

See Appendix A for a discussion of the conditions under which $\tilde{\mathbb{P}}$ is a probability measure, and under which $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$.

If we assume that these conditions are satisfied then we can rewrite (10) as

$$dS_t = \sigma(S_t)d\tilde{B}_t - (\tfrac{1}{2}\sigma(S_t)^2 + \lambda S_t)dt, \tag{11}$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$ Brownian motion. Then under $\tilde{\mathbb{P}}$ the discounted price process $e^{-rt}P_t$ is a martingale, and we can apply the elegant martingale pricing theory of Harrison and Kreps (1979) and Harrison and Pliska (1981) to conclude that the option price can be written

$$f(P_t, S_t, T-t) = e^{-r(T-t)}\tilde{\mathbb{E}}[q(P_T)|\mathcal{F}_t]. \tag{12}$$

Subject to integrability conditions on $q(.)$, the Brownian martingale representation Theorem implies that the contingent claim can be replicated using a previsible trading strategy.

By the Feynman-Kac formula (Karatzas and Shreve (1988, p366)) f satisfies the partial differential equation

$$0 = (rpf_p - rf - \lambda s f_s - f_t) + \left(-\tfrac{1}{2}f_s + \tfrac{1}{2}p^2 f_{pp} + \tfrac{1}{2}f_{ss} + pf_{sp}\right)\sigma(s)^2 \tag{13}$$

subject to the boundary condition

$$f(p, s, 0) = q(p). \tag{14}$$

Assuming that $\tilde{\mathbb{E}}(|q(P_T)|) < \infty$, the recipe (12) defines a solution to the partial differential equation (13) with boundary condition (14), the necessary smoothness of the f so defined being assured by Hörmander's Theorem (for a statement, see, for example, Rogers and Williams (1987, Theorem V.38.16)). In general uniqueness requires a further argument, but in the sequel we consider a situation which can be reduced using put-call parity to the case where q is bounded, and then the optional sampling theorem guarantees that all bounded solutions can be expressed via (12) and it is clear that there is a unique bounded solution to the PDE.

Note that if the stock volatility is constant so that $d\langle Z\rangle_t/dt \equiv \sigma(S_t)^2 = \sigma^2$ and f depends only on P and t we recover

$$0 = rpf_p - rf - f_t + \tfrac{1}{2}\sigma^2 p^2 f_{pp}$$

which is the standard pde for the Black-Scholes option price.

4.2 A Specific Example: Smiles and Skews

The purpose of this section is to calculate the option price by numerically solving the partial differential equation (13) subject to the boundary condition (14). This results in an option price surface and, to facilitate comparison with the standard Black-Scholes model, Black-Scholes implied volatilities are calculated. A numerical solution is necessary because even in the simple example presented below it seems difficult to derive properties of the solution analytically; see however the end of this section for a few observations.

The specific example considered is potentially one of the simplest non-trivial cases, namely to price a European call with the interest rate r taken to be zero (or equivalently working with forward prices) and dynamics for the discounted log-price process Z given by

$$\sigma(s) = \eta\sqrt{1+\epsilon s^2} \wedge N, \tag{15}$$

for some large constant N. The function σ has the useful properties of being even (see the remarks at the end of this section) and bounded. While more complicated functions σ depending on higher order offset functions could be studied, it will be demonstrated below that even the simple example specified by (15) effectively accounts for the possibilities of smiles and skews.

The intuition that we hope to capture with this model is that if the current price differs greatly from a past average, then the volatility is high. Moreover, by basing $\sigma(s)^2$ on a quadratic in s with non-zero linear coefficient (so that $\sigma(s) = \eta\sqrt{1+\delta s+\epsilon s^2} \wedge N$), then we could model markets in which volatility changes are correlated with price changes. Thus for example, it is possible to construct a model in which volatility is higher when the current (log)-price is below the past average, than when it is above the past average by the same amount. Finally note that for suitable choice of the drift μ, the offset S is mean reverting (see (10)), and this property will be inherited by the volatility σ.

By assumption μ and σ are Lipschitz continuous and time-homogeneous so that the SDEs (10) and (11) each have a pathwise unique strong solution. Moreover, since $\sigma(S)$ is both bounded above and bounded away from zero, S is non-explosive under both of the probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$, and by Theorem 7.19 in Liptser and Shiryayev (1977) these measures are equivalent. Thus the pricing theory of the previous section applies. Note that if the volatility specification (15) was replaced by $\sigma(s) = \eta\sqrt{1+\epsilon s^2}$, then the diffusion S would explode with positive probability under the candidate pricing measure $\tilde{\mathbb{P}}$, thus contradicting the assumption that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent.

The option valuation formula becomes a function of the current price P_t, the current offset S_t and the time to go $(T-t)$, as well as the parameters K (the strike), η, ϵ and λ. Taking (for the moment) $K=1$, and using the transformation $P \equiv e^Z$, $U \equiv Z - S$, the option pricing problem simplifies to solving for $V \equiv V(Z_t, U_t, T-t) \equiv V(Z_t, U_t, T-t; \sigma, \epsilon, \lambda)$, where V satisfies the partial differential equation

$$V_t = \tfrac{1}{2}\sigma(z-u)^2[V_{zz} - V_z] + \lambda(z-u)V_u \tag{16}$$

subject to the boundary condition

$$V(Z,U,0) = (e^Z - 1)^+. \tag{17}$$

Since the numerical solution of (16) subject to (17) is calculated over finite regions of the z and u variables, the choice of N does not affect the solution V. Furthermore, in order to guarantee existence and uniqueness of the solution it is better to solve (16) subject to the boundary condition $V(Z,U,0) = (1-e^Z)^+$. The price of a call can then be deduced using put-call parity.

Now V gives the price of a call with fixed *unit* strike as a function of the asset price at time 0, but it is a simple exercise using scaling to then calculate the price $\tilde{V} \equiv \tilde{V}(K,S,T)$ of a call with arbitrary strike assuming, as we shall from now on, that the initial asset value satisfies $P_0 = 1$. (Here the analysis depends critically on the fact that in our example σ is a function of the offset S alone, and is otherwise independent of the price level.) Denote by $p_{BS}(\sigma)$ the Black-Scholes option price as given by

$$p_{BS}(\sigma) = \Phi[(-\ln K + \sigma^2 T/2)/(\sigma\sqrt{T})] - K\Phi[(-\ln K - \sigma^2 T/2)/(\sigma\sqrt{T})]. \tag{18}$$

Then we can define a model implied volatility via

$$\sigma_{BS}(K,S,T) = p_{BS}^{-1}(\tilde{V}(K,S,T)).$$

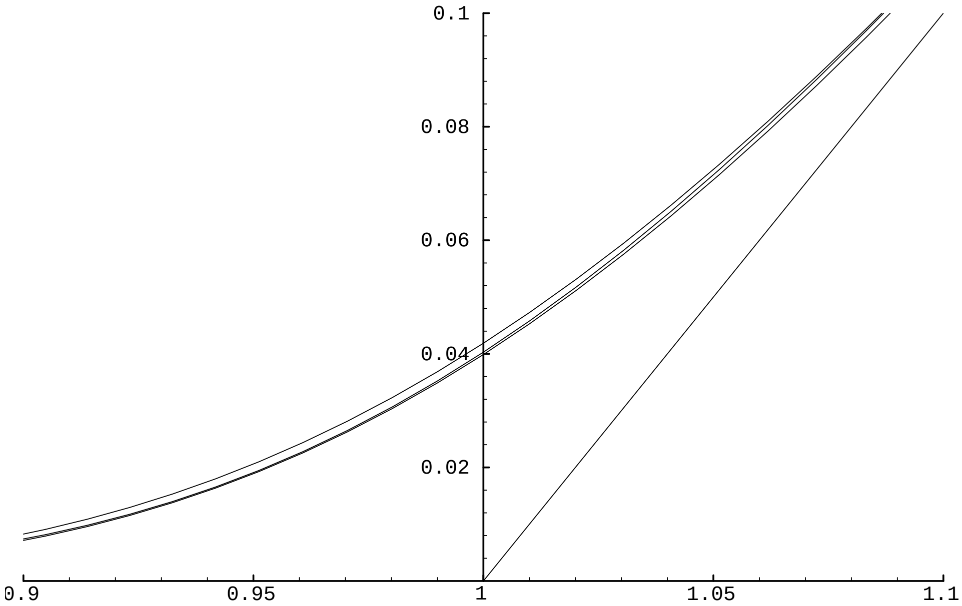

Figure 2: European Call Prices: For an option with time to maturity $T = 0.25$, and strike $K = 1$ the upper curve gives the Black-Scholes price for a volatility of 0.21, the lower curve the Black-Scholes price for a volatility of 0.2, and the middle curve the price from the model proposed in (13) with $\eta = 0.2$, $\epsilon = 5$, $\lambda = 1$ and $Z_0 - U_0 \equiv S_0 = 0.1$. The intrinsic value of the option is also shown.

By definition $\sigma_{BS}(K, S, T)$ is the volatility which, when substituted into the Black-Scholes pricing formula (18) together with the strike K and the time to exercise, gives the calculated option price.

Implied volatility provides a convenient measure for expressing the price of derivative securities in a language which can be applied accross different security payoffs. The results presented below are expressed in terms of the implied volatility σ_{BS} which is a function of the in-the-moneyness of the option (given by the strike K; recall that the current value of the asset is unity), the time to maturity T and the initial value of the offset S_0. To begin with we set T to be 0.25, though later we consider implied volatilities as a function of time.

Figure 3 shows the calculated implied volatility as a function of the first order offset S_0 and the strike K, for parameter values $\eta = 0.2$, $\epsilon = 5$ and $\lambda = 5$. The immediate conclusion is that the presence of the non-constant volatility term has the effect of increasing implied volatility, and that this effect is strongest when the initial offset is non-zero, and the option is not at-the-money.

At any moment in time there may trade a family of options with different degrees of in-the-moneyness, or correspondingly different strikes. Thus it is illuminating to consider cross-sections of this implied volatility surface, each cross-section corresponding to a different value of S_0. In principle, with the benefit of historical data, the value of S_0 is an observable so that the options trader will know which of the possible regimes for S_0 describes the current situation. In practice it may be that the options trader will use current options prices to infer the value of S_0.

For all values of S_0 the implied volatility is a convex function of the strike. Moreover skew effects are also plainly visible: the smile has a pronounced positive skew for positive values of S_0 and a negative skew when the current log-price is below a past average. This is in agreement with an intuitive understanding of the behaviour of the asset process.

Consider the situation of an out-of-the-money option with initial offset $S_0 > 0$. This option will be worthless until the asset price rises to the strike price, and the larger S_0, the sooner this will happen. Given that the strike price is reached before exercise, the offset at that time will always be positive, and typically at least as large as S_0. The volatility will then be quite high, inflating the option price. If we now consider what happens to this argument as we allow S_0 to get smaller, and then go negative, we see that it becomes more difficult (at least assuming that $-S_0$ is not too large) for the price of the asset to reach the strike before exercise, since the volatility has got smaller. Also, if $-S_0$

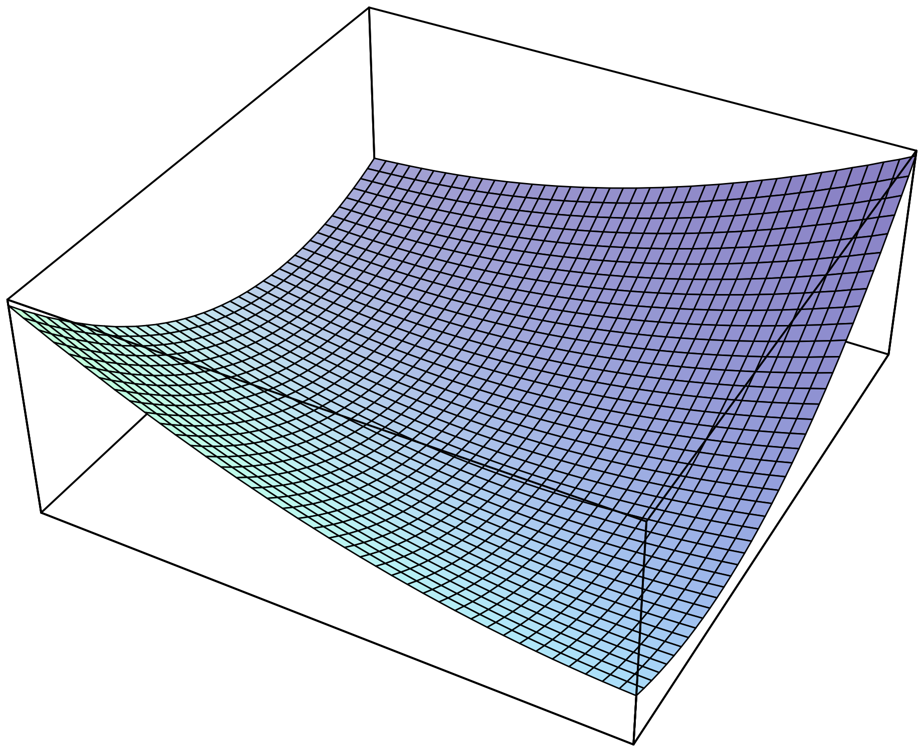

Figure 3: The Volatility Surface: The volatility surface is a function of the initial value of the offset, here ranging from -0.2 to +0.2, and shown from left to right, and the logarithm of the strike, which varies from -0.2 (in the money) in the foreground, to 0.2 (out of the money) at the back. The vertical scale showing the implied volatility ranges from 0.2 to 0.218.

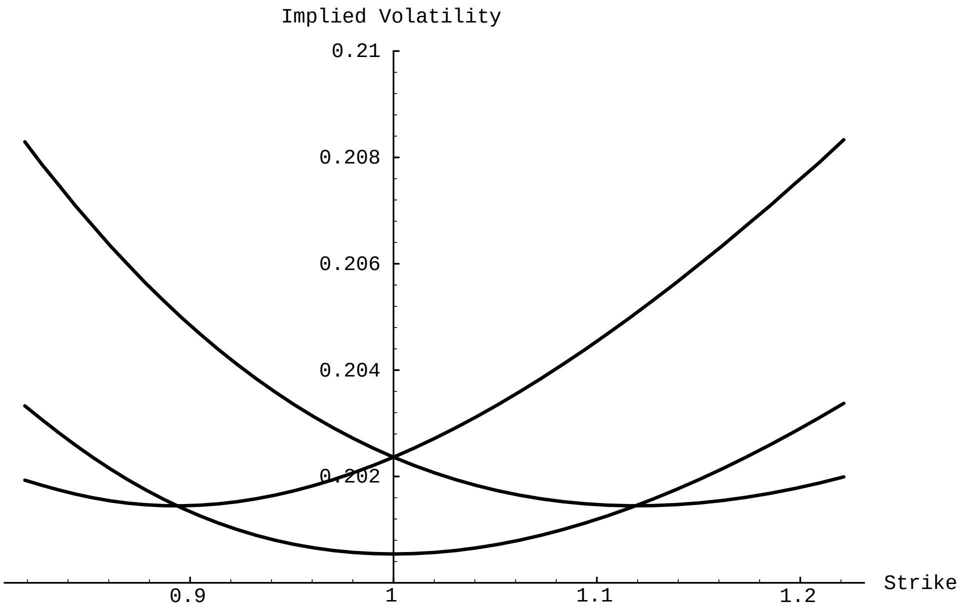

Figure 4: Sectional Smiles: A plot of Implied Volatility versus Strike; for the example in Figure 3 with the three curves representing $S_0 = -0.1$, $S_0 = 0$ and $S_0 = 0.1$ taken from top to bottom on the left. In each case there is an observable smile, and the skew varies with the initial offset.

is not too large, once the price reaches the strike, S will be nearer zero; the instantaneous volatility will therefore be low, and so the price of the option will be reduced. Of course, if $-S_0$ gets to be large then $-S$ will still be large when the exercise price is reached, and so the option price will again be larger. This gives a qualitative explanation of the curves seen in Figure 4.

To date the analysis has concentrated on options with maturity $T = 0.25$. We now relax this condition. Figure 5 displays plots of implied volatility as a function of both time to maturity and strike. There are two graphs corresponding to different initial values of the offset. An immediate observation is that the magnitude of the skews and smiles decreases with time. (Note that the main reason why the smile appears more pronounced when the initial offset is zero, is that the vertical scales in the two plots are different.) In both plots there are cross sections of constant strike along which the implied volatility increases with time, and cross sections along which it decreases with time. Both these observations are a corollary of the fact that the implied volatility is a measure of the average instantaneous volatility over the lifetime of the option, and this averaging leads to smoothing effects.

To complete the discussion of this numerical example we consider the sensitivity of the implied volatilities to changes in the parameters of the model.

There is a strong qualitative similarity between the families of smiles for each parameter pair (ϵ, λ) as illustrated by Figure 6. However parameter choice is seen to affect the magnitude of the smiles; the size of the smiles is directly related to ϵ and is inversely related to λ. The first of these relationships is immediate from (15). The explanation for the second observation is that large values of λ are associated with a shorter half-life for the lookback period in the definition of the past average. Typically this will decrease the values of the offset function S_t, (see (9)), which in turn reduces the volatility as defined via (15). Thus the nature of dependence on both the parameters ϵ and λ follows from the particular specification of the dynamics for the price process.

Finally we make some observations concerning the shape of the option price surface $V \equiv V(Z, U, T)$. If $\sigma(S)$ is an even function then it is easy to verify from the partial differential equation (16) and the boundary condition (17) that

$$V(z, u, t) = e^z V(-z, -u, t) + (e^z - 1).$$

This property is shared by the Black-Scholes formula $p_{BS}(z, t)$:

$$p_{BS}(z, t) = e^z p_{BS}(-z, t) + (e^z - 1)$$

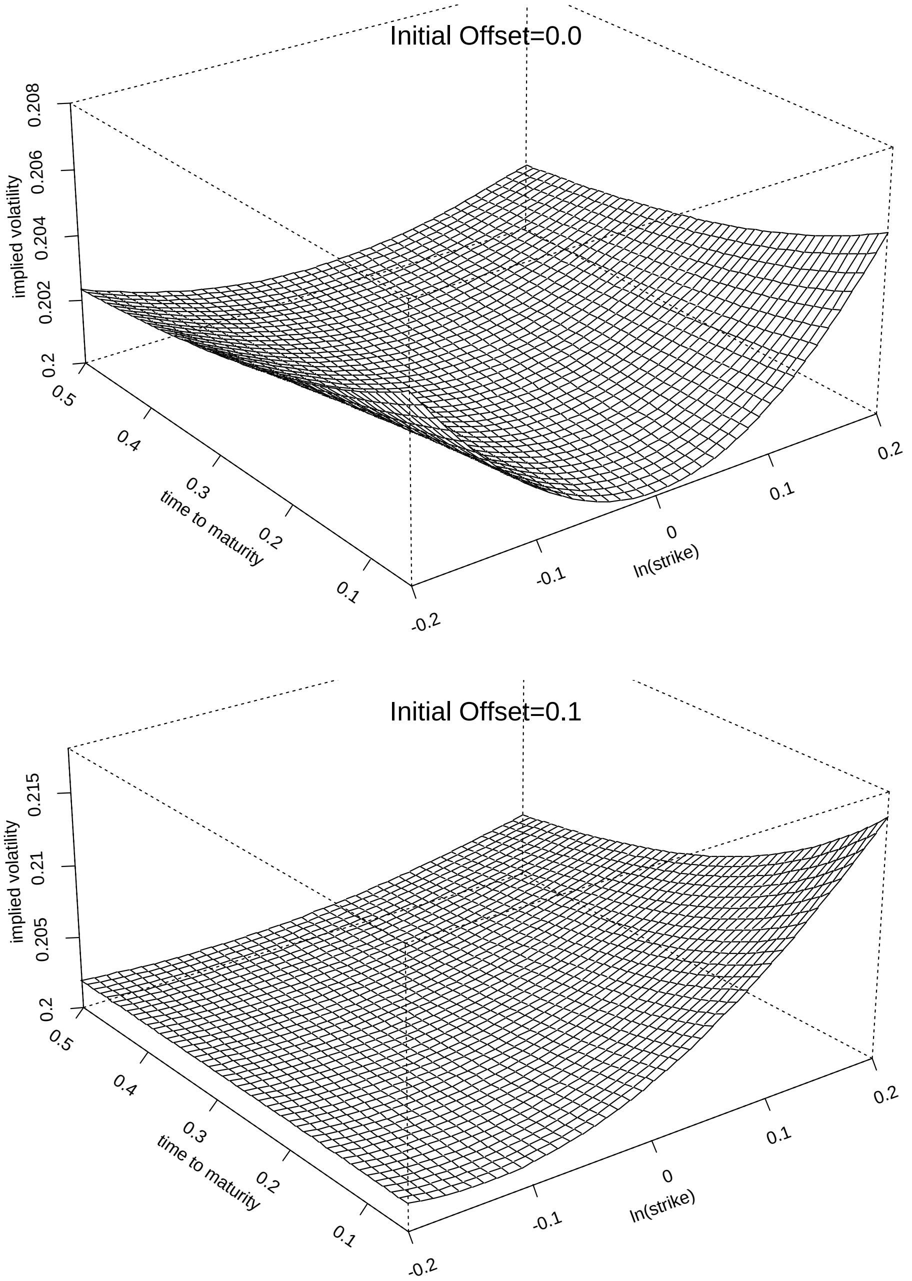

Figure 5: Term Structure of Volatility: two plots of implied volatility as a function of the logarithm of the strike and time to maturity of the option, with different initial values of the offset S_0.

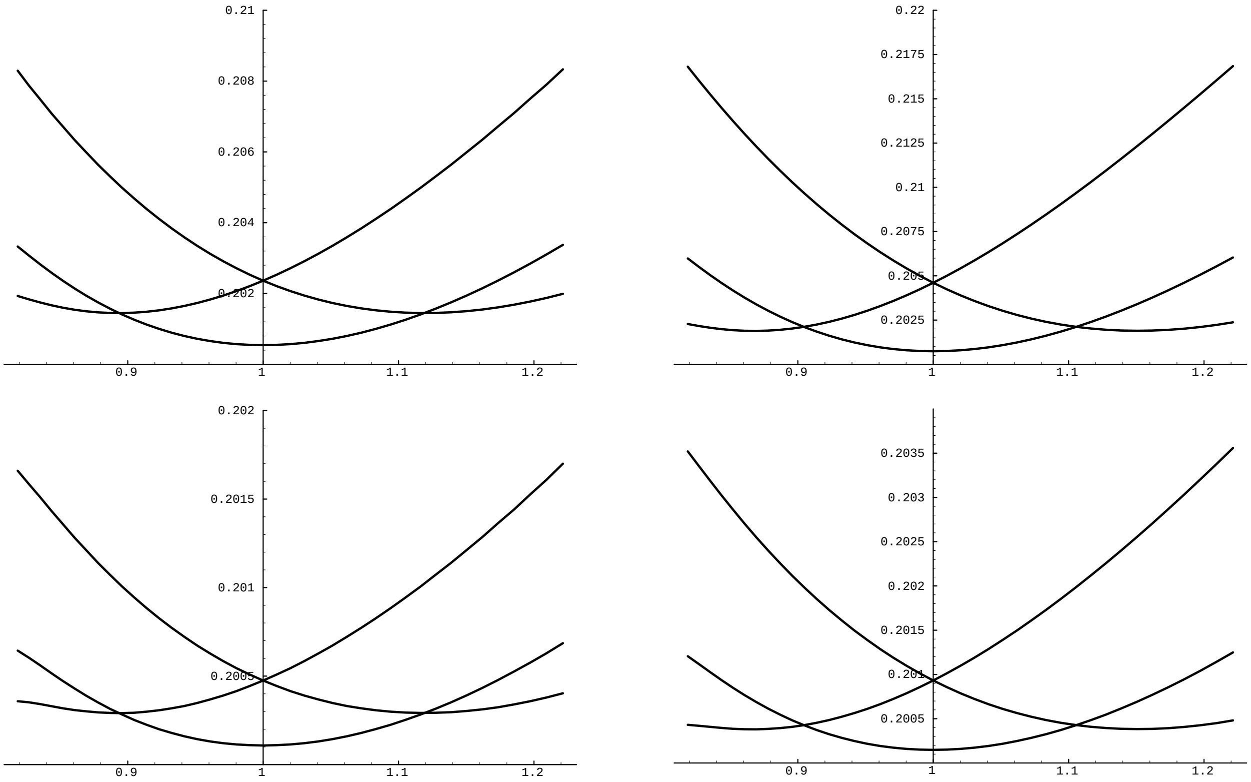

Figure 6: Sectional Smiles for different values of the parameters ϵ and λ. Note the changes in magnification of the y-scale. Each of the four plots displays a triple of volatility smiles for the initial offsets $S_0 = -0.1,\ 0,\ 0.1$, in a manner similar to Figure 4. The parameter ϵ has the value 5 for the upper two graphs, and one for the lower pair; λ has value 5 on the left and one on the right.

where p_{BS} is considered as a function of the current log-price z, and time to go t, assuming a unit strike. As a corollary, (see Renault and Touzi (1995) for details of this argument in a similar context),

$$\sigma_{BS}(K, S, T) = \sigma_{BS}(K^{-1}, -S, T).$$

Thus the implied volatility of an in-the-money option is equal to that of an out-of-the-money option subject to the sign of the current first-order offset being switched. This explains why in Figures 4 and 6 the skewed volatility smiles corresponding to $S_0 = \pm 0.1$ intersect at unit strike, and why in Figure 5 we have not shown a plot with initial offset $S_0 = -0.1$.

5 Summary

In this paper we have introduced a new class of models for asset prices. The chosen dynamics of the price processes are motivated by a desire to have a model which satisfies two criteria. Firstly the instantaneous volatility should be related to the recent history of the price process, and secondly, through the mechanism of options replication, there should be preference-independent contingent-claim prices.

The new class of complete models with stochastic volatility proposes a causal link between current asset price movements and future volatility. In contrast in the level-dependent volatility models, the volatility depends solely upon the asset price process, and in the stochastic volatility models the volatility is frequently taken to be an autonomous process. The class of level-dependent volatility models may be thought of as a special case of the new class of complete models with stochastic volatility. It shares the property of preference-independent options prices, but there is no opportunity for shocks in volatility to persist through time. In this sense the new class of models is similar in spirit to the class of stochastic volatility models.

Smiles and skews in the implied volatility of traded options are a common phenomenon. By considering one of the simplest non-trivial examples we have shown that smiles and skews arise naturally through the new model. For the parameters used in §4.2 the magnitude of the smile and skew of a three month option ranges from almost nothing to 10%. These values agree with the magnitudes of smiles found in empirical tests, for example by Fung and Hsieh. Moreover without changing the underlying dynamics of the price process the shape of the smile may change as the offset changes.

In the model used in §4.2 there is a simple qualitative relationship between the shape of the smile and the recent history of the price process. In particular there is a negative skew if and only if the asset price is below its recent average value. With a more complicated specification of the volatility in (15) other more complicated relationships between the shape of the implied volatility curve and the values of the first and higher order offsets will arise. Such volatility models have the potential to describe and explain both the prevailing implied volatility smiles and the dynamic changes of these smiles over time.

A Appendix

Recall that in Section 4.1 we introduced a new probability measure $\tilde{\mathbb{P}}$ under which the discounted price process was a martingale, and we *assumed* that $\tilde{\mathbb{P}}$ was equivalent to $\mathbb{P}$. The purpose of this appendix is to discuss that assumption.

Suppose that $\mathbb{P}_0$ is a probability measure on $(\Omega, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{F}_T)$ and suppose that we attempt to define a probability measure $\mathbb{P}_1$ by

$$\frac{d\mathbb{P}_1}{d\mathbb{P}_0} \equiv Y_t$$

where Y is a non-negative continuous $\mathbb{P}_0$-local martingale with $Y_0 = 1$. We require conditions on Y which will guarantee that Y is a true martingale, and hence that $\mathbb{P}_1$ is well defined.

Lemma A.1 *Define* $\tau_n = \inf\{u : Y_u > n\} \wedge T$*, and set* $Y_t^n = Y(t \wedge \tau_n)$*. Define the probability measures* $\mathbb{P}_1^n$ *via* $(d\mathbb{P}_1^n/d\mathbb{P}_0)|_{\mathcal{F}_t} = Y_t^n$ *so that for* $A \in \mathcal{F}_t$, $\mathbb{P}_1^n[A] = \mathbb{E}_0[Y_T^n I_A]$*. Then the following are equivalent:*

(i) Y *is a martingale;*

(ii) $\mathbb{E}_0(Y_T) = 1$*;*

(iii) $\mathbb{P}_1^n(\tau_n < T) \to 0$ *as* $n \uparrow \infty$*.*

If any of these conditions holds then $\mathbb{P}_1$ *is well defined and* $\mathbb{P}_1$ *is absolutely continuous with respect to* $\mathbb{P}_0$*.*

Proof

First let us remark that by Fatou's Lemma Y is a supermartingale and $\mathbb{E}_0(Y_T) \leq 1$. Clearly (i) implies (ii) and the converse is easily shown using the supermartingale property and proof by contadiction.

For the remaining implications observe that Y^n is a true martingale, and

$$\begin{aligned} 1 = \mathbb{E}_0[Y_T^n] &= \mathbb{E}_0[Y_T; \tau_n = T] + \mathbb{E}_0[Y_T^n; \tau_n < T] \\ &= \mathbb{E}_0[Y_T; \tau_n = T] + \mathbb{P}_1^n[\tau_n < T]. \end{aligned}$$

Now if (iii) holds then $\mathbb{E}_0[Y_T; \tau_n = T] \to 1$ so that $\mathbb{E}_0[Y_T] = 1$. Conversely if $\mathbb{E}_0[Y_T] = 1$ then from Doob's submartingale inequality we deduce that $\mathbb{P}_0[\tau_n < T] \to 0$ and hence $\mathbb{E}_0[Y_T; \tau_n = T] \to 1$.

Finally for $A \in \mathcal{F}_T$ we can define $\mathbb{P}_1(A) = \mathbb{E}_0[Y_T I_A]$ and if $\mathbb{P}_0(A) = 0$ then necessarily $\mathbb{P}_1(A) = 0$ also. $\square$

There is little more to be said in such a general setting, but when $\mathbb{P}_0$ and $\mathbb{P}_1$ are the laws of the solutions of stochastic differential equations there is hope of further progress.

Consider the special case $\Omega = C([0,T], \mathbb{R})$ as our sample space, with canonical process $(X_t)_{0 \leq t \leq T}$, and canonical filtration $\mathcal{F}_t^\circ \equiv \sigma(\{X_u : u \leq t\})$. For $i = 0, 1$ consider the SDE

$$dx_t = \sigma(t, x_t) dB_t + \mu_i(t, x_t) \tag{19}$$

where $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mu_i : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable, and σ is everywhere positive. More generally we could consider a d-dimensional process X_t, and a d-dimensional SDE with σ everywhere invertible, but the one-dimensional case has the twin advantages of notational simplicity and sufficiency for the example we have in mind.

Assume in addition that for each $i = 0, 1$ the SDE (19) has a pathwise unique strong solution; a sufficient condition for this assumption to hold is that σ and μ_i are time homogeneous and Lipschitz continuous, and hence our example from Section 4.2 fits into this setting. See Rogers and Williams (1987, Chapter V) for a discussion of concepts of solutions of SDEs.

Now for $i = 0, 1$, let $\mathbb{P}_i$ be the law of the solution of (19) with a given initial value $x_0 \in \mathbb{R}$. Consider $\mathbb{P}_i$ as a law on $(\Omega, \mathcal{F}_T^\circ)$ and form the usual augmentation $(\mathcal{F}_t)_{0 \leq t \leq T}$ of $(\mathcal{F}_t^\circ)_{0 \leq t \leq T}$ with respect to $\frac{1}{2}(\mathbb{P}_0 + \mathbb{P}_1)$.

Define $\theta_t = \sigma(t, X_t)^{-1}\{\mu_1(t, X_t) - \mu_0(t, X_t)\}$.

Proposition A.1 *The following are equivalent:*

(i) $\mathbb{P}_1$ *is absolutely continuous with respect to* $\mathbb{P}_0$*;*

(ii) $\mathbb{P}_1[\int_0^T \theta_s^2 ds < \infty] = 1$.

By interchanging the roles of $\mathbb{P}_0$ and $\mathbb{P}_1$ we immediately obtain the following corollary:

Corollary A.1 *$\mathbb{P}_0$ and $\mathbb{P}_1$ are equivalent if and only if $\int_0^T \theta_s^2 ds$ is finite almost surely under both $\mathbb{P}_0$ and $\mathbb{P}_1$*

Proof of Proposition A.1

For $i = 0, 1$ define

$$dW_t^i = \sigma(t, X_t)^{-1}\{dX_t - \mu_i(t, X_t)\}.$$

Then for each i, W^i is a $(\mathbb{P}_i, \mathcal{F}_t)$-Brownian motion, and the two are related by

$$dW_t^0 = dW_t^1 + \theta_t dt.$$

Suppose that (i) holds. Then we can define the Radon-Nikodym derivative Y_t via

$$\begin{aligned} \left.\frac{d\mathbb{P}_1}{d\mathbb{P}_0}\right|_{\mathcal{F}_t} \equiv Y(t) &= \exp\left\{\int_0^t \theta_u dW_u^0 - \tfrac{1}{2}\int_0^t \theta_u^2 du\right\} \\ &= \exp\left\{\int_0^t \theta_u dW_u^1 + \tfrac{1}{2}\int_0^t \theta_u^2 du\right\}. \end{aligned}$$

Now Y_t is a non-negative $\mathbb{P}_0$-local martingale so that $\mathbb{P}_0(\sup_t Y_t < \infty) = 1$, and by our assumption of absolute continuity, $\mathbb{P}_1(\inf_t (Y_t)^{-1} > 0) = 1$. Moreover Y_t^{-1} is a $\mathbb{P}_1$-local martingale; indeed

$$Y_t^{-1} = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t)$$

where $M_t = \int_0^t \theta_s dW_s^1$ and thus we can conclude that $\mathbb{P}_1[\inf_t\{M_t - \frac{1}{2}\langle M\rangle_t\} > -\infty] = 1$. Hence $\mathbb{P}_1[\sup_t \langle M\rangle_t < \infty] = 1$ which is condition (ii).

For the converse define

$$Y_t = \exp\left\{\int_0^t \theta_u dW_u^0 - \tfrac{1}{2}\int_0^t \theta_u^2 du\right\}.$$

Then $Y_t^{-1} = \exp\left\{-\int_0^t \theta_u dW_u^1 - \frac{1}{2}\int_0^t \theta_u^2 du\right\}$ and since by assumption $\mathbb{P}_1[\int_0^T \theta_s^2 ds < \infty] = 1$ we have that $\mathbb{P}_1[\inf_t Y_t^{-1} = 0] = 0$. Now let τ_n and $\mathbb{P}_1^n$ be defined as in Lemma A.1 Under $\mathbb{P}_1^n$ the canonical process X solves the SDE

$$dX_t = \sigma(t, X_t)dB_t + (\mu_1(t, X_t)I_{\{t \le \tau_n\}} + \mu_0(t, X_t)I_{\{t > \tau_n\}})dt.$$

Pathwise uniqueness implies that

$$\mathbb{P}_1^n(\tau_n < T) = \mathbb{P}_1(\inf_{0\le t\le T} Y_t^{-1} < 1/n) \downarrow 0$$

so that by Lemma A.1, Y is a true $(\mathbb{P}_0)$-martingale, such that $(d\mathbb{P}_1^n/d\mathbb{P}_0)|_{\mathcal{F}_t} = Y(t \wedge \tau_n)$ and in particular $\mathbb{P}_1$ is absolutely continuous with respect to $\mathbb{P}_0$. $\square$

For many examples it is straightforward to check the condition

$$\int_0^T \theta_s^2 ds < \infty \qquad \mathbb{P}_i \text{ almost surely.}$$

For example if the functions σ, μ_0 and μ_1 do not depend on t then the additive functional $A_t = \int_0^t a(X_u)du$ of the one-dimensional diffusion X_t does not explode if and only if a is locally integrable with respect to the speed measure m of X.

Finally, for the specific example we considered in Section 4.2, let S_t, the first order offset, be the canonical process. Then σ, as given by (15) is bounded above and below, μ_0 is Lipschitz (by Assumption 3.1 and Equation (10)), and μ_1 given by $\mu_1 \equiv -(\frac{1}{2}\sigma^2 + \lambda s)$ is again Lipschitz. As a corollary θ is bounded above and below on compact intervals and S_t is non-explosive under both $\mathbb{P}_0$ and $\mathbb{P}_1$, or rather $\mathbb{P}$ and $\tilde{\mathbb{P}}$ in the notation of Section 4.2. Hence by Corollary A.1, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent as desired.

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