Robust Hedging via Coupling

David G. Hobson
School of Mathematical Sciences, University of Bath,
Claverton Down, Bath, BA2 7AY. UK.
Email: `D.G.Hobson@maths.bath.ac.uk`

15 June 1997

Abstract

Consider the performance of an options writer who mis-specifies the dynamics of the price process of the underlying asset by over-estimating asset price volatility. When does he overprice the option? If he follows the hedging strategy suggested by his model, when does the terminal value of his strategy dominate the option payout?

We show that both these events happen if the option payoff is a convex function of the price of the underlying at maturity. The proofs involve the simple, powerful and intuitive techniques of coupling.

1 Robust hedging and super-replication

The standard approach in mathematical finance, and particularly in the pricing of derivative securities, is to begin by writing down a stochastic model which is assumed, without comment, to correctly and precisely specify the dynamics of the underlying asset. Arbitrage theory, backed by change of measure techniques and martingale representation theorems, then allows options to be priced and hedged. The fairness of the price and the success of the hedge depend crucially on the truth of the underlying model.

The purpose of this article is to consider the implications of a mis-specification of the dynamics of the asset price process. In particular, if the options writer uses an incorrect model, when does he overcharge for the option? Further, if he attempts to hedge using this incorrect model, can he still replicate (or rather super-replicate) the option payoff? (A super-replicating strategy is a dynamic hedging strategy which generates a terminal wealth which stochastically dominates the option payout.)

It is well known that the price of a call option in the Black-Scholes model is an increasing function of the volatility parameter, and that this monotonicity property extends to all European options with convex payoff profiles. We generalise this result to prove a price comparison theorem for pairs of models with stochastic volatilities. We have in mind the scenario where one model is used by the options writer to price and hedge options, and the second model represents the (unknown) truth.

The substantive results of this paper are not new and can be found in El Karoui et al [7]. Theorem 3.1 can be traced back to Bergman et al [4]. Instead the aim of this paper is to give proofs of the main results which are shorter and which in the opinion of the author, are both more natural and more intuitive. This improved clarity leads to an extension of the main results to non-diffusion processes, see Remark 2.3, and to an understanding of the counter-examples in [7], see Remark 2.4. A second justification for this paper is that the results are extensions of those proved in Bergman et al [4] and EL Karoui et al [7] in the sense that they require weaker conditions. For example the bounded derivative condition, see Remark 3.4, is not needed for the proof the convexity theorem, Theorem 3.1, and the price process need only be a H^1 martingale rather than L^2. See Section 3.

The key tool we use here is that of the coupling of stochastic processes; those unaware of the joys of coupling are referred to Lindvall [15]. This contrasts with the use of stochastic flows (El Karoui et al [7]) or the analysis of the partial differential pricing equation (Bergman et al [4]). As a first example of coupling (taken from the introduction to [15]) let X be a continuous Markov process and h an increasing function. Let a superscript denote the initial value. Given independent realisations X^x and X^y with $x < y$ define $\tau = \inf_u\{X_t^y \leq X_t^x\}$. Define a new process $\tilde{X}_s^x$ via

$$\tilde{X}_s^x \equiv \begin{cases} X_s^x & 0 \leq s \leq T \wedge \tau, \\ X_s^y & T \wedge \tau \leq s \leq T. \end{cases}$$

Then $\tilde{X}^x$ has the same law as X^x and moreover $\tilde{X}_s^x \leq X_s^y$ for all s by construction. Hence $\mathbb{E}(h(X_T^x)) \equiv \mathbb{E}(h(\tilde{X}_T^x)) \leq \mathbb{E}(h(X_T^y))$. See Remark 3.3 for a financial interpretation of this simple result.

Throughout this article we will consider a continuous-time model for the economy with a finite horizon T. Markets are frictionless with no transaction costs or taxes, assets are infinitely divisible and their prices are semi-martingales, and the prices of options are given by the (discounted) expected payoff of the option under the risk-neutral probability measure. This will be true if the market is complete, which in turn will follow given our other assumptions if the filtration is generated by the asset price process. The fundamental problem is to price a derivative security which at time T has non-negative payoff profile h.

For the sections on European options we will further assume that interest rates are zero. This has the advantage of minimising the amount of notation required, and thus facilitates maximum insight. All the results extend easily to the case of deterministic interest rates. Indeed, provided that the interest rates are adapted to the filtration generated by the asset price process and the market is complete, non-deterministic interest rates can also be accomodated. In the consideration of American options one of the motivations for early exercise is to capture profits which might otherwise depreciate over time. Hence it would be restrictive to limit attention to the case of zero interest rates, however for economy of notation we assume a constant rate of interest. Again the extension to deterministic interest rates is easy.

There are two main results in this article which we call option price monotonicity and the super-replication property. Suppose that there are two candidate models for the behaviour of the underlying asset; P_0 is given and either the price process solves

$$dP_s = P_s\hat{\sigma}(P_s, s)dB_s. \tag{1}$$

or

$$dP_s = P_s \tilde{\sigma}(P_s, s) dB_s \tag{2}$$

(For the purposes of this introduction we implicitly assume that the price process is a diffusion, we comment later on when this assumption can be weakened.)

Option price monotonicity

For options with a convex payoff profile $h(P_T)$, if for all p and s we have $\hat{\sigma}(p,s) \geq \tilde{\sigma}(p,s)$, then the option price is greater under the model (1) than under the model (2).

Super-replication property

Suppose h is convex and as before $\hat{\sigma}(p,s) \geq \tilde{\sigma}(p,s)$. Then if an options writer uses the model (1) to price and hedge options, but in reality the price process solves (2), then the sum of the initial option price and the gains from the (incorrect) hedging strategy dominate the option payout.

The super-replication property implies option price monotonicity.

The super-replication property is of great importance since it means that the option writer's hedging strategy is robust to model mis-specification of the dynamics of the underlying asset. The acknowledgement of the issue of model risk has led to several new modelling developments and strategies. Dupire [5, 6] has argued that by considering calls as liquid trading instruments it is possible to price exotic options in a model-free fashion. See also Hobson [11] for a related approach to robust hedging. The special case where the model risk is restricted to a mis-specification of volatility has also been the subject of recent attention. Stochastic volatility models fall into this area — see Shephard [20] for a recent review. Frey and Stremme [9] show how a non-constant volatility can arise in a model in which hedging demand affects the price dynamics of the underlying asset. Avellaneda et al [2] and Lyons [16] consider the situation where volatility is unconstrained except to lie in an interval. Lyons gives an example of a barrier option which illustrates that if the option is non-convex then the option price may not increase monotonically with volatility. All the papers cited in this paragraph are predicated upon a super-replication philosophy: for an alternative utility based approach to pricing and hedging with mis-specified volatility, see Ahn et al [1].

This paper is organised as follows. In Section 2 we provide a direct proof of the monotonicity of option prices with respect to volatility using a coupling argument. (This direct proof is original and shows that option price monotonicity does not require a diffusion assumption.)

In Section 3 we prove the super-replication property, and hence derive a second proof of option price monotonicity. The key step is to prove that for a diffusion model the option value at intermediate times is a convex function of the asset price. Again the proof is based on a coupling story. Finally, Section 4 contains an extension of the main results to American options; yet again there is a natural intuitive proof involving a coupling-style argument.

2 Option price monotonicity via coupling

Suppose that the price of an asset is a diffusion process and is given by the solution to the stochastic differential equation

$$dP_s = P_s \sigma_s dB_s \tag{3}$$

for a fixed initial value $P_0 > 0$, and for an adapted volatility process σ_s. Suppose that there are two candidate models for σ, namely $\sigma = \hat{\sigma}$ or $\sigma = \tilde{\sigma}$, and suppose that there is an ordering on these candidate volatilities in the sense that $\hat{\sigma}(x,s) \geq \tilde{\sigma}(x,s)$ for all asset values x and times s.

Theorem 2.1 *In a complete market the (non-path-dependent) option with convex payoff h is more expensive under the model with volatility $\hat{\sigma}$ than under the model with volatility $\tilde{\sigma}$.*

Notation and assumptions. When we define quantities τ, A and P with respect to σ, this should be taken as implicitly defining pairs of quantities, the elements of which are denoted in a natural fashion with the superscripts $\hat{}$ and $\tilde{}$. As a first example, for a Brownian motion W with $W_0 = P_0$, define τ to be the solution of the ordinary differential equation

$$\frac{d\tau_t}{dt} = \frac{1}{W_t^2 \sigma(W_t, \tau_t)^2}. \tag{4}$$

Assume that, almost surely, this ordinary differential equation has a unique strictly increasing solution which explodes when W first reaches zero.

Proof.

The main idea is that in each model P_t is a continuous local martingale so we can hope to write P as a time-change of Brownian motion. By constructing both models from the same Brownian motion we can hope to deduce comparison theorems for option prices from comparison theorems for the time changes.

Begin with a Brownian motion W started at P_0. (Note that since W is not defined with respect to σ we are thinking of a single Brownian motion rather than a pair.) Use this Brownian motion to define a (pair of) increasing functional(s) τ. Denote the inverse to τ by A_t. Now define a process P via $P_t \equiv W(A_t)$; then P is a local-martingale with quadratic variation

$$\frac{dA_t}{dt} \equiv W(A_t)^2 \sigma(W(A_t), \tau(A_t))^2 \equiv P_t^2 \sigma(P_t, t)^2.$$

In particular, we can represent P as a (weak) solution of the SDE $dP_t = P_t \sigma(P_t, t) dB_t$.

If $\hat{P}_t = \tilde{P}_t$ then $d\hat{A}_t \geq d\tilde{A}_t$ and this is certainly true at $t = 0$. Moreover if $\hat{A}_t = \tilde{A}_t$ then by construction $\hat{P}_t \equiv \tilde{P}_t$ (since both processes have been constructed as a time-change of the Brownian motion W) and hence $\hat{A}_t \geq \tilde{A}_t$ for all t.

Finally $h(\hat{P}_T) \equiv h(W(\hat{A}_T)) \equiv h(W(\tilde{A}_T) + (W(\hat{A}_T) - W(\tilde{A}_T))$ so that the claimed option price monotonicity follows by a conditional version of Jensen's inequality. $\square$

Remark 2.2 This argument proves price monotonicity, but it does not seem possible to extend this coupling argument to show that the options trader using the wrong model will super-replicate. This is the subject of the next section. However the above provides a general philosophy for showing the price monotonicity property: construct solutions $\hat{P}_t = W(\hat{A}_t)$ and $\tilde{P}_t = W(\tilde{A}_t)$ with respect to the same Brownian motion W, in such a way that $\hat{A}_T \geq \tilde{A}_T$ almost surely, and then apply Jensen's inequality. Thus for example if both $\hat{\sigma}$ and $\tilde{\sigma}$ are functions of time alone then a simple sufficient condition for option price monotonicity is that

$$\int_0^T \hat{\sigma}(t)^2 dt \geq \int_0^T \tilde{\sigma}(t)^2 dt. \tag{5}$$

Remark 2.3 In fact it is not necessary to restrict to diffusion models so that $\hat{\sigma}$, $\tilde{\sigma}$ can be functions of the whole price history provided that τ and its inverse can be defined, and that for all p and for all pairs of paths $\{\hat{p}_u\}_{0 \leq u \leq s}$, $\{\tilde{p}_u\}_{0 \leq u \leq s}$ with $\hat{p}_s \equiv p \equiv \tilde{p}_s$ we have that $\hat{\sigma}(\{\hat{p}_u\}_{0 \leq u \leq s}, s) \geq \tilde{\sigma}(\{\tilde{p}_u\}_{0 \leq u \leq s}, s)$.

Remark 2.4 El Karoui et al [7] give an example (due in part to Marc Yor) which at first sight appears to contradict the claim in Remark 2.3. In the example two volatility processes are given for which there is an ordering of the volatilities, but such that the prices of call options decrease with apparently increasing volatility. The point to note is that the Yor construction defines the price in each model as the strong solution of the SDE (3) with respect to the same Brownian motion B. Upon time change this means that the processes must be defined relative to different Brownian motions W and the coupling argument cannot be applied. For financial modelling and derivative pricing it is the law of the price process that is important, and not the relationship with a particular driving Brownian motion. Hence option price monotonicity does hold in the sense of Remark 2.3. If volatility is path-dependent then the 'correct' notion of a volatility comparison condition is that given in Remark 2.3.

3 Super-replication of European Contingent Claims

The purpose of this section is to provide a further proof that for European options with convex payoffs the option price is increasing in the volatility. Moreover an options trader who prices and hedges according to the model with the higher volatility will super-replicate the option payout if the true underlying model is the alternative regime.

The key result in this section, which is of independent interest, is that for diffusion models of the price of the underlying asset the prices of options with convex payoffs are a convex function of the underlying. This is Theorem 3.1 below, and again there is a simple and intuitive coupling proof of this result. The super-replication result (Theorem 3.5) then follows.

Theorem 3.1 *For a (complete) diffusion model of the asset price, and a European option with convex payoff h, the option value at intermediate times is convex in the price of the underlying.*

Assumptions. Suppose the model is $dP_s = P_s \sigma(P_s, s) dB_s$, then in a complete market the option price is given by $v(P_t, t) = \mathbb{E}(h(P_T)|P_t)$, at least under the assumption that $\mathbb{E}(|h(P_T)|) < \infty$. We assume that σ has sufficient continuity properties to ensure that the solution to the SDE is unique in law (for example a Lipschitz condition on $p\sigma(p, s)$; see Rogers and Williams [19, Remark V.16.4]), and that P is a true martingale. A simple sufficient condition for this second property is that $\sigma(p, s)$ is bounded, more generally the problem translates into one of verifying that the Doléans exponential is a true martingale; see Revuz and Yor [18, Section 3.5].

Proof.

The set up: Suppose $0 < z < y < x$ and for independent Brownian motions α, β, γ define X, Y, Z via

$$\begin{aligned} dX_s &= X_s \sigma(X_s, s) d\alpha_s \quad X_0 = x \\ dY_s &= Y_s \sigma(Y_s, s) d\beta_s \quad Y_0 = y \\ dZ_s &= Z_s \sigma(Z_s, s) d\gamma_s \quad Z_0 = z. \end{aligned}$$

The 'coupling': Let $H_x \equiv \inf_u\{X_u = Y_u\}$, and similarly $H_z \equiv \inf_u\{Y_u = Z_u\}$. Now define $\tau = H_x \wedge H_z \wedge T$. Then, by symmetry, on $\tau = H_x$

$$(X_T - Z_T)h(Y_T) \stackrel{\mathcal{D}}{=} (Y_T - Z_T)h(X_T)$$

so that

$$\mathbb{E}[(X_T - Z_T)h(Y_T); \tau = H_x] = \mathbb{E}[(Y_T - Z_T)h(X_T); \tau = H_x]$$

Also $\mathbb{E}[(X_T - Y_T)h(Z_T); \tau = H_x] = 0$. Similarly

$$\mathbb{E}[(X_T - Z_T)h(Y_T); \tau = H_z] = \mathbb{E}[(X_T - Y_T)h(Z_T); \tau = H_z] + \mathbb{E}[(Y_T - Z_T)h(X_T); \tau = H_z].$$

Finally, on $\tau = T$, we have that $Z_T < Y_T < X_T$, and by convexity of h

$$(X_T - Z_T)h(Y_T) \leq (X_T - Y_T)h(Z_T) + (Y_T - Z_T)h(X_T).$$

Taking expectations and adding gives

$$\mathbb{E}[(X_T - Z_T)h(Y_T)] \leq \mathbb{E}[(X_T - Y_T)h(Z_T)] + \mathbb{E}[(Y_T - Z_T)h(X_T)],$$

so that by the independence of X, Y, and Z

$$(x - z)\mathbb{E}[h(Y_T)] \leq (x - y)\mathbb{E}[h(Z_T)] + (y - z)\mathbb{E}[h(X_T)].$$

□

Remark 3.2 It is clear that this proof cannot be extended to non-diffusion models because then there can be no identity in law between, for example, X_T and Y_T on $\tau = H_x$.

Remark 3.3 The argument given in the introduction shows that if h is increasing then the option price function v must also be increasing. This simple result, Theorem 1 in Bergman *et al* [4], was described as 'the intuitive link between a diffusion process and properties of options prices'.

Remark 3.4 If h has bounded derivative on $[0, \infty]$, and bound C say, then it is clear that $h(0) - CP_t \leq v(P_t, t) \leq h(0) + CP_t$. Given also that v is convex, and hence that the space derivative is increasing, it must follow that

$$|v'(P_t, t)| \leq C. \tag{6}$$

One of the motivations for this paper was to provide a simple, direct and general proof of Theorem 3.1 using probabilistic methods. The result in Theorem 3.1 has appeared elsewhere (see Bergman et al [4], El Karoui et al [7] and Frey [8]), though the proofs in these papers use different methods and require more restrictive technical conditions. Bergman et al [4] and Frey [8] differentiate the partial differential equation for the option price v, with respect to space, to deduce a partial differential equation for v'. This requires the diffusion co-efficient $p^2\sigma(p,s)^2$ to be differentiable. El Karoui et al [7] again require continuous differentiability in order to use a method of stochastic flows. Both methods require the payoff function h to have a bounded derivative. In each case the proof uses the Feynmac-Kac formula to deduce a representation for the derivative of the option payoff in the form $v'(P_t,t) = \mathbb{E}^*(h(P_T)|P_t)$. Here the superscript * denotes the fact that P solves a different SDE to (3) in this representation.

Theorem 3.1 was used by El Karoui et al [7] as a key step in the proof of the main result of this paper, namely that an options trader who uses a model which systematically over-estimates volatility is following a robust pricing and hedging strategy. Here we reproduce the essence of their argument.

Suppose that the options trader uses a diffusion model

$$dP_s = P_s\sigma(P_s,s)dB_s$$

under which options prices are given by

$$v(P_s,s) = \mathbb{E}(h(P_T)|P_s) \qquad v(P_T,T) = h(P_T).$$

Suppose further that the true volatility is given by $\tilde{\sigma}(p,s)$ and that the model volatility dominates the true volatility in the sense that $\sigma(p,s) \geq \tilde{\sigma}(p,s)$ for all asset values p and intermediate times s.

Theorem 3.5 *If the model volatility dominates the true volatility, and if the option payoff function h is convex, then the options trader who prices and hedges according to the model will super-replicate the option payout.*

Assumptions. We need to apply Itô's formula to the option price function $v(P_s,s)$ in the proof below. Sufficient conditions for $v \in C^{2,1}((0,\infty)\times[0,T))$ are that σ is positive and Hölder continuous. For further details see El Karoui et al [7, Hypothesis 6.1], Karatzas and Shreve [14, Section 5.7], or Friedman [10].

Proof

Under the option writer's model, since the market is complete the payout can be decomposed into the initial price and the gains from trade:

$$h(P_T) = v(P_0, 0) + \int_0^T v'(P_t, t)dP_t.$$

Moreover since $v(P_t, t)$ is a martingale we must have that v solves the partial differential equation

$$\tfrac{1}{2}p^2\sigma^2(p, s)v'' + \dot{v} = 0. \tag{7}$$

Consider the effect of following the strategy specified by $\theta_s = \theta_s(p, s) = v'(p, s)$ in the real (or tilded) world. Start with initial fortune $v(P_0, 0)$. By time T this has increased to

$$v(P_0, 0) + \int_0^T \theta_s d\tilde{P}_s = v(P_0, 0) + \int_0^T v'(\tilde{P}_s, s)d\tilde{P}_s, \tag{8}$$

and by assumption this stochastic integral is well defined. For suitable paths α we have $v(\alpha_T, T) - v(\alpha_0, 0) = \int_0^T dv(\alpha_s, s)$ so that if $\alpha_0 = P_0 = \tilde{P}_0$ and $\alpha_s = \tilde{P}_s$, then

$$h(\tilde{P}_T) = v(\tilde{P}_0, 0) + \int_0^T dv(\tilde{P}_s, s). \tag{9}$$

Substituting $v(P_0, 0)$ for $v(\tilde{P}_0, 0)$, and combining (8) and (9) we obtain

$$v(P_0, 0) + \int_0^T \theta_s d\tilde{P}_s = h(\tilde{P}_T) + \int_0^T v'(\tilde{P}_s, s)d\tilde{P}_s - \int_0^T dv(\tilde{P}_s, s).$$

Moreover, by Itô's formula, and then using (7),

$$\begin{aligned} dv(\tilde{P}_s, s) &= v'(\tilde{P}_s, s)d\tilde{P}_s + \tfrac{1}{2}v''(\tilde{P}_s, s)(d\tilde{P}_s)^2 + \dot{v}(\tilde{P}_s, s)ds \\ &= v'(\tilde{P}_s, s)d\tilde{P}_s + \left[\tfrac{1}{2}\tilde{P}_s^2 v''(\tilde{P}_s, s)\tilde{\sigma}(\tilde{P}_s, s)^2 + \dot{v}(\tilde{P}_s, s)\right] ds \\ &= v'(\tilde{P}_s, s)d\tilde{P}_s + \left[\tfrac{1}{2}\tilde{P}_s^2 v''(\tilde{P}_s, s)\left\{\tilde{\sigma}(\tilde{P}_s, s)^2 - \sigma(\tilde{P}_s, s)^2\right\}\right] ds. \end{aligned}$$

Thus

$$v(P_0, 0) + \int_0^T \theta_s d\tilde{P}_s = h(\tilde{P}_T) + \int_0^T \tfrac{1}{2}\tilde{P}_s^2 v''(\tilde{P}_s, s)\left\{\sigma(\tilde{P}_s, s)^2 - \tilde{\sigma}(\tilde{P}_s, s)^2\right\} ds \tag{10}$$

and under our volatility comparison assumption

$$v(P_0, 0) + \int_0^T \theta_s d\tilde{P}_s \geq h(\tilde{P}_T). \tag{11}$$

□

The quantity $e_t = \int_0^t \frac{1}{2}\tilde{P}_s^2 v''(\tilde{P}_s, s)\left\{\sigma(\tilde{P}_s, s)^2 - \tilde{\sigma}(\tilde{P}_s, s)^2\right\} ds$ was labelled the tracking error by El Karoui et al [7]. Super-replication is equivalent to $e_T \geq 0$, almost surely.

Corollary 3.6 *The model option price is greater than the fair price in the true world.*

Assumptions. We need that the local martingale $\int_0^t \theta_s d\tilde{P}_s$ is a true martingale. If h has bounded derivative then by Remark 3.4 we may restrict attention to the case of bounded integrands. Assume that $\mathbb{E}[(\int_0^T \tilde{P}_s^2 \tilde{\sigma}(\tilde{P}_s, s)^2 ds)^{1/2}] < \infty$, so that $\tilde{P}$ is an element of the Hardy space H^1 of martingales, then by the Burkholder-Davis-Gundy inequalities (Rogers and Williams [19, Theorem IV.42.1]), the stochastic integral is a true martingale. Exercise IV.4.22 in Revuz and Yor [18] shows that at this level of generality we cannot hope to do better.

Proof

This follows on taking expectations in (11).

Remark 3.7 Clearly the proof can be applied in reverse in the sense that if the option has been bought for a price based on a model with volatility σ which is dominated by the true volatility, then by using the model hedge the option purchaser can guarantee that his income (the option payoff), dominates his obligations (the sum of the initial premium and the losses from his model hedging strategy).

Remark 3.8 Whilst it is crucial that the option writer's model is a diffusion process so that we can apply the convexity result, there is no requirement that the true process is a diffusion, merely that at all times s, and for all price histories $\{p_u\}_{0 \le u \le s}$ we have that $\sigma(p_s, s) \ge \tilde{\sigma}(\{p_u\}_{0 \le u \le s}, s)$.

Remark 3.9 In Remark 2.2 we gave a simple condition (5) for option price monotonicity. Note that even the the stronger condition

$$\int_0^s \sigma(t)^2 dt \ge \int_0^s \tilde{\sigma}(t)^2 dt \qquad \forall s \in [0, T]$$

is not sufficient to guarantee the super-replication property.

4 American options via coupling

We next turn our attention to American contingent claims. We assume that the claim is an entitlement to receive $h(P_t)$ at time t for some time-independent payoff profile h where the (stopping) time t is at the discretion of the option holder.

With American contingent claims, since the timing of payments can vary, it is not possible to completely remove discount factors from the analysis by a change of numeraire. For notational simplicity we assume a fixed and constant rate of interest $r \geq 0$, though the extension to deterministic interest rates is easy.

Suppose that the market is complete and that the (under the risk-neutral measure) the price process solves the SDE $dP_s = P_s \sigma_P(P_s, s) dB_s + rP_s ds$. Let Q_s denote the discounted price process $Q_s = e^{-rs} P_s$; then Q solves the SDE

$$dQ_s = Q_s \sigma(Q_s, s) dB_s \tag{12}$$

where $\sigma(Q_s, s) \equiv \sigma_P(e^{rs} Q_s, s)$. Discounted into time 0 prices, the payoff of the option, if exercised at time t, is $e^{-rt} h(P_t) \equiv e^{-rt} h(e^{rt} Q_t)$.

Theorem 4.1 *For a (complete) diffusion model of the asset price, and an American option with convex payoff h, the option value at intermediate times is convex in the price of the underlying.*

Assumptions: We assume that σ_P has sufficient continuity properties to ensure that the solution to the SDE for the price process (or equivalently (12)) is unique in law, that Q is a true martingale, and that for all stopping times ρ, $\mathbb{E}(|h(P_\rho|) < \infty$.

Proof.

Bensoussan [3], Karatzas [12, 13] and Myneni [17] have shown that the price v of an American option is given by a Snell Envelope:

$$v(q, s) = \text{ess sup}_{\rho \in \Gamma(s)} e^{-r\rho} \mathbb{E}[h(e^{r\rho} Q_\rho) | Q_s = q]$$

where $\Gamma(s)$ is the set of all stopping times which take values in the interval $[s, T]$. The optimal stopping time or exercise time is $E = \inf\{u : v(Q_s, s) = e^{-rs} h(e^{rs} Q_s)\}$. In general $v(Q_s, s)$ is a supermartingale; it is a martingale on the continuation region $v(q, s) > e^{-rs} h(e^{rs} q)$.

The set up: As in the proof of Theorem 3.1 let $0 < z < y < x$ and for independent Brownian motions α, β, γ define discounted price processes X, Y, Z via

$$\begin{aligned} dX_s &= X_s \sigma(X_s, s) d\alpha_s \quad X_0 = x \\ dY_s &= Y_s \sigma(Y_s, s) d\beta_s \quad Y_0 = y \\ dZ_s &= Z_s \sigma(Z_s, s) d\gamma_s \quad Z_0 = z. \end{aligned}$$

The 'coupling': Let $E \equiv E(y)$ be the optimal exercise time for the process started at y. Define $\tau = H_x \wedge H_z \wedge E$ where as before $H_x \equiv \inf_u \{X_u = Y_u\}$, and $H_z \equiv \inf_u \{Y_u = Z_u\}$.

On $\tau = E$, we have that $e^{-r\tau}h(e^{r\tau}Y_\tau) \equiv v(Y_\tau, \tau)$ and, by convexity,

$$\begin{aligned}(X_\tau - Z_\tau)v(Y_\tau, \tau) &\leq (Y_\tau - Z_\tau)e^{-r\tau}h(e^{r\tau}X_\tau) + (X_\tau - Y_\tau)e^{-r\tau}h(e^{r\tau}Z_\tau) \\ &\leq (Y_\tau - Z_\tau)v(X_\tau, \tau) + (X_\tau - Y_\tau)v(Z_\tau, \tau).\end{aligned}$$

By symmetry, on $\tau = H_x$,

$$(X_\tau - Z_\tau)v(Y_\tau, \tau) = (Y_\tau - Z_\tau)v(X_\tau, \tau)$$

and $0 = (X_\tau - Y_\tau)v(Z_\tau, \tau)$. By playing a similar trick on the set $\tau = H_z$ we get that always

$$(X_\tau - Z_\tau)v(Y_\tau, \tau) \leq (Y_\tau - Z_\tau)v(X_\tau, \tau) + (X_\tau - Y_\tau)v(Z_\tau, \tau)$$

Now taking expectations, and using independence,

$$(x - z)\mathbb{E}[v(Y_\tau, \tau)] \leq (y - z)\mathbb{E}[v(X_\tau, \tau)] + (x - y)\mathbb{E}[v(Z_\tau, \tau)].$$

But $v(Y_t, t)$ is a martingale on $t \leq \tau$, so $\mathbb{E}[v(Y_\tau, \tau)] = v(y, 0)$ and $v(X_t, t)$ and $v(Z_t, t)$ are supermartingales so that, for example, $\mathbb{E}[v(X_\tau, \tau)] \leq v(x, 0)$. The convexity property for $v(., 0)$ now follows. □

Corollary 4.2 *Option price monotonicity and the super-replication property hold for American options.*

Proof

Subject to minor modifications the proof follows Theorem 3.5. The main change involves considering the gains from trade up to an intermediate time t. We have that

$$e^{-rt}h(P_t) \leq v(Q_t, t) = v(Q_0, 0) + \int_0^t v'(Q_s, s)dQ_s.$$

Moreover, assuming that Itô's formula applies, we have by the supermartingale property that $\frac{1}{2}q^2\sigma^2 v'' + \dot{v} \leq 0$. In conclusion

$$\begin{aligned}v(Q_0, 0) + \int_0^t \theta_s d\tilde{Q}_s &\geq v(\tilde{Q}_t, t) + \tfrac{1}{2}\int_0^t \tilde{Q}_s^2 v''(\tilde{Q}_s, s)[\sigma(\tilde{Q}_s, s)^2 - \tilde{\sigma}(\tilde{Q}_s, s)^2]ds \\ &\geq e^{-rt}h(\tilde{P}_t)\end{aligned}$$

provided that $\sigma(q, s) \geq \tilde{\sigma}(q, s)$ everywhere. In order to deduce option price monotonicity we need that $\int_0^t \theta_s d\tilde{Q}_s$ is a true martingale; again if h has bounded derivative then θ is bounded and a sufficient condition is $\mathbb{E}[(\int_0^T \tilde{Q}_s^2 \tilde{\sigma}(\tilde{Q}_s, s)^2)^{1/2}] < \infty$. □

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DAVID G. HOBSON

SCHOOL OF MATHEMATICAL SCIENCES,

UNIVERSITY OF BATH,

CLAVERTON DOWN,

BATH, BA2 7AY.

UK.