Perpetual American Options in Incomplete Markets: The Infinitely Divisible Case †

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Abstract

We consider the exercise of a number of American options in an incomplete market. In this paper we are interested in the case where the options are infinitely divisible. We make the simplifying assumptions that the options have infinite maturity, and the holder has exponential utility. Our contribution is to solve this problem explicitly and we show that, except at the initial time when it may be advantageous to exercise a positive fraction of his holdings, it is never optimal for the holder to exercise a tranche of options. Instead the process of option sales is continuous; however, it is singular with respect to calendar time. Exercise takes place when the stock price reaches a convex boundary which we identify.

1 Introduction

In this paper we consider the problem of determining the optimal exercise strategy for a number of American options in an incomplete market. The problem is incomplete since the agent is restricted from trading in the underlying asset itself. In a complete market, the optimal exercise time of American options does not depend on the quantity of options held. However, in an incomplete market this is no longer the case and we show the holder of a number of American options in an incomplete market would prefer to exercise options intertemporally, rather than exercising all options at one time.

We provide an explicit analysis of the situation where options are infinitely divisible, so that the agent can exercise fractions of options continuously over time. Our assumptions that the holder of the options has exponential utility and the options are infinite maturity have the advantage of enabling us to solve the problem in closed-form. We derive the optimal exercise boundary and show that it is a convex function of the asset price. The optimal policy is to exercise just enough options to stay below the boundary.

Our model lends itself naturally to the study of exercising executive stock options, where the recipient of American call options is restricted from trading in the stock of his own company. Many papers including Carpenter [3] and Detemple and Sundaresan [7] have considered this problem with only a single option, or equivalently have assumed that all options must be exercised at the same time. It is also relevant for modelling real investment decisions (see Dixit and Pindyck [8], McDonald and Siegel [19] and in an incomplete market, Henderson [10]). In both of these applications, our assumption of an infinite maturity is a reasonable approximation. Executive stock options most commonly have a ten year maturity, whilst canonical problems in real options are solved with an infinite horizon (see Dixit and Pindyck [8]).

Jain and Subramanian [16] were the first to analyse the exercise strategy when a number of American options are held by an executive who is unable to trade the underlying stock. They observed numerically in a simple binomial framework that options should be exercised intertemporally. See also Grasselli [9]. Rogers and Scheinkman [23] extend the setting to a continuous time framework. They approximate the

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continuous time problem with a grid (in the time and quantity of options dimensions) and show that in the limit the solution converges to the continuous problem. All these papers use numerical approaches to study options of a finite maturity. Since it is well known that the standard complete market American option does not have a closed-form solution, the incomplete version of the problem certainly will not.

In the special case of complete markets, the perpetual American option problem was solved by McKean [20] (see also Merton [21]). Furthermore, under the assumption of infinite maturity, exponential utility, and incomplete markets, Henderson [10] solves in closed-form the American option exercise problem with perfect indivisibility (or equivalently, the situation with only one option). In Henderson [11] she studies the problem where the options are finitely divisible (the options can only be exercised in whole units). This can be solved in closed-form and can be thought of as an approximation of the problem in this paper.

Unlike Jain and Subramanian [16] and Rogers and Scheinkman [23], but as in Henderson [10, 11] and Grasselli [9], we also consider the situation in which the agent has access to a financial market on which there trades an instrument which is correlated with the underlying asset. (For stock options it is appropriate to assume that the executive is forbidden contractually from trading on his own stock, but it is also realistic to assume that he might both wish and have the opportunity to hedge his option risk using a market index.) We show that by a suitable transformation the problem can be reduced to the case without a hedging asset. Thus it is easy to deduce the value function, exercise strategy and hedging strategy in this case.

The problem we consider in this paper is a stochastic control problem with singular control (the exercise of options). When there is a financial hedging asset, the problem also has continuous control. Similar problems arise in the literature on optimal consumption and investment problems with proportional transactions costs, see Davis and Norman [5]. In the transactions costs models, the non-decreasing processes for the cumulative purchase and sale of stock are the singular controls, and transactions only take place to take you to the boundaries of a wedge, and to keep you on these boundaries. Once the financial asset is introduced, the problem we treat in this paper is of mixed control/stopping type. Other examples of this type in mathematical finance include Davis and Zariphopoulou [6], Karatzas and Kou [17], and Karatzas and Wang [18]. Problems with multiple stopping have also been studied in the context of energy derivatives, in particular, swing options, see Carmona and Touzi [2].

2 The Model

We consider an agent who holds θ units of an American style claim with payoff upon exercise of $C(Y)$ per-unit claim, where Y denotes the price of an underlying. The claim is infinitely divisible and the act of exercising is irreversible. By assumption the agent is not able to trade in the asset Y itself, either for contractual reasons (in the case of executive stock options) or because it is not a financial asset. Since standard hedging arguments cannot be used, the agent faces an incomplete market. We specialise to the perpetual option situation by assuming the claim has infinite maturity. The agent's objective is to maximise the expected utility of wealth, where wealth accrues from the gains received upon option exercise. Denote by Θ_t the number of options remaining at time t, and suppose $\Theta_0 = \theta$. The utility maximising agent with initial wealth x solves

$$\max_{(\Theta_t)\in\mathcal{S},\Theta_0=\theta} \mathbb{E}U\left(x+\int_{t=0}^{\infty} C(Y_t)|d\Theta_t|\right) \tag{1}$$

where $\mathcal{S}$ is the set of positive decreasing processes $(\Theta_t)_{t\geq 0}$. The problem in (1) can equivalently be written as

$$\max_{(\tau_\phi)_{0\leq\phi\leq\theta},\tau_\phi\in\mathcal{T}} \mathbb{E}U\left(x+\int_{\phi=0}^{\theta} C(Y_{\tau_\phi})d\phi\right)$$

where $\mathcal{T}$ is the family of decreasing stopping times parameterised by the quantity ϕ which represents the number of unexercised options, so that $\tau_\phi = \inf\{t : \Theta_t \leq \phi\}$.

We will assume exponential utility of the form $U(x) = -e^{-\gamma x}/\gamma$. Throughout the paper we work with zero interest rates for simplicity, which is equivalent to taking the risk-less bond as numeraire. We are interested in the American call option, so we take $C(Y) = (Y-K)^+$, and we assume that the underlying asset Y follows exponential Brownian motion

$$\frac{dY}{Y} = \nu dt + \eta dW$$

with constant drift ν and volatility η. We suppose that $\nu \leq \eta^2/2$, else Y_t grows to infinity almost surely, and the problem is degenerate.

Note that in the formula for the option payoff, K is a constant strike with respect to the bond numeraire. This may be less realistic for stock options, but is appropriate in the real options setting, and is one of the features of the model that allows us to progress to an explicit solution.

We do not include any additional subjective discount factor in our formulation above. To do so would not change the nature of the solution other than to provide additional encouragement for the agent to exercise early. Note also that we do not allow for intermediate consumption in our problem, the agent is simply selecting how to exercise his allocation of options. Related problems involving consumption and (perfectly indivisible) option exercise in incomplete markets include Miao and Wang [22] and Henderson and Hobson [12].

The intuition behind the form of the solution is that retaining ownership of in-the-money options involves risk that the price of the underlying asset falls. However this risk decreases as the option holdings decrease, and therefore as his holdings decrease we expect the agent to wait for a higher payoff for each option. For this reason we expect the optimal strategy to consist of a waiting or continuation region and an exercise region for which the boundary is given by a decreasing function of the quantity of options remaining. In our model, since the options are perpetual, this boundary is independent of time.

3 The Solution

Let

$$V = \max_{\tau_\phi \in \mathcal{T}} \mathbb{E} U \left(x + \int_0^\theta C(Y_{\tau_\phi}) d\phi \,|Y_0 = y, \Theta_0 = \theta \right) \tag{2}$$

Note that by the Markov property, $V = V(x, y, \theta)$. By the properties of exponential utility, we expect that the value function factorises, so that $V = -\frac{1}{\gamma} e^{-\gamma x} \Lambda(y, \theta)$ for some function Λ. We present the solution to the agent's problem in the following theorem, which relies on the definition of a key function.

Definition 1 *Let $\beta = 1 - 2\nu/\eta^2$ and suppose $\beta > 0$. For $\beta > 1$ define $E(\beta) = \beta/(\beta-1)$, and set $E(\beta) = \infty$ otherwise. For $1 < y < E(\beta)$ define*

$$I(y) = \frac{2}{(y-1)} - (1+\beta) \ln \left(\frac{y}{y-1} \right) + i_{(\beta>1)} \left[(1+\beta) \ln \beta - 2(\beta - 1) \right],$$

where i is the indicator function, and for $\beta > 1$ and $y \geq E(\beta)$ set $I(y) = 0$. Finally, let J be the inverse to I with $J(0) = E(\beta)$ for $\beta > 1$ and $J(0) = \infty$ otherwise.

Theorem 2 *Suppose $\beta > 0$. For $0 < y < \infty$ and $0 \leq \theta < \infty$ define*

$$\Lambda(y, \theta; \gamma, K) = \Lambda(y, \theta) = \begin{cases} 1 - y^\beta J(\gamma\theta K)^{-(\beta+1)} K^{-\beta} (\beta - (\beta-1) J(\gamma\theta K)) & y \leq K J(\gamma\theta K) \\ \beta e^{-(y/K-1)(\gamma\theta K - I(y/K))} (1 - K/y) & K J(\gamma\theta K) < y < K E(\beta) \\ e^{-\gamma(y-K)\theta} & K E(\beta) \leq y \text{ (if } \beta > 1). \end{cases} \tag{3}$$

Then

$$V = V(x, y, \theta) = -\frac{1}{\gamma} e^{-\gamma x} \Lambda(y, \theta) \tag{4}$$

and the optimal strategy is to take

$$\Theta_t = \frac{1}{\gamma K} I \left(\frac{1}{K} \max_{0 \leq s \leq t} Y_s \right) \tag{5}$$

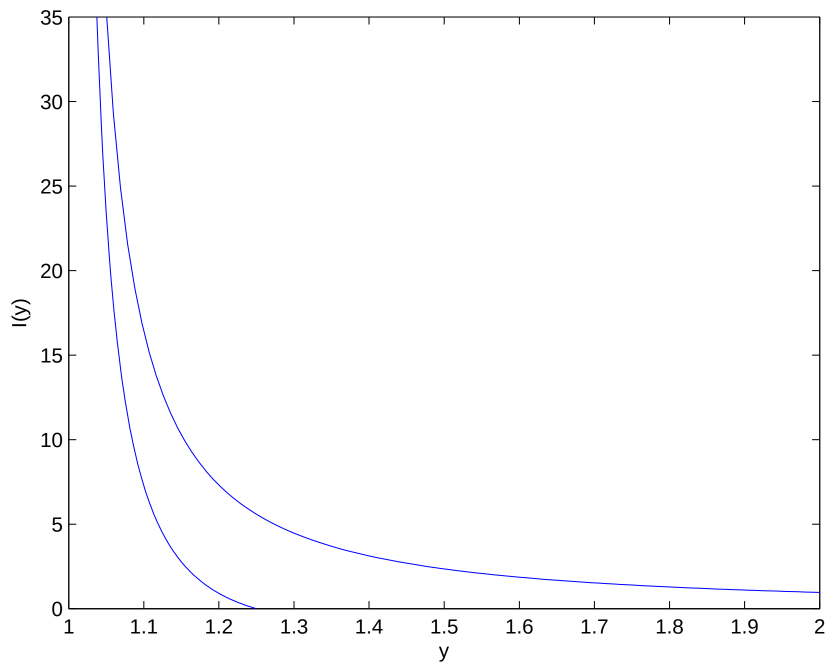

Figure 1: *Plots of $I(y)$ in the two cases $\beta > 1$, and $0 < \beta \leq 1$. The lower line corresponds to $\beta = 5$ and the upper line $\beta = 0.5$. In both cases I is a decreasing convex function defined on $y > 1$. (In general $I(y)$ is monotonic in β, and the convexity in y follows on differentiation.) When $\beta > 1$ we have $I(y) = 0$ for $y \geq E(\beta) = 1.25$. For $\beta \leq 1$ we have $I(y) > 0$ for $y > 1$. The optimal exercise boundary can be deduced from $J \equiv I^{-1}$, and a rescaling to allow for $K \neq 1$ and $\gamma \neq 1$.*

Remark 3
(i) If the initial option holdings θ satisfy $\theta > \frac{1}{\gamma K} I(Y_0/K)$ then the optimal strategy involves exercising a tranche of the initial holdings to reduce the holdings to $\Theta_{0+} = \frac{1}{\gamma K} I(Y_0/K)$. Thereafter the optimal strategy is to exercise just enough options to remain in the region $\Theta_t \leq \frac{1}{\gamma K} I(Y_t/K)$. Since Θ has to be decreasing, this means that Θ_t is given by (5). As a result the optimal exercise strategy is a singular control.
(ii) For $\beta > 1$, $KE(\beta)$ is the threshold for the perpetual American call option problem with strike K for a risk neutral agent. The threshold used by a risk neutral agent is independent of quantity. For $\beta < 1$ it is never optimal for the risk neutral agent to exercise options and the problem is degenerate. Thus, when we introduce incompleteness and risk aversion into the model, the set of parameter values for which we get a non-degenerate problem (with finite exercise thresholds) expands. A similar phenomenon was found in Henderson [10] for the perfectly indivisible case.

Proof of Theorem 2: Given the conjectured value function $-\frac{1}{\gamma} e^{-\gamma x} \Lambda(y, \theta)$, the proof that this is the true value function follows by a standard verification argument.

Consider $M_t = e^{\gamma \int_0^t (Y_s - K)^+ d\Theta_s} \Lambda(Y_t, \Theta_t)$. The idea is to show that M is a martingale under the optimal strategy, and a sub-martingale in general. Then

$$\Lambda(y, \theta) = M_0 \leq \mathbb{E}[M_\infty] = \mathbb{E}[e^{\gamma \int_0^\infty (Y_s - K)^+ d\Theta_s}]$$

where we use that $Y_\infty = 0$ (recall $\beta > 0$) and $\Lambda(0, \theta) = 0$. It follows that

$$V(x, y, \theta) = -\frac{1}{\gamma} e^{-\gamma x} \Lambda(y, \theta) \geq -\frac{1}{\gamma} \mathbb{E}[e^{-\gamma(x - \int_0^\infty (Y_s - K) d\Theta_s)}]$$

as required, with equality for the optimal strategy.

Note that $0 \leq \Lambda \leq 1$. Then we also have $0 \leq M \leq 1$ (recall that any admissible Θ is decreasing) so that in order to prove that M is a uniformly integrable martingale under the optimal strategy it is sufficient to prove that it is a local martingale.

By Ito's formula,

$$dM_t = e^{\gamma \int_0^t (Y_s - K)^+ d\Theta_s} \left\{ \frac{\eta^2}{2} L^Y \Lambda dt + (\dot{\Lambda} + \gamma \Lambda (Y - K)) d\Theta_t + d\tilde{M}_t \right\}$$

where $\tilde{M}_t = \int_0^t \eta Y_s \Lambda'(Y_s) dW_s$ and

$$L^Y \Lambda = y^2 \Lambda'' + y(1 - \beta) \Lambda'$$

and y derivatives have been written as primes and theta derivatives as dots.

To prove the theorem it is sufficient to show that Λ solves (see Rogers and Scheinkman [23, Equations (10)—(12)] for comparison)

$$\min\{L^Y \Lambda, -\dot{\Lambda} - \gamma \Lambda (y - K)\} = 0 \tag{6}$$

with value matching and smooth fit on the boundary:

$$\Lambda, \Lambda', \dot{\Lambda} \text{ are continuous at } y = KJ(\gamma \theta K), \tag{7}$$

(and, if $\beta > 1$, Λ and Λ' are continuous at $KE(\beta)$).

Further we should have $L^Y \Lambda = 0$ for $y \leq KJ(\gamma \theta K)$ and $-\dot{\Lambda} - \gamma \Lambda (y - K) = 0$ for $\theta < \frac{1}{\gamma K} I(y/K)$. These last two conditions are immediate. It is also immediate that if the middle expression in (3) is extended to $y = KJ(\gamma \theta K)$ then it agrees with the upper expression, and if $\beta > 1$ and the middle expression is extended to $y = E(\beta)$ then it agrees with the lower expression. Hence we have value matching. Proofs of the remaining parts of the verification are given in Lemma 8 in the Appendix.

□

4 The Optimal Boundary

In this section we will show how we derived the functional forms of the value function and boundary. Rather than solve the HJB equation (6) subject to (7) - which appears to be a non-trivial exercise since it involves determining the form of both the optimal boundary, and the value function on the boundary - we solve for the value function for an arbitrary exercise boundary and then use calculus of variations to determine the optimal boundary.

Consider the strategy of exercising the infinitesimal θth (to go) unit of option, the first time, if ever, that the underlying asset price exceeds $h(\theta)$. Here h is a decreasing function such that $h(\theta) \geq K$. This last condition reflects the fact that it is never optimal to exercise a call which is out of the money. More precisely, let $\Theta_t = h^{-1}(\max_{0 \leq s \leq t} Y_s)$ for some function h. We assume that h is continuous and differentiable, and therefore has a well-defined inverse.

First, suppose first that $\theta \equiv \Theta_0 \leq h^{-1}(Y_0)$. The total revenue from exercising the options is

$$R = \int_0^\theta d\phi (h(\phi) - K)^+ i_{(\max_{0 \leq t \leq \infty} Y_t \geq h(\phi))}$$

We are interested in $\mathbb{E}[e^{-\gamma R}]$ for some fixed exercise boundary h.

Second, suppose $\Theta_0 > h^{-1}(Y_0) > 0$. In this case we perform an initial lump-sum exercise to reduce holdings to $h^{-1}(Y_0)$. Thereafter option exercises are such that Θ is kept as large as possible without violating the condition $\Theta_t \leq h^{-1}(Y_t)$. For this strategy, in the region $\theta > h^{-1}(y)$ we have that total revenue is

$$R = (y - K)(\theta - h^{-1}(y)) + \int_0^{h^{-1}(y)} d\phi (h(\phi) - K)^+ i_{(\max_{0 \leq t \leq \infty} Y_t \geq h(\phi))}$$

and

$$\mathbb{E}^{y,\theta}[e^{-\gamma R}] = e^{-\gamma(y-K)(\theta - h^{-1}(y))} \mathbb{E}^{y, h^{-1}(y)}[e^{\gamma \int_0^{h^{-1}(y)} (Y_s - K)^+ d\Theta_s}].$$

This means the problem reduces to finding $\mathbb{E}[e^{-\gamma R}]$ in the region $y \leq h(\theta)$.

Third, suppose that $y \geq h(0)$. In this case we exercise all the options. The value function associated with this strategy is $-e^{-\gamma x - \gamma\theta(y-K)}/\gamma$.

We will use the convention that h (and f) denote general (decreasing, continuous) functions, and that H (and F) denote the optimal boundary.

Proposition 4 *For $y \leq h(\theta)$*

$$\mathbb{E}^{y,\theta}[e^{-\gamma R}] = 1 - y^\beta D_h(\theta)$$

where

$$D_h(\theta) = \gamma \int_0^\theta d\phi \; h(\phi)^{-\beta}(h(\phi) - K) e^{-\gamma \int_\phi^\theta d\psi (h(\psi) - K)}.$$

Proof of Proposition 4

Let $S = \max_{0 \leq t < \infty} Y_t$ and let $R(s)$ denote the total revenue from exercising options conditional on $S = s$:

$$R(s) = \int_0^\theta d\phi (h(\phi) - K) i_{(s \geq h(\phi))} = \int_{h^{-1}(s)}^{\theta \vee h^{-1}(s)} d\phi (h(\phi) - K).$$

Then, with $R \equiv R(S)$,

$$\begin{aligned}
\mathbb{E}[e^{-\gamma R}] &= \int_y^\infty \mathbb{P}(S \in ds) e^{-\gamma R(s)} \\
&= \left[-\mathbb{P}(S \geq s) e^{-\gamma R(s)} \right]_y^\infty - \gamma \int_y^\infty \mathbb{P}(S \geq s) R'(s) e^{-\gamma R(s)} ds \\
&= 1 - \gamma y^\beta \int_{h(\theta)}^{h(0)} \frac{1}{s^\beta} \left| \frac{d}{ds} h^{-1}(s) \right| C(s) e^{-\gamma R(s)} ds \\
&= 1 - \gamma y^\beta \int_0^\theta \frac{1}{h(\phi)^\beta} d\phi C(h(\phi)) e^{-\gamma \int_\phi^\theta d\psi C(h(\psi))}
\end{aligned}$$

where we use the fact that $\mathbb{P}^y(S \geq s) = (y/s)^\beta$.

□

We now observe that provided, as must be the case, that the optimal strategy is of the conjectured form, then agent's problem in (2) can be written as

$$V = \max_{h \geq K} \mathbb{E}U(x+R) = -\frac{1}{\gamma}e^{-\gamma x} \min_{h \geq K} \mathbb{E}e^{-\gamma R} = -\frac{1}{\gamma}e^{-\gamma x}[1 - y^\beta \max_{h \geq K} D_h(\theta)] \tag{8}$$

using Proposition 4, at least when $y \leq h(\theta)$. The following lemma (which works with dimensionless quantities) and subsequent corollary (which translates the result to the original variables) give expressions for

$$\max_{h \geq K} D_h(\theta)$$

which will allow us to find the optimal boundary H and the associated value function V. Recall $\beta > 0$.

Lemma 5 *Suppose $\alpha > 0$. Define*

$$A(\alpha) = \max_{f \geq 1} \int_0^1 dx f(x)^{-\beta}(f(x)-1)e^{-\alpha \int_x^1 dz(f(z)-1)} \tag{9}$$

Then $A(\alpha) = J(\alpha)^{-(\beta+1)}[\beta - (\beta-1)J(\alpha)]/\alpha$, where $J(z)$ is the function defined in Definition 1. The maximum in (9) is attained at $F(x) = J(\alpha x)$.

The following Corollary is immediate.

Corollary 6

$$\max_{h \geq K} D_h(\theta) = K^{-\beta} J(\gamma\theta K)^{-(\beta+1)}[\beta - (\beta-1)J(\gamma\theta K)]$$

where the maximum is attained at $H(\phi) = KJ(\phi\gamma K)$.

Proof of Lemma 5

Suppose first that $\alpha = 0$. Then it is easy to see that provided $\beta > 1$ the maximum is attained by $F(x) = \frac{\beta}{\beta-1}$, or $F(x) = \infty$ if $\beta \leq 1$.

More generally,

$$\begin{aligned}\int_0^1 dx\ f(x)^{-\beta}(f(x)-1)e^{-\alpha \int_x^1 dz(f(z)-1)} &= \int_\epsilon^1 dx\ f(x)^{-\beta}(f(x)-1)e^{-\alpha \int_x^1 dz(f(z)-1)} \\ &\quad + e^{-\alpha \int_\epsilon^1 dz(f(z)-1)} \int_0^\epsilon dx\ f(x)^{-\beta}(f(x)-1)e^{-\alpha \int_x^\epsilon dz(f(z)-1)}.\end{aligned}$$

Hence the optimiser over $[0, \epsilon]$ does not depend on the optimiser over $[\epsilon, 1]$ and moreover when we rescale the last integral we find that it can be rewritten as

$$\epsilon \int_0^1 \tilde{f}(y)^{-\beta}(\tilde{f}(y)-1)e^{-\alpha\epsilon \int_y^1 dz(\tilde{f}(z)-1)}$$

where $\tilde{f}(y) = f(\epsilon y)$. As $\epsilon \downarrow 0$ the optimum is given by $\tilde{f} \equiv \frac{\beta}{\beta-1}$ so we conclude that $f(0) = \frac{\beta}{\beta-1}$ for $\beta > 1$ and $f(0) = \infty$ otherwise.

Let $g(x) = \int_x^1 (f(z)-1)dz$. We want to maximise

$$-\int_0^1 dx(1-g'(x))^{-\beta}g'(x)e^{-\alpha g(x)}.$$

By calculus of variations (or equivalently by setting $g(x) = G(x) + \epsilon\eta(x)$ where G is the optimiser, and considering the first order expansion in ϵ) we find that the maximiser G satisfies (see Arfken [1], Equation 17.18)

$$(1-G'(x))^{-\beta}G'(x)e^{-\alpha G(x)} - G'\frac{\partial}{\partial G'}\left[(1-G'(x))^{-\beta}G'(x)e^{-\alpha G(x)}\right] = \text{constant}.$$

It follows that
$$(1-G'(x))^{-(\beta+1)}G'(x)^2e^{-\alpha G(x)} = \text{constant}.$$

Differentiating we find
$$\frac{(1+\beta)G''(x)}{1-G'(x)} + \frac{2G''(x)}{G'(x)} - \alpha G'(x) = 0$$
which simplifies to
$$F'(x) = \frac{-\alpha F(x)(1-F(x))^2}{(1+\beta)-(\beta-1)F(x)}$$
where $F(x) = 1-G'(x)$ is the optimiser in (9). Write $F(x) = J_0(\alpha x)$. Then J_0 solves
$$J_0'(z) = \frac{-J_0(z)(J_0(z)-1)^2}{(1+\beta)-(\beta-1)J_0(z)}$$
with $J_0(0) = E(\beta)$. It remains to show that $J_0 \equiv J$ where J is given in Definition 1.

Let $I_0 = J_0^{-1}$. Then, provided $y < E(\beta)$,
$$I_0'(y) = -\frac{[(1+\beta)-(\beta-1)y]}{y(y-1)^2}$$
and $I_0(E(\beta)) = 0$. For $\beta > 1$ and $1 \le y \le E(\beta)$,
$$\begin{aligned} I_0(y) &= \int_y^{\frac{\beta}{\beta-1}} \frac{(1+\beta)-(\beta-1)w}{w(w-1)^2}dw \\ &= \int_y^{\frac{\beta}{\beta-1}} \left\{\frac{2}{(w-1)^2} - \frac{1+\beta}{w-1} + \frac{1+\beta}{w}\right\} dw \\ &= \left[(1+\beta)\ln\left(\frac{w}{w-1}\right) - \frac{2}{w-1}\right]_y^{\frac{\beta}{\beta-1}} \\ &= (1+\beta)\ln\beta - 2(\beta-1) - (1+\beta)\ln\left(\frac{y}{y-1}\right) + \frac{2}{y-1}. \end{aligned}$$
If $\beta \le 1$ we get the simpler formula
$$I_0(y) = \frac{2}{y-1} - (1+\beta)\ln\left(\frac{y}{y-1}\right).$$
Hence $I_0(y) = I(y)$ as required.

It remains to find
$$A(\alpha) = \max_{f\ge 1}\int_0^1 dx\ \ f(x)^{-\beta}(f(x)-1)e^{-\alpha\int_x^1 dz(f(z)-1)} = \int_0^1 dx\ \ F(x)^{-\beta}(F(x)-1)e^{-\alpha\int_x^1 dz(F(z)-1)}.$$
Recall that for F we have
$$F(x)^{-(\beta+1)}(F(x)-1)^2e^{-\alpha\int_x^1 dz(F(z)-1)} = F(1)^{-(\beta+1)}(F(1)-1)^2$$
Then
$$\begin{aligned} \int_0^1 dxF(x)^{-\beta}(F(x)-1)e^{-\alpha\int_x^1 dz(F(z)-1)} &= \int_0^1 dx\frac{F(x)}{F(x)-1}\{F(x)^{-(\beta+1)}(F(x)-1)^2e^{-\alpha\int_x^1 dz(F(z)-1)}\} \\ &= F(1)^{-(\beta+1)}(F(1)-1)^2\int_0^1 dx\frac{F(x)}{F(x)-1}. \end{aligned}$$
But, if we use the substitution $y = J(z)$,

(10) $$\int_0^1 dx\frac{F(x)}{F(x)-1} = \frac{1}{\alpha}\int_0^\alpha dz\frac{J(z)}{J(z)-1} = \frac{1}{\alpha}\int_{J(\alpha)}^{J(0)} dy\frac{[(\beta+1)-(\beta-1)y]}{(y-1)^3} = \frac{\beta-(\beta-1)J(\alpha)}{\alpha(J(\alpha)-1)^2}.$$

Collecting the terms together we find that,

$$A(\alpha) = J(\alpha)^{-(\beta+1)}(\beta - (\beta - 1)J(\alpha))/\alpha.$$

Note $A(\alpha)$ is well defined for all $\beta > 0$ and not just $\beta > 1$. □

Thus we have shown how to evaluate the value function for a strategy based on a boundary from a given family, and how to choose the optimal member of that family. It simply remains to collate the various formulæ in order to show how the candidate value function in (4) was derived.

It follows from (8) that for the optimal boundary and for $y \leq H(\theta)$ we have the candidate value function $V = -\frac{1}{\gamma}e^{-\gamma x}\Lambda(y,\theta)$ where Λ is as given in Theorem 2. Further, by the remarks before Proposition 4, for $H(\theta) < y < E(\beta)$ we have

$$V = -\frac{1}{\gamma}e^{-\gamma x}e^{-\gamma(y-K)(\theta-H^{-1}(y))}\Lambda(y, H^{-1}(y)).$$

But, now we note that

$$\begin{aligned}\Lambda(y, H^{-1}(y)) &= 1 - \frac{y^\beta}{J(\gamma K H^{-1}(y))^{\beta+1}K^\beta}(\beta - (\beta-1)J(\gamma K H^{-1}(y))) \\ &= \beta(1 - K/y),\end{aligned}$$

where we use the fact that $J(\gamma K H^{-1}(y)) = y/K$. Again, we find that the value function for the optimal boundary is as given in Theorem 2. Finally, for $\beta > 1$ and $y \geq E(\beta)$ the candidate optimal strategy is to exercise all the options in a single tranche, and then

$$V = -\frac{1}{\gamma}e^{-\gamma x}e^{-\gamma(y-K)\theta}.$$

5 Hedging with a Correlated Asset

In previous sections we have treated the option exercise problem under the assumption the agent is not able to trade in the underlying asset. In fact, we have assumed that the option exercise problem can be studied in isolation, and that there are no other assets available for investment. In this section we continue to assume the agent is forbidden or unable to trade the underlying, but we assume there is a correlated asset available for hedging. More generally, the case with several correlated hedging assets can be reduced to this case.

Recall that the underlying asset price is given by

$$\frac{dY}{Y} = \nu dt + \eta dW,$$

and suppose there is a second correlated asset P with price process

$$\frac{dP}{P} = \mu dt + \sigma dB$$

where B and W are correlated Brownian motions. We assume the agent is free to invest in P so his wealth process (with bond as numeraire) is given by

$$X_t = x + \int_0^t \pi_s \frac{dP_s}{P_s}.$$

The choice facing the agent is to choose the optimal exercise boundary $h(\theta)$ - it is clear that for exponential utility wealth factors out and the agent should exercise options at a threshold which is a function of the number of remaining options alone - and the hedging strategy $(\pi_s)_{s\geq 0}$. This model - the non-traded assets model - has been extensively studied in related contexts, see Davis [4] and Henderson and Hobson [14] for references.

In what follows we assume $\mu = 0$. For the finite expiry problem, the general case can easily be reduced to this case by an equivalent change of measure. Over the infinite horizon some care is needed both in the statement and interpretation since this change of measure is no longer equivalent, see Henderson and Hobson [13].

We show the Hamilton-Jacobi-Bellman equation can be reduced to that in our original model. Our problem is to find

$$\inf_{\pi,\Theta\in\mathcal{S},\Theta_0=\theta} \mathbb{E}[e^{-\gamma x-\gamma\int_0^\infty \pi_s dP_s/P_s+\gamma\int_0^\infty d\Theta_s C(Y_s)}] \equiv e^{-\gamma x}\Gamma(y,\theta).$$

From the fact that

$$e^{-\gamma\int_0^t \pi_s dP_s/P_s+\gamma\int_0^t d\Theta_s C(Y_s)}\Gamma(Y_t,\Theta_t)$$

should be a martingale under the optimal strategy and a submartingale otherwise, we find that

$$\min\left\{-\dot{\Gamma}-\gamma(y-K)\Gamma\ ;\ \inf_\pi\left(\frac{\eta^2}{2}L^Y\Gamma-\gamma\pi\sigma\rho\eta y\Gamma'+\frac{\pi^2}{2}\gamma^2\sigma^2\Gamma\right)\right\}\geq 0,$$

or, after minimising with respect to π,

$$\min\left\{-\dot{\Gamma}-\gamma(y-K)\Gamma\ ;\ \left(L^Y\Gamma-\frac{(\Gamma')^2}{\Gamma}\rho^2y^2\right)\right\}\geq 0 \tag{11}$$

Write $\Gamma = \Psi^\delta$ for $\delta = \frac{1}{1-\rho^2}$. (This transformation is the Hopf-Cole transformation, as used by Zariphopoulou [24]). The HJB equation (11) becomes

$$\min\left\{-\dot{\Psi}-\gamma(1-\rho^2)\Psi(y-K)\ ;\ y^2\Psi''+(1-\beta)y\Psi'\right\}\geq 0$$

with value matching and smooth fit conditions. It follows that $\Psi(y,\theta) = \Lambda(y,\theta)$ where Λ is given in Theorem 2 but with an effective risk aversion of $\gamma(1-\rho^2)$, so that

$$\Gamma(y,\theta) = \left(\Lambda(y,\theta;\gamma(1-\rho^2),K)\right)^{1/(1-\rho^2)}.$$

6 Discussion and Conclusions

In the main text we concentrated on determining the optimal selling strategy for a risk averse agent in possession of perpetual American claims. Now we briefly talk about the value of these options.

The value of the options can be obtained via utility indifference pricing (see Hodges and Neuberger [15] and the survey of Henderson and Hobson [14]) as the solution to $U(x+p) = V(x,y,\theta)$. From Theorem 2 we obtain:

Corollary 7 *The certainty equivalent value $p_t(\theta;y)$ for θ units of the perpetual American call is given by*

$$p_t(\theta;y) = -\frac{1}{\gamma}\ln\Lambda(y,\theta)$$

The superscript t refers to the fact that p_t is the total value of θ options. We can also define the marginal price:

$$p_m(\theta;y) = \frac{\partial}{\partial\theta}p_t(\theta) = -\frac{1}{\gamma}\frac{\dot{\Lambda}(y,\theta)}{\Lambda(y,\theta)}.$$

For $y < KJ(\gamma K\theta)$ we have

$$p_m(\theta;y) = \frac{\beta Ky^\beta(J(\gamma K\theta)-1)^2}{J(\gamma K\theta)^{\beta+1}K^\beta - y^\beta(\beta-(\beta-1)J(\gamma K\theta))}$$

and $p_m(\theta;y) = (y-K)$ otherwise.

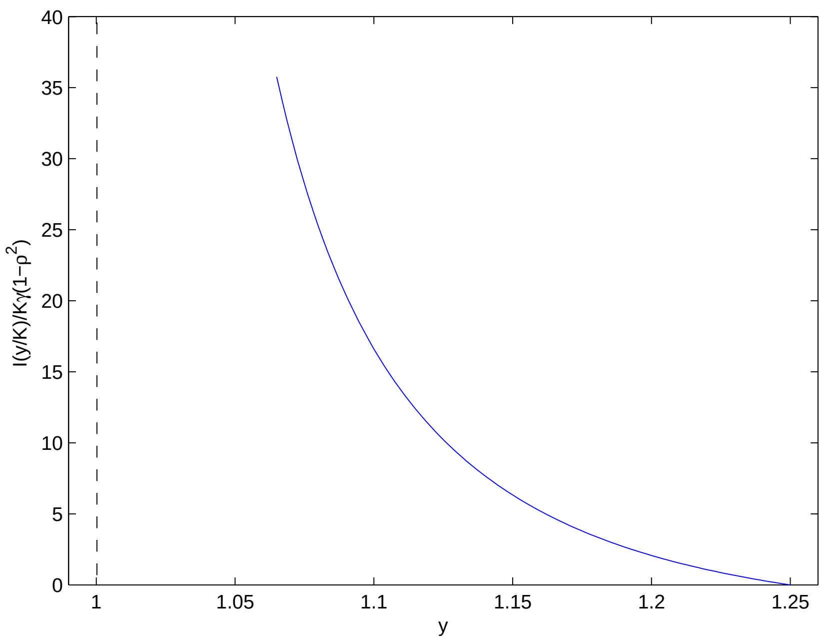

Figure 2: *The continuation region is given by* $\Theta \leq I(Y/K)/\gamma(1-\rho^2)K$*. This is plotted for* $\beta = 5$ *together with* $\gamma = 1$*,* $K = 1$ *and* $\rho = 0.75$*. In comparison with the* $\beta = 5$ *(lower) line in Figure 1 the effect of introducing the hedging instrument is to reduce the effective risk aversion, or equivalently to increase the continuation region.*

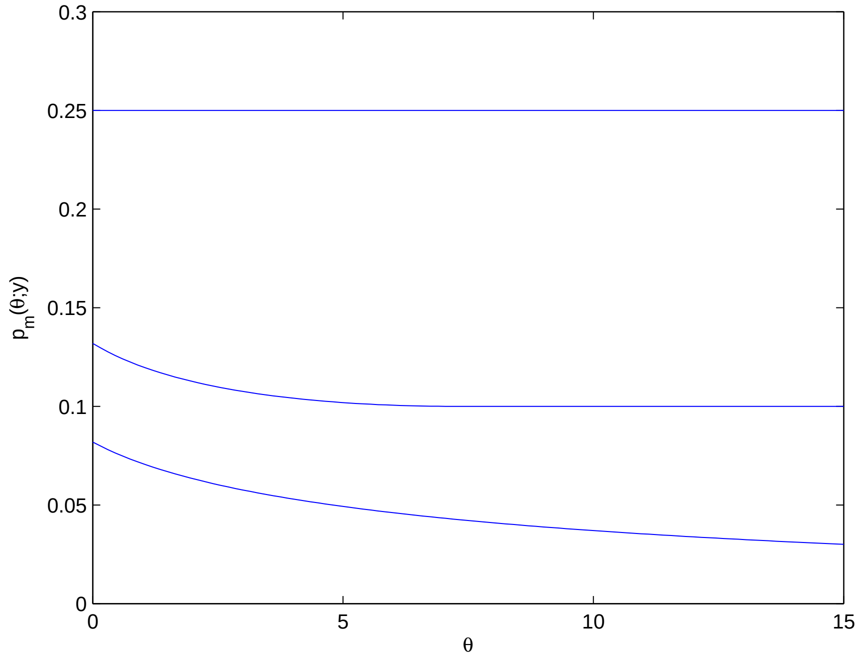

Figure 3: *The marginal utility indifference price plotted as a function of the number of remaining options θ, and as a function of three different values of the underlying asset price y, namely, and starting from the lowest, $y = 1, 1.1, 1.25$. We take $\beta = 5$. The call has strike $K = 1$ and the agent has unit risk aversion. The marginal price is increasing in y, decreasing in θ and satisfies $p_m(\theta; y) \geq (y-1)^+$. For $y = 1.1$ we have that for large enough θ the marginal price is given by the intrinsic value, but for sufficiently small θ the price exceeds the intrinsic value.*

We find that $p_m(\theta; y) \geq (y-K)^+$, is decreasing in θ, and the marginal price for the first units of claim is

$$p_m(0; y) = K^{1-\beta} y^\beta \beta^{-\beta} (\beta - 1)^{\beta - 1} \qquad \beta > 1$$

and $p_m(0; y) = \infty$ if $0 < \beta \leq 1$. Finally we have $\lim_{\theta \uparrow \infty} p_m(\theta, y) = (y - K)^+$. Thus the marginal value of the option is at least as great as the intrinsic value, but is decreasing in the number of options held. See Figure 3.

The main aim of this paper has been to give an explicit closed form solution to the problem of optimal exercise for American options. To this end we made various simplifying assumptions, including the fact that the options are perpetual. On the other hand, we are able to move beyond the unrealistic case where the agent is forbidden to trade in any assets, and we allow the agent to hedge using a correlated instrument.

Closed form solutions are very rare, but also very powerful for understanding the comparative statics of the problem. Jain and Subramanian [16], see also Rogers and Scheinkman [23] found in a discretised model that options were exercised over time. Our results echo this finding, and we are able to give an explicit form for the boundary.

The solution depends on a key parameter β which is related to the Sharpe ratio per unit of volatility for the underlying asset. There are three cases. If $\beta \leq 0$, the problem is degenerate, whereas the problem has an interesting solution in the case $\beta > 0$. This solution changes character at $\beta = 1$. For $0 < \beta \leq 1$ we find that the problem for exponential utility has a non-trivial solution whereas the corresponding problem for a risk neutral agent is degenerate. For $\beta \le 1$, it is never optimal for the agent to exercise all his options, whereas for $\beta > 1$ this event can happen with positive probability.

7 Appendix

We provide details of the verification argument[1] in this section.

Lemma 8 *(i) In the continuation region $y \le KJ(\gamma\theta K)$*

$$-\dot{\Lambda} - \gamma(y-K)\Lambda \ge 0.$$

(ii) In the exercise region $\theta > \frac{1}{\gamma K} I(y/K)$

$$L^Y \Lambda \ge 0.$$

(iii) At $y = KJ(\gamma\theta K)$, we have that Λ, Λ' and $\dot{\Lambda}$ are all continuous. Further if $\beta > 1$, then Λ and Λ' are continuous at $y = KE(\beta)$.

Proof of Lemma 8:

We take $\gamma = K = 1$. The general case can be recovered by appropriate scalings.

(i) For $y \le J(\theta)$ we have

$$\Lambda(y,\theta) = 1 - \frac{y^\beta}{J(\theta)^{\beta+1}}(\beta - (\beta-1)J(\theta)).$$

Differentiating, and substituting for $\dot{J}(\theta)$ we obtain

$$\dot{\Lambda} = -\frac{y^\beta \beta (J(\theta)-1)^2}{J^{\beta+1}(\theta)}.$$

Fix θ and write J as shorthand for $J(\theta)$. Let

$$\Sigma(y) = \dot{\Lambda} + (y-1)\Lambda = y - 1 + \frac{y^\beta}{J^\beta}\{(\beta+1) - \beta J\} - \frac{y^{\beta+1}}{J^{\beta+1}}(\beta - (\beta-1)J)$$

and consider $\Sigma(y)$ for $y \in (0,J)$. We have $\Sigma(0) = -1, \Sigma(J) = 0$ and $\Sigma'(J) = 0$. Further,

$$\Sigma''(y) = \beta \frac{y^{(\beta-2)}}{J^\beta}\left\{(\beta-1)[(\beta+1) - \beta J] - (\beta+1)\frac{y}{J}(\beta - (\beta-1)J)\right\}.$$

Then $\Sigma''(J) = -\frac{\beta(\beta+1-(\beta-1)J)}{J^2} < 0$ since $(\beta-1)J \le \beta$, and there is at most one root of $\Sigma''(y) = 0$ in $(0,J)$. Hence $\Sigma(y) \le 0$ as required.

(ii) Suppose $\beta < 1$. From Theorem 2, for $y > J(\theta)$, we have

$$\Lambda(y,\theta) = \beta e^{-(y-1)(\theta - I(y))}(1 - 1/y).$$

We want $L^Y\Lambda \ge 0$ where $L^Y\Lambda = y^2\Lambda'' + (1-\beta)y\Lambda'$. After some calculation, we find

$$\frac{\Lambda'}{\beta e^{-(y-1)(\theta-I(y))}} = -(\theta - I(y))\frac{y-1}{y} - \frac{(\beta-(\beta-1)y)}{y^2},$$

and

$$\frac{\Lambda''}{\beta e^{-(y-1)(\theta-I(y))}} = (\theta - I(y))^2\frac{y-1}{y} + 2(\theta - I(y))\frac{\beta-(\beta-1)y}{y^2} + \frac{(1-\beta)(\beta-(\beta-1)y)}{y^3}.$$

Finally we have

$$\frac{y^2\Lambda'' + (1-\beta)y\Lambda'}{\beta e^{-(y-1)(\theta-I(y))}} = (\theta - I(y))^2 y(y-1) + (\theta - I(y))[(\beta+1) - (\beta-1)y] \ge 0 \tag{12}$$

[1]This appendix is provided for completeness. We are happy for it to be omitted from any published version.

since $y > J(\theta) > 1$ and $\beta < 1$.

Now suppose $\beta > 1$. Then (12) is valid for $y < E(\beta)$, and so in this range, $y^2\Lambda + (1-\beta)y\Lambda' \geq 0$. Further, for $y > E(\beta)$, $\Lambda(y,\theta) = e^{-(y-1)\theta}$ and in this region $y^2\Lambda'' + (1-\beta)y\Lambda' = \theta y[\theta y + (\beta-1)]\Lambda > 0$.
(iii) At $y = J(\theta)_-$, $\Lambda = 1 - \frac{1}{J}(\beta - (\beta-1)J) = \beta(1-\frac{1}{J})$; at $y = J(\theta)_+$, $\Lambda = \beta(1-\frac{1}{J})$. Similarly, at $y = J(\theta)_-$, $\Lambda'(J(\theta)_-,\theta) = -\frac{\beta}{J^2}(\beta-(\beta-1)J)$, and at $y = J(\theta)_+$, $\Lambda'(J(\theta)_+,\theta) = -\frac{\beta}{J^2}(\beta-(\beta-1)J)$. Also we have $\dot{\Lambda} + (y-1)\Lambda = 0$ for $y > J(\theta)$. For $y \leq J(\theta)$, we have (from (i)) $\Sigma(y) \leq 0$ with $\Sigma(y) = 0$ at $y = J(\theta)$. Since we have value-matching, we must also have $\dot{\Lambda}$ matches at $y = J(\theta)$. We remark that smooth pasting in θ together with value-matching is sufficient to deduce smooth pasting in y: simply consider differentiating $\Lambda(J(\theta)_+,\theta) = \Lambda(J(\theta)_-,\theta)$.

Now suppose $\beta > 1$ and consider Λ, Λ' at $y = E(\beta)$. We have

$$\Lambda(E(\beta)_-,\theta) = \beta e^{-(E(\beta)-1)\theta}\left(\frac{\beta-(\beta-1)}{\beta}\right) = e^{-(E(\beta)-1)\theta} = \Lambda\left(E(\beta)_+,\theta\right).$$

Similarly, $\Lambda'(E(\beta)_-,\theta) = -\theta e^{-(E(\beta)-1)\theta} = \Lambda'(E(\beta)_+,\theta)$.

□

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