Optimal Liquidation of Derivative Portfolios†

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Abstract

We consider the problem facing a risk averse agent who seeks to liquidate or exercise a portfolio of (infinitely divisible) American style options. The optimal liquidation strategy is of threshold form and can be characterised explicitly as the solution of a calculus of variations problem. Apart from a possible initial exercise of a tranche of options, the optimal behaviour involves liquidating the portfolio in infinitesimal amounts, but at times which are singular with respect to calendar time. We consider a number of illustrative examples involving CRRA and CARA utility, stocks and portfolios of options with different strikes, and a model where the act of exercising has an impact on the underlying asset price.

1 Introduction

In this paper we consider the problem facing a risk-averse agent who seeks to liquidate or exercise a portfolio of American-style options. The agent cannot hedge and so faces an incomplete market. If the market were complete, the agent would exercise a tranche of options at a single stopping time. This is no longer true in our incomplete setting, and the key contribution of this paper is to allow the agent to exercise fractions of the portfolio over time. Indeed, the options are assumed to be perfectly divisible, and the (potentially different) constituents of the portfolio may be exercised at a family of stopping times.

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Our objective is to study the optimal liquidation problem in a reasonably general set-up with a time-homogeneous diffusion price process, general utility function, and option payoff. Our first result is to show that the optimal exercise strategy is of theshold form - the agent should exercise options the first time the asset price reaches a barrier level which depends on that part of the portfolio which has not yet been exercised. Just enough options are exercised to keep the portfolio holdings below the barrier, and thus the solution is a singular control.

This barrier can be characterised in an explicit form as the solution of a calculus of variations problem. We give three examples where we can perform explicit calculations, which are chosen to illustrate a variety of situations covered by our approach. The idea is to calculate the value function for any strategy of threshold type, and then to choose the best barrier. This is an alternative to the usual HJB approach, although once we have solved for the candidate barrier, we use the HJB approach for the verification. The advantage of this approach is that it decouples the problems of finding the value function and finding the optimal strategy, whereas the standard HJB approach finds both simultaneously. The disadvantage is that it requires that a solution for the value function can be found for a wide class of strategies, and not just the optimal strategy. Similar ideas have been tried on other stochastic control problems: contrast the solutions of Peskir [17] and Hobson [14] to an optimal stopping problem for Brownian motion.

There are several key features of the model which are crucial in allowing us to make progress in obtaining an explicit solution. Firstly, we work with respect to a single numeraire and assume that the characteristics of the option payoff and the agent's utility function can be expressed relative to this numeraire in such a way that they are time-independent. Secondly, we assume that the dynamics of the underlying asset are also time-homogeneous. Thirdly, we assume that the aim of the agent is to maximise utility of wealth, where wealth is calculated as revenue from the exercise of options over the infinite horizon. Finally, the options themselves are perpetual American style. All of these features combine to make the problem independent of time. However, we are still left with three dimensions - the asset price, wealth of the agent, and the quantity of options remaining.

The observation that partial exercise of American options may be optimal in an incomplete market appears in models considering identical options which are exercisable in discrete units. A natural setting for this problem is executive stock options, whereby managers receive American call options on the stock of their company, and are typically constrained from hedging, see Carpenter [6]. Jain and Subramanian

[15] treated the multiple option exercise problem in a binomial setting. Grasselli and Henderson [11] obtain closed-form solutions for successive exercise thresholds in a model of multiple perpetual American options. Also, Rogers and Scheinkman [18] and Leung and Sircar [16] have studied numerical solutions for exercise thresholds for options with a finite maturity. Another area where problems of American option exercise arise is that of real options (see Dixit and Pindyck [10]), and Henderson [12] considered the situation of a single perpetual American call option and dervied closed-form solutions for the threshold and utility indifference option value. Swing options also result in consideration of a multiple optimal stopping problem, see Carmona and Touzi [5].

We consider three representative examples which can be tackled using our results. In general, the optimal boundary depends upon the wealth of the agent. Our first example with power utility illustrates this dependence. The boundary is wealth and portfolio dependent and satisfies a consistency condition with regard to the initial endowment.

The second situation we consider is a portfolio of (call) options with different strikes. We begin with a lemma which under limited but natural circumstances describes the order in which options are exercised under optimal behaviour. We solve explicitly for the boundary when the portfolio has call options with two different strikes. The boundary has a jump at the point where the agent switches from exercising the low strike calls to the high strike calls. The solution is such that the agent waits for a higher asset price before exercising the low strike options than an agent with a portfolio containing the same total quantity of options but each with the same low strike. Henderson and Hobson [13] previously considered a special case of this example with identical options, exponential utility, and lognormal price dynamics.

In our third example, we incorporate a price impact such that exercising options or selling stock will have an effect on the asset price. Our example takes an exponential Brownian motion for the fundamental value of the asset, and appends a permanent price impact which is a function of the quantity sold or exercised. This mechanism is similar to those used in studies of the price impact of a large trader (see Bank and Baum [4] and references therein). The solution remains a singular control, however the price impact causes the boundary to shift such that the waiting region is larger. Intuitively, since exercising causes the price to fall, in order to compensate, the agent waits for a higher price level to start exercising. Various limiting situations are of interest. We show that for some parameter values, the agent will always retain part of her portfolio. The solution also has the feature that even with an arbitrarily large option position, it might be the case that the agent never exercises any options, even though they all start in-the-money.

The character of our solution as a singular control makes our example quite different to studies of liquidity involving a temporary price impact (Almgen [2], Almgen and Chriss [3], Cetin, Jarrow and Protter [7], Cetin and Rogers [8] and Rogers and Singh [19]). Usually the temporary impact plays the role of a transaction cost and there is a trade-off between unwinding a position quickly to reduce risk exposure and the accumulation of transactions costs. In particular, Scheid and Schoneborn [20] model stock liquidation by a risk averse agent with Brownian price dynamics and characterise the liquidation rate as the solution of a non-linear parabolic pde.

The paper is structured as follows. In the next section we describe the set-up of the model in detail. In Section 3 we describe how the problem can be translated into the problem of choosing an optimal boundary to minimise a certain functional, and how the optimum can be characterised via calculus of variations. In Section 4 we apply this general solution to several important examples. These examples form the most important feature of this study.

2 The model

We consider a risk-averse agent with an initial portfolio of options on a single underlying. Her aim is to maximise the expected utility of the total revenue achieved from the liquidation of her portfolio.

The portfolio consists of options with (potentially) different characteristics. The main example we have in mind is when the different types of options have different strikes. Her initial portfolio can be considered as a measure ρ on the space $\mathcal{K} \subseteq \mathbb{R}$ of possible types or labels. If ρ_t is the measure of un-exercised options at time t then $(\rho_t)_{t\geq 0}$ is a decreasing family of measures (on $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$) with $\rho_0 = \rho$ and $\rho_t \geq 0$. We denote by Θ_t the total number of remaining options so that $\Theta_t = \int_{\mathcal{K}} \rho_t(dk)$ and $\Theta_0 = \theta_0 = \int_{\mathcal{K}} \rho(dk)$: we will argue below that for the class of problems which we consider the general setting of a measure ρ_t can be reduced to a simpler case in which Θ_t is a sufficient statistic. This is clearly the case if the portfolio of options are all of the same type.

We assume that all the options are on the same underlying asset. We let X_t describe the price of this asset. We work on a filtration $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ supporting a Brownian motion B and assume that X_t follows a time-homogeneous diffusion[1] process with dynamics

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt \qquad\qquad X_0 = x$$

such that X is transient to a lower value $\underline{x}$ (typically 0). We denote the scale function of X by $\mathcal{S}$ which we normalise so that $\mathcal{S}(\underline{x}) = 0$, and the generator of X by L^X so that $L^X f = (1/2)\sigma(x)^2 f_{xx} + \mu(x) f_x$. In particular, $\mathcal{S}$ is a soluton to $L^X f = 0$.

All options are assumed to be perpetual American options, whose payoff does not depend explicitly on t. Instead the payoff from exercising a unit of option with label k is given by $C = C(x, \theta, k)$, where C is continuous. The total revenue from the liquidation of options by the agent is

$$R = \int_{t=0}^{\infty} \int_{\mathcal{K}} C(X_t, \Theta_t, k)(-d\rho_t(dk))$$

and the objective of the agent is to maximise

$$\mathbb{E}[U(w + R)]$$

for fixed initial wealth[2] w and concave (and continuously differentiable) utility function U. Here the maximisation is taken over adapted decreasing families of measures ρ_t.

2.1 Examples

Example 2.1 (Portfolio) The most important interpretation of the option label or type is as the strike of the option. The choice $\rho_0(dk) = \theta_0 \delta(k - K)dk$, where δ is the Dirac function, corresponds to a portfolio of identical call options each with strike K. If $\rho_0(dk) = \sum_i \theta^{(i)} \delta(k - K^{(i)})dk$ where $\sum_i \theta^{(i)} = \theta_0$ then the agent has a portfolio of call options with different strikes.

Example 2.2 (Dynamics) Typically we will assume that the dynamics of X are lognormal so that $dX_t = X_t(\sigma dB + \mu dt)$ (with $\mu < \sigma^2/2$). Then $\underline{x} = 0$ and $\mathcal{S}(x) = x^\beta$ with $\beta = 1 - 2\mu/\sigma^2$. Alternatively we may take X to be linear Brownian

[1] We need X to be time-homogeneous and Markovian but the continuity property of a diffusion is not strictly necessary. Rather we need the maximum process of X to be continuous. Other processes which fit this description include (continuous increasing functions of) spectrally Lévy negative processes.

[2] The initial wealth could be incorporated into the utility function U but it is convenient to include it directly within the formulation of the problem.

motion which might represent the logarithm of the traded price. It will be clear that provided the option payout is modified to reflect the modelling changes, the analysis is not sensitive to a re-parameterisation of this form.

Example 2.3 (Option payout) If the options are call options and x denotes the traded price, and if the label is the strike of the option then the payoff C is of the form $C(x,\theta,k) = (x-k)^+$. If x is the logarithm of the traded price then $C(x,\theta,k) = (e^x - k)^+$. We assume the agent receives the equivalent cash proceeds upon exercise. Clearly, shares can be included by considering an option with strike zero.

Example 2.4 (Price Impact) One interesting example arises when the act of exercising options or selling shares has a detrimental effect on the price of the underlying. Suppose that X has lognormal dynamics, and set $Y_t = X_t e^{-p(\theta_0 - \Theta_t)}$ so that $(dY_t/Y_t) = \sigma dB_t + \mu dt + p d\Theta_t$. (Note that Θ is a decreasing, finite variation process.) Then we might take $C(X_t, \Theta_t, k) = (X_t e^{-p(\theta_0-\Theta_t)} - k)^+ = (Y_t - k)^+$. The interpretation is that X_t represents the 'fundamental value' of the share, and Y_t the 'trading price' of the share, net of the impact of sales by the manager. The parameter p describes the (permanent) price impact of sales. The idea is that the market interprets exercises by the manager as a negative signal. Another application in this vein is the dilution effect arising from the issue of new shares to the manager.

3 The solution

The aim of this section is to provide a solution to the general problem described in the previous section. The idea is to use the form of the problem (and especially the time-homogeneity) to deduce that the optimal strategy involves exercising options the first time that the price crosses some wealth and portfolio dependent, but *time-independent* threshold. Then, for any strategy of this form we calculate the expected utility of the agent. Finally we use calculus of variations to determine the optimal set of thresholds. The method should be contrasted with the classical HJB equation approach in which the optimal strategy and value function are derived simultaneously. The advantage of decoupling the two problems is that individually the two problems may be simpler than the combined problem; however this technique only works if the value function can be characterised for an arbitrary threshold strategy.

We begin with a lemma which in limited but natural circumstances describes the order in which options are exercised under optimal behaviour. The important case is when $\Gamma(k) = k$ and the label is precisely the strike of the option.

Lemma 3.1 *Suppose $C(x, \theta, k) = (G(x, \theta) - \Gamma(k))^+$ where G is non-decreasing in θ and Γ is non-decreasing in the label k. For any strategy for which options with a high label are exercised before options with a low label, there is a modified version for which options are exercised in increasing label order which raises at least as much revenue.*

Proof: Consider a strategy, identified with $\hat{\rho}_t$, in which options are not exercised in increasing order of label. Let $\tilde{\rho}_t$ be a modified version of this strategy in which out-of-the-money options (ie those with $G(x, \theta) < \Gamma(k)$) are not exercised but otherwise options with the same labels are exercised at the same times, and let ρ'_t be a further modification of $\tilde{\rho}$ in which the same quantity of options are exercised at each time t, but they are exercised in order of increasing label. Then, using Θ to represent the total number of options remaining in each case, we have by construction that for each element of the sample space $\hat{\Theta}_t \leq \tilde{\Theta}_t = \Theta'_t$. From the monotonicity of G we have $G(X_s, \hat{\Theta}_s) \leq G(X_s, \tilde{\Theta}_s) = G(X_s, \Theta'_s)$.

The key property of the strategy ρ' is that for each t and element of the sample space, the measures $\rho_0 - \tilde{\rho}_t$ and $\rho_0 - \rho'_t$ have the same total mass, but $\int_{\mathcal{K}} \Gamma(k)(\rho_0 - \tilde{\rho}_t)(dk) \geq \int_{\mathcal{K}} \Gamma(k)(\rho_0 - \rho'_t)(dk)$. Then

$$
\begin{aligned}
&\int_0^t \int_{\mathcal{K}} (-d\hat{\rho}_s(dk))(G(X_s, \hat{\Theta}_s) - \Gamma(k))^+ \\
&\quad = \int_0^t \int_{\mathcal{K}} (-d\tilde{\rho}_s(dk))(G(X_s, \hat{\Theta}_s) - \Gamma(k)) \\
&\quad = \int_0^t G(X_s, \hat{\Theta}_s)(-d\tilde{\Theta}_s) - \int_{\mathcal{K}} \Gamma(k)(\rho_0 - \tilde{\rho}_t)(dk) \\
&\quad \leq \int_0^t G(X_s, \Theta'_s)(-d\Theta'_s) - \int_{\mathcal{K}} \Gamma(k)(\rho_0 - \rho'_t)(dk)
\end{aligned}
$$

This inequality holds at $t = \infty$, and hence if R with appropriate superscripts denotes the total revenue under the various strategies then we have $\hat{R} \leq R'$ as required. □

Henceforth we will make the following assumption:

Assumption 3.2 *We assume either that the payoffs of the option contracts are such that the hypotheses of Lemma 3.1 are satisfied, or that the order in which the options contracts must be exercised is predetermined.*

In the second case the label on the option might correspond to the order in which the options are exercised. In either case, implicit in the optimal strategy is a

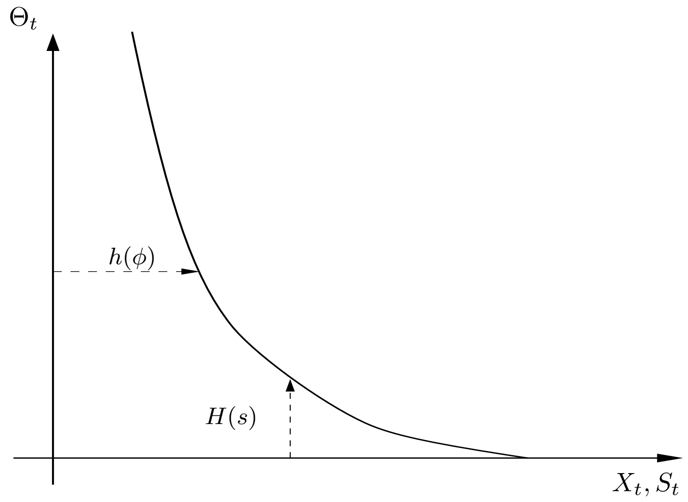

Figure 1: *A generic threshold $H(s)$ and inverse $h(\theta)$.*

fixed order in which options are exercised, and we can write the label of the θ-to go option as $J(\theta)$. Then $C(x, \theta, J(\theta))$ may be abbreviated to $C(x, \theta)$ and, given Θ_t, ρ_t can be reconstructed as

$$\rho_t(A) = \int_0^{\Theta_t} d\theta I_{\{J(\theta) \in A\}}$$

where I is the indicator function.

Given the time-homogeneity of the problem — in terms of price dynamics and option payoffs — it follows that if it is optimal not to exercise options at a given price level, then provided no options have been exercised in any intervening period, it will still be optimal to not exercise the options if the price returns to this level. Hence the exercise strategy must take the form of a set of thresholds, and must be $\Theta_t = H(S_t)$ where $S_t = \max_{s \leq t} X_s$. Here H is a necessarily non-increasing function, which we assume to be right-continuous, though we do not preclude jumps, nor intervals of constancy. Note that H may depend on w_0 the initial wealth of the agent. We denote by $h = h(\theta) = h(\theta; w_0)$ the left-continuous inverse of H, see Figure 1.

Given a non-increasing threshold function H (or equivalently, given h) we can define an exercise strategy as follows. If the initial number of options θ_0 is such that $\theta_0 > H(x_0)$ then the agent exercises $\theta_0 - H(x_0)$ options immediately. After this potential initial transaction, we insist that $\Theta_t = H(S_t)$. If H is continuous then apart from a possible exercise of a tranche of options at $t = 0$, the exercise strategy is continuous, though it is singular with respect to time since exercise only occurs when X_t is at its historical maximum. Singular control also appears in the transactions costs literature, see Davis and Norman [9].

Theorem 3.3 *For strategies of threshold type and for $\theta_0 \leq H(x_0)$ we have*

$$\mathbb{E}[U(w+R)] = U(w) + \mathcal{S}(x)\int_0^{\theta_0} \frac{C(h(\theta),\theta)}{\mathcal{S}(h(\theta))} U'\left(w + \int_\theta^{\theta_0} C(h(\phi),\phi)d\phi\right) d\theta$$

Proof: For strategies of this threshold type we have that the total revenue from exercise is

$$R = -\int_{t=0}^{\infty} C(X_t,\Theta_t)d\Theta_t = \int_0^{\theta_0} I_{\{S\geq h(\theta)\}} C(h(\theta),\theta)d\theta$$

where $S = S_\infty = \max_t X_t$, which is finite almost surely by the transience assumption on X_t. In particular, conditional on S,

$$R = R(S) = \int_{H(S)}^{\theta_0} C(h(\theta),\theta)d\theta$$

is deterministic. Then, conditioning on S, integrating by parts and using the change of variable $\theta = H(s)$,

$$\begin{aligned}
\mathbb{E}[U(w+R)] &= \int_x^{\infty} \mathbb{P}(S\in ds)U(w+R(s)) \\
&= [-U(w+R(s))\mathbb{P}(S\geq s)]_x^{\infty} + \int_x^{\infty} \mathbb{P}(S\geq s)U'(w+R(s))R'(s)ds \\
&= U(w) - \int_x^{\infty} ds \frac{\mathcal{S}(x)}{\mathcal{S}(s)}\frac{dH}{ds} C(s,H(s))U'(w+R(s)) \\
&= U(w) + \mathcal{S}(x)\int_0^{\theta_0} \frac{C(h(\theta),\theta)}{\mathcal{S}(h(\theta))} U'(w+R(h(\theta)))d\theta.
\end{aligned}$$

□

Thus, for a given threshold function we have calculated the value function for the agent. It remains to choose the optimal threshold, h^* say. First we need to find $h^*(0)$. For a general h set $\bar{h} = h(0)$, and for small θ consider

$$\frac{1}{\theta}\int_0^{\theta} \frac{C(h(\psi),\psi)}{\mathcal{S}(h(\psi))} U'\left(w + \int_\psi^{\theta} C(h(\phi),\phi)d\phi\right) d\psi.$$

By l'Hôpital's rule, provided C is continuous, this tends to

$$\frac{C(\bar{h},0)}{\mathcal{S}(\bar{h})}U'(w),$$

and the optimal $h^*(0)$ is chosen to maximise this expression. Effectively $h^*(0)$ is fixed by considering the problem for the risk-neutral agent.

Fix x_0, w_0 and θ_0. The aim is to choose a function $h^*(\phi) = h^*(\phi; w_0, \theta_0)$ to maximise

$$\int_0^{\theta_0} \frac{C(h(\theta),\theta)}{\mathcal{S}(h(\theta))} U'\left(w_0 + \int_\theta^{\theta_0} C(h(\phi),\phi)d\phi\right) d\theta. \tag{1}$$

The optimal threshold h can be characterised by calculus of variations.

Given C write c for the inverse $c = C^{-1}$ so that if $z = C(x, \phi)$ then $x = c(z, \phi)$. We assume that C and c are continuous and twice differentiable. Note that we only need this property locally in the region where exercise takes place, so that if $C(x, k) = (x - k)^+$ then it is satisfied since exercise only occurs when the option is in-the-money.

Theorem 3.4 *The optimal h satisfies*

$$h'(\phi) = -\frac{\left[c_\phi - A(h, \phi; w_0, \theta_0)C^2 c_z + 2C_\phi c_z + CC_\phi c_{zz} + Cc_{z\phi}\right]}{\left[2C_x c_z + B(\mathcal{S}, h(\phi))Cc_z + CC_x c_{zz}\right]} \tag{2}$$

where

$$A(h, \phi; w, \theta) = \frac{U''(w + \int_\phi^\theta C(h(\psi), \psi)d\psi)}{U'(w + \int_\phi^\theta C(h(\psi), \psi)d\psi)}; \qquad B(\mathcal{S}, h(\phi)) = \frac{\mathcal{S}''(h(\phi))}{\mathcal{S}'(h(\phi))} - 2\frac{\mathcal{S}'(h(\phi))}{\mathcal{S}(h(\phi))}$$

and (2) *is evaluated at $x = h(\phi)$ and $z = C(h(\phi), \phi)$.*

Proof:
Set $D(\phi) = -\int_\phi^{\theta_0} C(h(\psi), \psi)d\psi$. Then, $h(\phi) = c(D'(\phi), \phi)$ and the expression in (1) becomes

$$\int_{0,\theta_0} d\phi D'(\phi)\mathcal{D}(D'(\phi), \phi)U'(w_0 - D(\phi)) \tag{3}$$

where $\mathcal{D}(z, \phi) = 1/\mathcal{S}(c(z, \phi))$. Choosing the optimal threshold h is equivalent to choosing the optimal D. By Arfken [1, Equation 17.15] the optimal D solves

$$D'(\phi)U'(w - D(\phi))\frac{\partial \mathcal{D}}{\partial \phi}(D'(\phi), \phi) + \frac{d}{d\phi}\left[U'(w - D(\phi))D'(\phi)^2 \frac{\partial \mathcal{D}}{\partial D'}(D'(\phi), \phi)\right] = 0$$

which in terms of h becomes

$$c_\phi - A(w, \phi, h)C^2 c_z + 2c_z(h'C_x + C_\phi) + B(\mathcal{S}, h)h'Cc_z + C\left[(h'C_x + C_\phi)c_{zz} + c_{z\phi}\right] = 0$$

where we use the fact that $h(\phi) = c(D'(\phi), \phi)$ and $h'(\phi) = (c_z D'' + c_\phi)$. □

Corollary 3.5 *If $C(x, \theta) = C(x)$ is independent of θ (in particular if the options are identical, and there is no price impact) then h^* solves the simpler equation*

$$\frac{C(h(\phi))^2 \mathcal{S}'(h(\phi)}{C'(h(\phi))\mathcal{S}(h(\phi))^2} U'\left(w_0 + \int_\phi^{\theta_0} d\psi C(h(\psi))\right) = \textit{constant.}$$

Proof:
In this case $\frac{\partial \mathcal{D}(D'(\phi),\phi)}{\partial \phi} = 0$ and the $d/d\phi$ term can be integrated to leave $U'(w - D(\phi))D'(\phi)^2 \frac{\partial \mathcal{D}}{\partial D'}(D'(\phi), \phi) = \text{constant}$. □

Remark 3.6 Theorem 3.4 gives a candidate optimal threshold function h. As we shall see in the examples below this candidate gives the true optimal strategy. However, calculus of variations only gives that the candidate is associated with a turning point which could in principle be merely a local maximum, or a minimum. Therefore to complete the study of any given example further analysis is necessary, for example in the form of a verification lemma of the Hamilton-Jacobi-Bellman equation. Note that we have bypassed the problem of the construction of solutions to the HJB equation, and the task is merely to verify that the candidate solution has the necessary properties.

Remark 3.7 There is an implicit consistency condition in the form of any optimal threshold h^*; namely if the agent with initial wealth w_0 and options θ_0 uses h^* to determine a strategy which reduces her holdings to θ_1, then her optimal strategy thereafter coincides with that of a second agent with initial allocation θ_1 of options, (whose strikes correspond to the highest θ_1 strikes from the portfolio of the original agent), and initial wealth $w_1 = w_0 + \int_{\theta_1}^{\theta_0} C(\max(x, h(\theta)), \theta) d\theta$. It is clear from the form of $A(h, \phi; w_0, \theta_0)$ in Theorem 3.4 that this consistency condition holds. See the discussion in Section 4.1 for a direct verification of this consistency condition in an example.

4 Examples

4.1 Agent with CRRA utility

Suppose that the dynamics of X are exponential Brownian motion as in Example 2.2, that there is no price impact, $G(x, \theta) = x$, and we consider a stock portfolio so $C(x, \theta) = x$. We have $\mathcal{S}(x) = x^\beta$, with $\beta = 1 - \frac{2\mu}{\sigma^2}$. Suppose further that the agent has constant relative risk aversion $U(w) = w^{1-\alpha}/(1-\alpha)$ with $\alpha \in (0, \infty)\backslash\{1\}$.

By Corollary 3.5 the optimal h satisfies (for economy of notation in this Examples section we drop the superscript $*$ which we previously used to denote the optimal)

$$h(\phi)^{1-\beta} \left(w_0 + \int_\phi^{\theta_0} h(\psi) d\psi \right)^{-\alpha} = \text{constant} = \left(\frac{q\alpha}{1-\beta} \right)^{-\alpha}, \tag{4}$$

for some constant q, subject to the fact that $h(0)$ maximises $F(\bar{h}) = \bar{h}^{1-\beta}$. Indeed, if $\beta > 1$, $h(0) = 0$, the problem is degenerate and it is optimal to sell all stock immediately. Henceforth we restrict attention to the case $0 < \beta < 1$, in which case $h(0)$ is infinite.

On differentiating the defining equation for the optimal threshold we find that $\dot{h} = -h^{1+\eta}/q$ where $\eta = (\alpha + \beta - 1)/\alpha < 1$, which has solution $h(\phi) = (\eta\phi/q)^{-1/\eta}$. Since we require $h(0) = \infty$ the parameter values must be such that $\eta > 0$.

The constant q is fixed by the requirement that (4) holds:

$$\begin{aligned} w_0^{-\alpha}\left(\frac{\eta\theta_0}{q}\right)^{(\beta-1)/\eta} &= \lim_{\phi\downarrow 0}\left(\frac{\eta\phi}{q}\right)^{(\beta-1)/\eta}\left(w_0 + \left(\frac{\eta}{q}\right)^{-1/\eta}\int_\phi^{\theta_0}\psi^{-1/\eta}d\psi\right)^{-\alpha} \\ &= \left(\frac{1-\eta}{q}\right)^{\alpha} \end{aligned}$$

so that $q = \eta(\eta^{-1} - 1)^\eta w_0^\eta \theta_0^{1-\eta}$. Finally

$$h(\theta) = h(\theta; w_0, \theta_0) = \frac{w_0}{\theta_0}(\eta^{-1} - 1)\left(\frac{\theta_0}{\theta}\right)^{1/\eta}. \qquad (5)$$

Note that h satisfies a consistency condition with regard to the initial endowment. Consider an agent with endowment (w_0, θ_0) who follows the optimal policy, and suppose the path of X is such that she has sold $\theta_0 - \theta_1$ stock. Then, from the form of $h(\theta; w_0, \theta_0)$ it follows that her wealth is $w_1 = w_0(\theta_0/\theta_1)^{1/\eta-1}$, and then, for $\theta \leq \theta_1$,

$$h(\theta; w_0, \theta_0) = \frac{w_0}{\theta_0}(\eta^{-1} - 1)\left(\frac{\theta_0}{\theta}\right)^{1/\eta} = \frac{w_1}{\theta_1}(\eta^{-1} - 1)\left(\frac{\theta_1}{\theta}\right)^{1/\eta} = h(\theta; w_1, \theta_1)$$

so her thresholds for $\theta \leq \theta_1$ are consistent with those of an agent with initial endowment (w_1, θ_1).

The candidate value function $V(w, x, \theta) = \mathbb{E}[U(w + R)|X_0 = x, \Theta_0 = \theta]$ takes two distinct forms depending on whether the initial endowment is such that $x < h(\theta_0; w_0, \theta_0)$ or $x \geq h(\theta_0; w_0, \theta_0)$. Let $\mathcal{C}$ denote the continuation region $x < h(\theta; w_0, \theta_0)$ and let $\mathcal{E}$ denote the exercise region $x \geq h(\theta; w_0, \theta_0)$.

By direct calculation based on the candidate optimal threshold we have that in $\mathcal{C}$

$$V(w, x, \theta) = \frac{w^{1-\alpha}}{1-\alpha} + w^{1-\alpha}\left(\frac{\theta x}{w}\right)^{\beta}\left(\frac{1-\eta}{\eta}\right)^{1-\beta} \qquad (6)$$

Conversely, if the initial holdings are such that $x > h(\theta_0; w_0, \theta_0)$ then optimal behaviour is to sell a strictly positive amount of stock instantly at time zero. This amount should be chosen so that the new stock holdings θ_1 satisfy

$$x = h(\theta_1; w_0 + (\theta_0 - \theta_1)x, \theta_1) = \frac{w_0 + (\theta_0 - \theta_1)x}{\theta_1}\frac{(1-\eta)}{\eta}$$

so that $\theta_1 = (w_0 + \theta_0 x)(1-\eta)/x$. For example, if initial wealth w_0 is zero then $\theta_1 = \theta_0(1-\eta)$. In the sale region $x > h(\theta_0; w_0, \theta_0)$ the value function is

$$V(w, x, \theta) = V(w + (\theta - \theta_1)x, x, \theta_1) = \frac{(1 - \alpha + \eta\alpha)}{\eta(1-\alpha)}\left(\eta(w + \theta x)\right)^{1-\alpha}. \qquad (7)$$

We now have a full candidate solution for the problem, including expressions for the optimal exercise thresholds and the value function. Though we shall not do so here, these expressions can be used to calculate further quantities of interest such as the indifference value of her portfolio to the agent. The remaining step is a verification lemma to show that the solution we have constructed via calculus of variations gives a true optimum. (This step is also necessary in Examples 4.2 and 4.3, but given the method is standard, we shall only discuss verification here.)

For a function $f = f(w, x, \theta)$ recall that the generator L^X is given by $L^X f = \sigma(x)^2 f_{xx}/2 + \mu(x) f_x$, where a subscript denotes a partial derivative, which in our case yields $L^X f = (\sigma^2/2)[x^2 f_{xx} + (1-\beta) x f_x]$. Define the operator L^Θ by $L^\Theta f = f_\theta - x f_w$.

Lemma 4.1 *In $\mathcal{C}$ the value function V solves $L^X V = 0$, and $L^\Theta V \geq 0$. In $\mathcal{E}$ we have $L^X V \leq 0$, and $L^\Theta V = 0$. There is value matching on the boundary $x = h(\theta; w, \theta)$.*

Proof: It is trivial that $L^X V = 0$ in $\mathcal{C}$ and $L^\Theta V = 0$ in $\mathcal{E}$. Also at $x = h(\theta; (w, \theta))$ both (6) and (7) yield

$$V\left(w, \frac{w}{\theta}(\eta^{-1} - 1), \theta\right) = w^{1-\alpha} \frac{\beta\alpha}{(1-\alpha)(\alpha+\beta-1)}.$$

After some calculation we find that in $\mathcal{C}$,

$$L^\Theta V = x w^{-\alpha} z^{\beta-1} \left[\beta + (1-\beta) z - z^{1-\beta}\right]$$

where $z = (\theta x \eta)/(w(1-\eta))$, and then the boundary corresponds to $z = 1$. Hence $L^\Theta V \geq 0$ in $\mathcal{C}$.

In $\mathcal{E}$ we have $L^X V \leq 0$ provided $-\alpha\theta^2 x^2 + (1-\beta) x\theta(w + \theta x) \leq 0$ and this is equivalent to $z \geq 1$. □

Proposition 4.2 *$V(w - \int_0^t X_s d\Theta_s, X_t, \Theta_t)$ is a super-martingale in general and a martingale under the strategy $\Theta_t = H(S_t; w - \int_0^t X_s d\Theta_s)$, where H is the inverse of the optimal h given in (5). Hence the threshold strategy determined by h is optimal, and the expressions in (6) and (7) give the value function.*

Proof: By Itô's formula,

$$dV_t = L^\Theta V d\Theta_t + L^X V dt + dM_t \leq dM_t$$

where the local martingale M_t is given by $M_t = \int_0^t \sigma X_s V_x dB_s$. Then $V_t \leq V_0 + M_t$ and since for a fixed initial endowment V is bounded below we have that $M_t \geq$

$V_t - V_0$ is a supermartingale. Since $dV_t \leq dM_t$ this property is inherited by V. Then $\mathbb{E}[U(w+R)] = \mathbb{E}[(w - \int_0^\infty X_s d\Theta_s)^{1-\alpha}/(1-\alpha)] \leq V(w,x,\theta)$. Conversely, the reverse inequality follows from direct calculation for the threshold strategy h. □

4.2 A Portfolio of options with different strikes

Suppose the agent has exponential utility $U(w) = -e^{-\gamma w}/\gamma$ and that she has a portfolio of perpetual American call options such that she starts with a measure $\rho_0(dk)$ of options with strike k. As described in Section 3, assuming that she behaves optimally and exercises the low-strike options first, we can parameterise the strike of the call options in terms of the number of remaining options, so that $C(x,\theta) = (x - J(\theta))^+$ for a decreasing function $J(\theta)$.

Suppose that the price of the asset X_t follows a Bessel process of dimension $2-\delta$ with $\delta > 0$. Then X hits zero with probability one, and we make zero an absorbing boundary point. The process X is given by

$$dX_t = dB_t - \frac{(\delta-1)}{2X}dt; \qquad X_0 = x > 0$$

and the scale function is $\mathcal{S}(x) = x^\delta$.

Note that this scale function is identical to that of exponential Brownian motion. (This also follows from the fact that one process is a stochastic time-change of the other.) Hence the analysis of this section translates immediately to the exercise of option portfolios in the exponential Brownian motion model for stock prices subject to replacing δ with β.

If J is differentiable then we can apply Theorem 3.4 to find that the first-order optimal threshold h satisfies

$$h'(\phi) = -\frac{h(\phi)[\gamma(h(\phi) - J(\phi))^2 - J'(\phi)]}{2h(\phi) - (h(\phi) - J(\phi))(1+\delta)}. \tag{8}$$

Perhaps more realistic is the case where the agent has a portfolio of discrete strikes. We cover the case where the agent has θ_1 options with strike k_1 and $\theta_2 - \theta_1$ options with strike $k_2 < k_1$. (More complicated portfolios can be incorporated by extension.) By Lemma 3.1 we know that the optimal policy involves exercising the low strike options first, and then $J(\phi) = k_1$ for $\phi \leq \theta_1$ and $J(\phi) = k_2$ for $\theta_1 < \phi \leq \theta_2$.

From (8) we can conclude that for $\phi < \theta_1$ the optimal h solves

$$h'(\phi) = -\frac{\gamma(h(\phi) - k_1)^2 h(\phi)}{k_1(1+\delta) + (1-\delta)h(\phi)}. \tag{9}$$

The case with identical options and exponential utility is studied in Henderson and Hobson [13]. Equation (9) can be solved by considering the inverse function

$H = h^{-1}$, so that $dH/dz = -(k_1(1+\delta)+(1-\delta)z)/(\gamma z(z-k_1)^2)$. It is easily verified (see [13] for details) that in the case $\delta \le 1$, $h(0) = \infty$ and

$$H(x) = \frac{2}{\gamma(x-k_1)} + \frac{(1+\delta)}{\gamma k_1} \ln\left(\frac{x-k_1}{x}\right). \tag{10}$$

The case $\delta > 1$ is similar except that $h(0) = k_1\delta/(\delta-1) < \infty$ and then H must be modified by a constant to allow for the new boundary condition. We find

$$H(x) = \frac{2}{\gamma(x-k_1)} - \frac{2(\delta-1)}{\gamma k_1} + \frac{(1+\delta)}{\gamma k_1} \ln\left(\frac{\delta(x-k_1)}{x}\right). \tag{11}$$

Recall (1) and observe that for exponential utility, wealth factors out of the problem. For $\theta < \theta_1$, and for the optimal threshold, we can now evaluate

$$\begin{aligned} \Lambda(\theta) &:= \int_0^\theta \frac{C(h(\phi),\phi)}{\mathcal{S}(h(\phi))} U'\left(\int_\phi^\theta (h(\psi)-k_1)d\psi\right) d\phi \\ &= \int_0^\theta \frac{(h(\phi)-k_1)}{h(\phi)^\delta} e^{-\gamma\int_\phi^\theta (h(\psi)-k_1)d\psi} d\phi \end{aligned} \tag{12}$$

and after several lines of algebra this can be shown to give

$$\Lambda(\theta) = h(\theta)^{-(1+\delta)} \left(\delta k_1 + (1-\delta)h(\theta)\right)/\gamma. \tag{13}$$

Note that for exponential utility $\Lambda(\theta)$ solves

$$\frac{d\Lambda}{d\theta} = \frac{C(h(\theta),\theta)}{\mathcal{S}(h(\theta))} - \gamma C(h(\theta),\theta)\Lambda(\theta),$$

so that in seeking to maximise $\Lambda(\theta)$ it is appropriate to take

$$h(\theta) = \operatorname*{argmax}_x \left\{C(x,\theta)\left[\mathcal{S}(x)^{-1} - \gamma\Lambda(\theta)\right]\right\}. \tag{14}$$

(We have already seen a special case of this result in calculating the threshold at $\theta = 0$. Furthermore, given the formula for Λ in (13) it is easily checked that this approach is consistent with previous methods.) In particular, if $\bar{x} = \lim_{\theta\uparrow\theta_1} h(\theta)$ then $\bar{x} = \text{argmax}\{(x-k_1)(x^{-\delta} - \gamma\Lambda(\theta_1))\}$. Note that the explicit value of $\bar{x}$ is fixed as the inverse to H in (10) or (11).

The issue is to extend the range of the definition of the optimal threshold h to include $\theta \in [\theta_1, \theta_2]$. In the interior of this region h solves (9) with k_2 substituted for k_1, and since this can again be solved by considering the inverse function, the remaining task is to find $\hat{x} := h(\theta_1+)$. Using (14) we have

$$\bar{x} = \operatorname*{argmax}_x \left[\frac{x-k_1}{x^\delta} - \Lambda(\theta_1)(x-k_1)\right] \qquad \hat{x} = \operatorname*{argmax}_x \left[\frac{x-k_2}{x^\delta} - \Lambda(\theta_1)(x-k_2)\right]$$

This means that $\bar{x}$ and $\hat{x}$ are related via

$$\frac{\delta k_1 + (1-\delta)\bar{x}}{\bar{x}^{1+\delta}} = \frac{\delta k_2 + (1-\delta)\hat{x}}{\hat{x}^{1+\delta}}$$

where both sides equal $\Lambda(\theta_1)$. Finally then, for $\theta > \theta_1$, the optimal h is given by the inverse to H where

$$H(x) = \theta_1 + \frac{2}{\gamma(x-k_2)} - \frac{2}{\gamma(\hat{x}-k_2)} + \frac{(1+\delta)}{\gamma k_2} \ln\left(\frac{(x-k_2)\hat{x}}{x(\hat{x}-k_2)}\right).$$

Since $k_2 < k_1$ it follows that $\hat{x} < \bar{x}$, and there is a jump in the threshold at θ_1. Further, if for $i = 1, 2$ we let $\{h_i(\theta)\}_{0 \le \theta \le \theta_2}$ be the optimal threshold for a portfolio consisting of θ_2 options, all with strike k_i, then a comparison of the appropriate variants of the defining ordinary differential equations, see (8) and (9) above, shows that $h_2(\theta) < h(\theta) \le h_1(\theta)$ for $0 < \theta \le \theta_2$, see Figure 2. Specifically, an agent with a portfolio of call options, some of which have higher strikes will wait for a higher asset value before exercising the low strike options than an agent with a portfolio consisting of the same total quantity of options, each of which has the same low strike. Put another way, an agent with a mixture of high and low strike options acts as if the high strike options were replaced with a smaller number of low strike options, at least until all her low strike options are exercised. Here we make crucial use of the fact the agent has exponential utility.

4.3 A Model with Price Impact

In our final example we suppose the asset price follows exponential Brownian motion, but that the act of exercising (or selling) has a permanent impact on the asset price. Consider an agent with exponential utility and a portfolio of call options.

Let X_t denote the fundamental value of the stock and suppose $dX_t/X_t = \sigma dB_t + \mu dt$ with $X_0 = x_0$. We define the traded stock price

$$Y_t = G(X_t, \Theta_t),$$

where $G(x,\theta) = xe^{-p(\theta_0-\theta)}$, and $p > 0$ is a parameter capturing the permanent price impact. We have that $dY_t/Y_t = dX_t/X_t + pd\Theta_t$. The call options give the right to buy a stock worth Y_t in exchange for the strike k, hence $C(x,\theta) = (xe^{-p(\theta_0-\theta)} - k)^+$. (An alternative specification might take $G(x,\theta) = x/(N + \theta_0 - \theta))$ where N is the initial number of shares, and then Y_t models the dilution effect, whereby the existence of warrants results in an increase in the number of stocks but does not change the total firm value, as represented by X_t.)

In the notation of Section 3, $\mathcal{S}(x) = x^\beta$, with $\beta = 1 - 2\mu/\sigma^2$, $c(z,\theta) = (z + k)e^{p(\theta-\theta_0)}$ and $\mathcal{D}(z,\theta) = (z+k)^{-\beta}e^{-\beta p(\theta_0-\theta)}$, so that if $D(\phi) = -\int_\phi^{\theta_0} C(h(\psi),\psi)d\psi = -\int_\phi^{\theta_0}(h(\psi)e^{-p(\theta_0-\psi)} - k)d\psi$ then the fundamental problem is to maximise (from

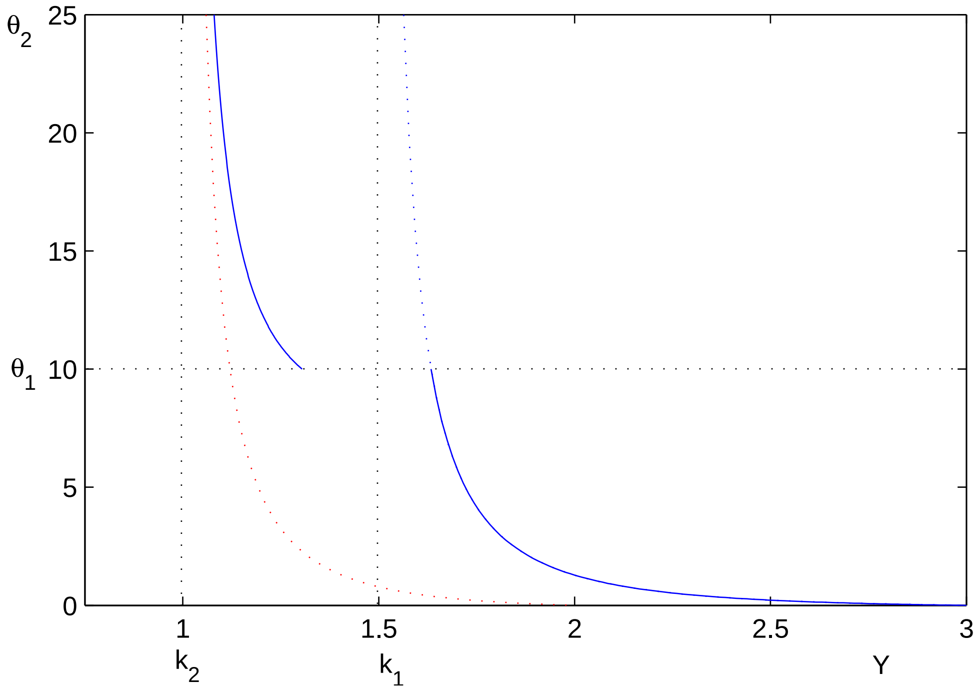

Figure 2: *The solid lines are the thresholds for agent with* $\theta_1 = 10$ *options with strike* $k_1 = 1.5$ *and* $\theta_2 - \theta_1 = 15$ *options with strike* $k_2 = 1$*. Computations give* $\bar{x} = 1.63$ *and* $\hat{x} = 1.3$*. Also shown (dotted lines) are* h_1 *and* h_2 *which satisfy* $h_2 \leq h \leq h_1$*. Other parameters are* $\delta = 2$, $\gamma = 1$.

(3))

$$e^{-\gamma w}\int_0^{\theta_0} d\phi D'(\phi)(D'(\phi)+k)^{-\beta}e^{\beta p(\phi-\theta_0)}e^{\gamma D(\phi)}. \tag{15}$$

The integrand in (15) depends on ϕ, but if we set

$$E(\phi) = D(\phi) + \frac{\beta p}{\gamma}(\phi - \theta_0)$$

then $E'(\phi) = D'(\phi) + \beta p/\gamma$ and omitting the wealth component $e^{-\gamma w}$ from (15) the problem reduces to maximising

$$\int_0^{\theta_0} d\phi \left(E'(\phi) - \frac{\beta p}{\gamma}\right)\left(E'(\phi) + k - \frac{\beta p}{\gamma}\right)^{-\beta} e^{\gamma E(\phi)}.$$

Now there is no explicit ϕ dependence, so we deduce (see Arfken [1, Equation 17.18]) that the optimal h is such that

$$e^{\gamma E(\phi)}\left(E'(\phi)+k-\frac{\beta p}{\gamma}\right)^{-(\beta+1)}\left[E'(\phi)^2 - \frac{p}{\gamma}(1+\beta)E'(\phi) - \frac{p}{\gamma}\left(k - \frac{\beta p}{\gamma}\right)\right]$$

is constant.

Write $g(\psi) = e^{-p(\theta_0-\psi)}h(\psi)$ and abbreviate p/γ to ξ, so that ξ measures the relative importance of the price impact and the risk aversion. Then $E(\phi) = -\int_\phi^{\theta_0}[g(\psi) - k + \beta\xi]d\psi$. Taking logs and differentiating, after some algebra we find

$$g'(\theta) = \frac{-\gamma g\left(g^2 + g(\xi(\beta-1) - 2k) + k(k-\beta\xi)\right)}{g(1-\beta) + (\beta+1)k} \tag{16}$$

with initial condition $g(0) = e^{-p\theta_0}\bar{h}$ where $\bar{h} = \operatorname{argmax} h^{-\beta}(he^{-p\theta_0} - k)$.

As in Section 4.2, (16) can be solved to give an explicit formula for the inverse. However, the formula are not compact, and bring little insight beyond the differential equation formulation, and the study of the stock ($k = 0$) case below. We will however plot the solution $g(\theta)$ to (16) in Figure 3.

Note that when $p = 0$ we recover (recall (9))

$$g' = \frac{-\gamma g(g-k)^2}{g(1-\beta) + (\beta+1)k}.$$

Conversely, if $p > 0$ but $k = 0$, so the agent holds stocks rather than calls, then (16) simplifies to

$$g' = -g\gamma\left(\frac{g}{1-\beta} - \xi\right).$$

In this case, if $\beta \geq 1$, the problem is degenerate in the sense that the stock price is a supermartingale, and all shares are sold instantly. If $0 < \beta < 1$ then $g(\theta) = \xi(1-\beta)/(1-e^{-p\theta})$. Finally,

$$h(\theta) = \frac{e^{p(\theta_0-\theta)}\xi(1-\beta)}{(1-e^{-p\theta})} = \frac{e^{p\theta_0}p\theta}{(e^{p\theta}-1)}h_0(\theta),$$

where $h_0(\theta) = (1-\beta)/(\gamma\theta)$ is the solution in the absence of price impact.

Note that $X_t = h(\Theta_t)$ if and only if $Y_t = g(\Theta_t)$ so that g can be considered as the threshold boundary for the stock Y_t. Then $g(\theta) = h_0(\theta)p\theta/(1-e^{-p\theta}) > h_0(\theta)$. The effect of the price impact is to increase the boundary. The agent waits for a higher stock price to begin selling in order to counteract the loss due to the price impact.

An interesting feature of the solution is the fact that

$$g(\infty) := \lim_{\theta\uparrow\infty} g(\theta) = p\frac{(1-\beta)}{\gamma} > 0; \qquad 0 < \beta < 1$$

so that, unlike in the zero price impact case, it is possible even with arbitrarily large initial holdings that the agent never sells any stock. (In the case with no price impact the agent immediately reduces her holdings to $(1-\beta)/\gamma x$ if her initial holdings exceed this value).

The story is similar for option holdings. Figure 3 plots the boundary $g(\theta)$ for price impact level $p = 0.05$ and the equivalent boundary when there is no price impact. We see the price impact shifts the boundary to the right - reflecting that the agent waits for higher price levels to exercise than in the absence of price impact.

The large θ limit of g is given by the (largest) positive root[3] of $F(g) = 0$ where

$$F(g) = g^2 + (\xi(\beta-1) - 2k)\, g + k\,(k - \beta\xi)\,.$$

Thus

$$g(\infty) := \lim_{\theta\uparrow\infty} g(\theta) = k + \frac{\xi}{2}\left[\left((\beta-1)^2 + \frac{4k}{\xi}\right)^{1/2} - (\beta-1)\right].$$

For the parameter values in Figure 3 we have $g(\infty) = 1.2$ and we note that the boundary converges quickly to this limit. Indeed, for $p > 0$, it can be shown that g converges exponentially fast to the limiting value, whereas when $p = 0$ the convergence is at rate θ^{-1}.

In the limit $\xi \downarrow 0$ we find $g(\infty) = k + (\xi k)^{1/2} + O(\xi)$. In the limit as $\xi \uparrow \infty$ we find

$$\begin{aligned} g(\infty) &= \xi(1-\beta) + k\frac{(2-\beta)}{(1-\beta)} + O(1/\xi); \quad \beta < 1 \\ g(\infty) &= \frac{k\beta}{(\beta-1)} + O(1/\xi); \quad \beta > 1 \end{aligned}$$

Thus, in the case $\beta > 1$, as the price impact becomes very large relative to risk aversion, then the agent waits to exercise at the risk-neutral level $k\frac{\beta}{\beta-1}$. When $\beta < 1$ the risk neutral agent never exercises, and again this is captured in the large ξ limit of $g(\infty)$.

[3]It is always the case that F has two real roots, at least one of which is positive.

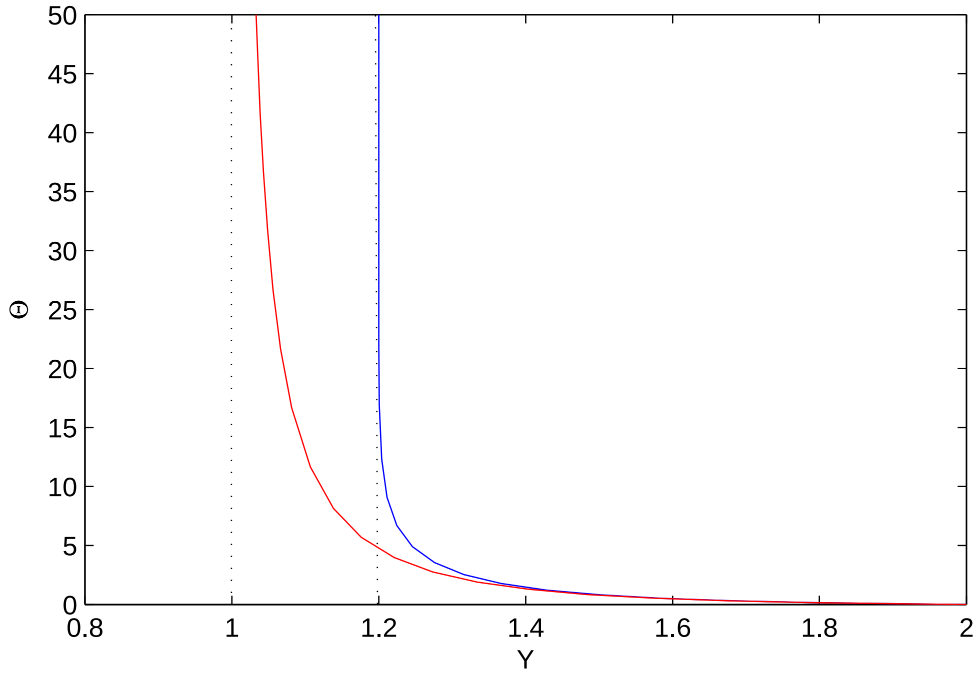

Figure 3: *Exercise boundaries for options with strike $k = 1$. Other parameters are $\beta = 2$ and $\gamma = 1$. The rightmost boundary uses price impact parameter $p = 0.05$ and for these parameters, $g(\infty) = 1.2$. The leftmost boundary has no price impact and hence $g(\infty) = k = 1$. Both boundaries have $g(0) = k\frac{\beta}{\beta-1} = 2$.*

5 Conclusion

In this paper we have presented a method for determining the optimal exercise boundary for a risk-averse agent seeking to liquidate an option portfolio. A key assumption is that the option portfolio is infinitely divisible. Clearly this assumption is not satisfied in practice, but may be apprporiate for large portfolios, where it may bring considerable insight. Apart from a possible initial exercise of a tranche of options, the optimal behaviour involves liquidating the portfolio in infinitesimal amounts, but at times which are singular with respect to calendar time.

In the paper we have considered three examples involving CRRA and CARA utility, stocks, options, and portfolios of different options, and models where the act of exercising impacts on the price. These examples are not meant to be exhaustive, but rather to indicate the variety of situations where our approach may be fruitfully applied.

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