# Robust Hedging of the Lookback Option

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**Abstract**

The aim of this article is to find bounds on the prices of exotic derivatives, and in particular the lookback option, in terms of the (market) prices of call options. This is achieved without making explicit assumptions about the dynamics of the price process of the underlying asset, but rather by inferring information about the potential distribution of asset prices from the call prices. Thus the bounds we obtain and the associated hedging strategies are model independent. The appeal and significance of the hedging strategies arises from their universality and robustness to model mis-specification.

# 1 Call Options as Traded Assets

Trading in call options is now so liquid that some authors (including Dupire [5, 6]) have argued that calls should no longer be treated as derivative assets to be priced using a Black-Scholes model (or otherwise). Instead calls should be treated as primary assets, whose prices are fixed exogenously by market sentiments. It is only more complicated contingent claims which should be treated as derivative securities. These latter claims should be priced in a fashion consistent with call prices so as to preclude arbitrage.

The development of this theory reflects the new perspective that term structure models have brought to the pricing of bonds and interest rate derivatives. In the traditional viewpoint of, for example, Vasiček [21], there was a single state variable (typically a short term interest rate) from which all bond prices could be calculated. Subsequently Heath, Jarrow and Morton [10] argued that the whole of the current yield curve is needed to summarise current information, so that bond prices must be inputs to, and not outputs from, an interest rate model. Similarly here we are progressing from a model in which the asset price is the sole state variable, to one which incorporates the prices of call options.

If the dynamics of the asset price process are specified (for example if the asset price is assumed to follow the solution of a stochastic differential equation), then it is possible, at least in principle, to calculate the finite-dimensional distributions of the price process. If moreover the market is complete then all derivatives can be replicated. Standard arguments then imply that the prices of contingent claims can be expressed as the (discounted) expected payoff of the claim under the equivalent martingale measure (EMM). Thus prices of many derivative securities can be calculated via a two-stage process: firstly calculate the marginal distributions of the asset price (under the EMM), and then simply use this result to calculate the average payoff. However the accuracy of these prices depends on the veracity of the underlying model.

Here we reverse the analysis with the aim of determining prices for exotic options which are robust to model mis-specification. The only modelling assumption that we make is that the market is arbitrage-free, so that call prices have been derived as a discounted expected payoff under an EMM. From these call prices we *infer* marginal distributions for the asset price process. These distributions can be used to price any derivative security whose payoff is contingent only on the terminal value of the asset. Such derivative securities can be hedged via investment in calls (see Ross [18] and Breeden and Litzenberger [3]). When we couple these marginal distributions with the martingale pricing property we can hope to make statements about the prices of path-dependent derivatives.

In particular this article is primarily concerned with the lookback option, a security whose payoff is the maximum value attained by the asset. The aim is to price the lookback option with reference to the prices of call options. (We assume that there trades a family of call options whose maturity matches the terminal date of the lookback.) The lookback option was priced in a pioneering paper by Goldman, Sosin and Gatto [9] in which the authors descibe a hedging strategy involving investment in calls. The resulting price is fair, and the associated hedging strategy succesful, provided that (and only if) the asset price follows an exponential Brownian motion with *known* parameters. Our approach, which is not dependent on knowledge of the law of the underlying asset price process is to apply the mathematics literature (Blackwell and Dubins [2], Dubins and Gilat [4], Kertz and Rösler [13], Rogers [17], Hobson [11]) on the possible laws of the maxima of a martingale to place bounds on the price of a lookback.

We have in mind the scenario of an option writer, hereafter called you, who is both extremely sceptical and risk averse. Your risk aversion is such that you are not prepared to write a derivative security unless you receive in payment sufficient funds to finance a super-replicating strategy. A super-replicating strategy is a hedging strategy which generates a terminal fortune which dominates the option payout with probability one. Moreover your scepticism means that you are not prepared to make any assumptions about how asset prices will behave. Conversely you will only be prepared to buy a derivative security if, perhaps with the help of a hedging strategy, you can generate a payoff which with probability one exceeds the price you paid.

The bounds we calculate on the price of the lookback depend on the prices of call options. The upper bound represents the lowest price at which you are prepared to sell a lookback option, and the lower bound the highest price which you are prepared to pay. Of course other options traders may be prepared to trade within these limits, but the bounds represent limits on the possible price of the lookback which are necessary for the absence of arbitrage.

The philosophy of pricing and hedging in incomplete markets through super-replication of the option payoff has appeared elsewhere in the mathematical finance literature, for example in the work of El Karoui and Quenez [7], Kramkov [14] and Föllmer and Kabanov [8]. However in all these papers it is assumed that the model for the underlying asset price process is known, and the novelty in our approach lies in our scepticism or non-belief of a particular specified law for the asset price process. Thus whereas Kramkov proves that for every model there is a super-replicating strategy, we prove that there is a strategy which super-replicates in every model.

In his ground-breaking article on robust and model independent option pricing Dupire [6] was prepared to make additional assumptions about the market and consequently was led to stronger conclusions. He assumed that the underlying price process is a diffusion, and that there are traded call options with all possible maturities as well as all possible strikes. Under regularity conditions this completely specifies the diffusion process, so that in principle all other derivatives can be priced and hedged. The advantage of the bounds on prices calculated in this article is that they rely on weaker assumptions about the underlying price process, and they yield extremely simple hedging strategies. Whereas Dupire ambitiously calculates precise derivative prices which should be used with care, we cautiously find wide bounds which can be applied with imprudence.

In calculating the bounds for the price of a lookback option we assume that the call prices are derived within a complete market without transaction costs. However it is sufficient that call prices are consistent with such a model. If so, then it is irrelevant whether this model is a true representation of the market, for the same analysis still generates a super-replicating strategy, and bounds on the lookback price. (In contrast the zero-transaction cost assumption is crucial for hedging in Dupire [6].)

In recent work, Soner, Shreve and Cvitanić [20] have shown that in a market with non-zero transaction costs there is no non-trivial hedging portfolio for pricing call options. (The trivial hedging portfolio is to charge the full price of the asset for the option, to use these funds to purchase one unit of asset, and to hold it to maturity. This strategy is certain to super-replicate.) A natural extension is to ask whether there is any super-replicating strategy for lookback options in this model. It is a plausible conjecture that there is no such strategy with finite initial investment. This conjecture assumes that hedging is restricted to trading in the underlying asset. However, the results of this article imply that if investment in call options is a permissible component of a hedging strategy then, even with transaction costs, it is possible to find a super-replicating strategy with finite initial investment.

The remainder of this paper is structured as follows. Section 2 contains a solution to the main problem we consider in this article; namely to find an upper bound on the price of a lookback which is valid in all complete markets which are consistent with a given set of prices for calls. Coupled with this price bound there is a super-replicating strategy. In Section 2 it is assumed that interest rates are zero, but this condition is relaxed in the next section. In Section 4 we find bounds on the prices of other derivative securities. The common feature of the securities that we study is that the payoff is contingent on the value of the maximum price attained by the price process of the underlying asset. Section 5 summarises and concludes the paper.

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## 2 Bounds on the Price of a Lookback

The goal of this section is to calculate upper and lower bounds on the price at which a lookback option can trade. If the market price lies outside these theoretical bounds then there is a simple strategy for generating arbitrage profits.

We assume for the moment that interest rates are zero so that under the equivalent martingale measure the price process of the underlying asset is a martingale. This allows us to apply the mathematical theory to full effect. The case of non-zero interest rates is covered in the next section. Note however that even when the interest rates are non-zero, the forward price, the amount which it is agreed now will be exchanged for the asset on the contract date *is* a martingale under the $T$-forward measure. Thus the bounds that we find in this section can be applied directly to a lookback option written on a forward price.

Denote the price for the underlying asset by $(P_t)_{t\geq 0}$. Without loss of generality we set $P_0 \equiv 1$. By definition the payoff of the lookback option with exercise date $T$ is given by

$$S_T \equiv \sup_{0\leq t\leq T} P_t.$$

We are assuming continuous updating of the maximum. If the asset price is only observed on a discrete time set, then the payoff of the lookback can only be reduced, and the upper bound we calculate here remains valid. It will become clear that, provided $T$ is one of the updating points, then the lower bound is also valid.

Denote by $C(k)$ the initial price of a European call option with strike $k$ and maturity $T$. These prices are market prices rather than the outputs from any model.

The bounds on the price of the lookback are predicated on the following pairs of assumptions:

**Assumption 2.1 (Continuous Market)**

AI–i There is a family of call options traded with maturity $T$, one for each of the continuum of potential strike prices.

AI–ii Calls are available for purchase in arbitrary amounts, including fractions, and are sold at the same price independent of the quantity.

**Assumption 2.2 (Regularity of Call Prices)**

AII–i For $k \geq 0$, $C(k)$ is a decreasing, convex function of $k$ satisfying $C(0) = 1$ and $\lim_{k \uparrow \infty} C(k) = 0$.

AII–ii A European call will, if sold before expiry, always realise the intrinsic value of the corresponding American-style call; thus at time $t < T$ a European call with strike $k$ can be sold for at least $(P_t - k)^+$.

Clearly the continuous market assumptions are violated in practice. However it might be hoped that they are a close approximation to a realistic situation. The regularity assumptions on call prices are very mild. The assumption that $C(0) = 1$ merely states that investors are prepared to pay 1, the current market price of the asset, for receipt of the asset at time $T$. Moreover, if the payoff of one portfolio of assets dominates the payoff of a second then, in the absence of transaction costs, arbitrage considerations mean that the first portfolio cannot trade at the lower price. Every investor will always prefer $k$ and a call option with strike $k$ to a unit of asset; hence AII–ii follows. Similarly, for all $p$ and for all $\lambda \in [0,1]$,

$$\lambda(p - k_1)^+ + (1-\lambda)(p - k_2)^+ \geq (p - (\lambda k_1 + (1-\lambda)k_2))^+$$

so that the call price function $C(k)$ must be convex and AII–i follows. If transaction costs are non-zero, then assumptions AII–i-ii are not automatically valid, but remain very plausible.

**Lemma 2.3** *There is a 1-1 correspondence between call price functions $C(.)$ satisfying AII-i–ii and probability measures $\nu$ with unit mean and support contained in $[0,\infty)$.*

**Proof.** $\nu$ and $C$ are related by $\nu((k,\infty)) = -C'(k+)$ where $C'(\cdot+)$ denotes the right derivative, and $C(k) = \int_0^\infty (y-k)^+ \nu(dy)$. AII-ii ensures that $C'(0+) \geq -1$; if the inequality is strict then to make $\nu$ into a probability measure we insist that $\nu(\{0\}) = 1 + C'(0+)$. $\square$

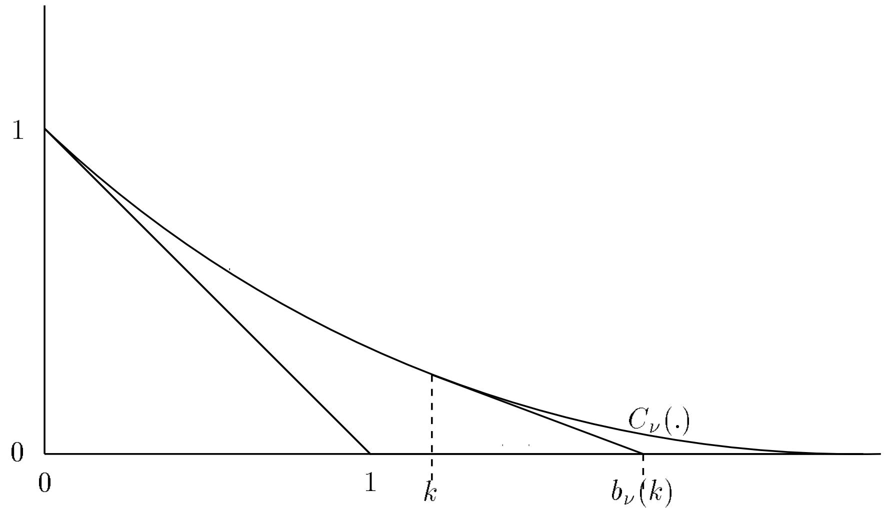


Figure 1: The barycentre function $b_\nu$, for a probability measure $\nu$ with unit mean and support contained in $[0,\infty)$.

Let us now have a brief digression about Hardy-Littlewood transforms and barycentre functions. Define the barycentre function $b_\nu$ by

$$b_\nu(x) = \frac{1}{\nu((x,\infty))} \int_{(x,\infty)} y \, \nu(dy).$$

If $X$ is a random variable with continuous distribution $\nu$ then $\nu^*$, the Hardy-Littlewood transform of $\nu$, is the law of $b_\nu(X)$. Azéma and Yor [1] and Rogers [16] give a pictorial representation of $b_\nu$, which was adapted by Hobson [11] to a more suitable form for this context. Specifically $b_\nu(k)$ is the $x$-coordinate of the point where the tangent at $k$ to the function $C_\nu(k) = \int_{\mathbb{R}} (y-k)^+ \nu(dk)$ intersects the $x$-axis, and $\nu^*((b_\nu(k),\infty))$ is (the modulus of) the slope of this tangent. See Figure 1.

If $\nu$ has atoms then the barycentre function is discontinuous and some care is needed. The pictorial representation gives a clue as to how we can define $\nu^*$: given a point $x$ on the $x$-axis, define $\nu^*((x,\infty))$ to be (the modulus of) the slope of the tangent to $C_\nu$ which intersects the $x$-axis at $x$. If there is an atom of $\nu$ at $k$ then there are multiple tangents to $C_\nu$ at $k$, but for each $x$ on the $x$-axis there is a unique tangent passing through $x$, so that $\nu^*((x,\infty))$ is well defined. For further discussion of the measure $\nu^*$ and the barycentre function $b_\nu$ see Azéma and Yor [1] or Hobson [11].

Having completed our digression, let us return to the problem of pricing the lookback option given the call price functions $C(k)$, which by assumption satisfy AII-i–ii. By Lemma 2.3, we can associate with $C$ a probability measure $\mu$ with unit mean. An interpretation of $\mu$ is given after Remark 2.6. Denote the Hardy-Littlewood transform of $\mu$ by $\mu^*$. The probability measure $\mu^*$ has support contained in $[1,\infty)$.

**Theorem 2.4** *Under the assumptions AI–i-ii and AII–i-ii the universal upper bound for the price of a lookback is $L^+ \equiv \int x\mu^*(dx)$. Specifically:*

- *it is possible to purchase a portfolio of call options at time 0, costing $L^+$, and to trade this portfolio to generate at least $S_T$ at time $T$.*
- *there is a market for which the (unique) price of the lookback option is $L^+$.*

*The universal lower bound for the price of a lookback is $L^- \equiv C(1) + 1$. Specifically*

- *it is possible to sell a portfolio of call options at time 0, of value $L^-$, and to trade this portfolio in such a way that the revenue from the lookback, $S_T$, is sure to exceed the obligations which arise from the portfolio of calls at time $T$.*
- *there is a market for which the (unique) price of the lookback option is $L^-$.*

**Remark 2.5** $L^+$ denotes the lowest price at which you, the risk averse, sceptical investor, are prepared to sell the lookback option; $L^-$ denotes the highest price that you are prepared to pay. No-arbitrage considerations then imply that the price of a lookback option must lie in the interval $[L^-, L^+]$.

**Proof of Upper Bound: Super-replicating Strategy.** To show that you would be prepared to sell the option for the price $L^+$ it is sufficient to show that with initial fortune $L^+$ you can super-replicate the payoff $S_T$ of the lookback.

The idea behind the replicating strategy is as follows. The initial fortune $L^+$ is used to purchase a portfolio of call options. These options are gradually sold over time according to a rule which depends on the behaviour of the asset. When the time of expiry is reached any remaining options are exercised. The key feature of the strategy is that the revenue generated from these combined sales is sufficient to cover the obligation (namely $S_T$). In a worst case scenario, the unsold calls expire out-of-the-money and the revenue generated from the sales of the call options exactly covers the obligation with zero surplus. In other circumstances, some additional profits are generated.

The first step in the hedging strategy is to purchase an initial portfolio of call options. Given $s \in [1, \infty)$, let $k = b_\mu^{-1}(s)$, and purchase $ds/(s-k)$ calls with strike $k$. The cost of this initial investment can be rewritten as

$$\int_{s \geq 1} \frac{C(b_\mu^{-1}(s))}{s - b_\mu^{-1}(s)} ds = \int_{s \geq 1} \mu^*((s, \infty)) ds = L^+ - 1. \tag{1}$$

At time $t = 0$ the maximum attained by $P$ is unity, which matches the funds remaining after the initial purchase of the call portfolio. As this maximum rises the strategy is to sell calls. In particular, when the maximum rises from $s - ds$ to $s$, those calls with strike $b_\mu^{-1}(s)$ are sold.

If $s = b_\mu(k)$ then there are $ds/(s-k)$ such calls. (If the martingale jumps to a new maximum $s$ then the strategy is to sell a tranche of calls consisting of all unsold calls with strikes less than $b_\mu^{-1}(s)$.) Suppose the calls with strike $k$ are sold for a unit price of $\lambda(k)$, then by assumption $\lambda(k) \geq s - k \geq s - b_\mu^{-1}(s)$ and, the total cash generated as the maximum rises from 1 to $S_T$ is

$$\int_1^{S_T} \frac{\lambda(b_\mu^{-1}(s))}{s - b_\mu^{-1}(s)}\, ds \geq \int_1^{S_T} ds = S_T - 1.$$

If we add in the unit of cash which was not spent on call options, then the current cash holdings are at least sufficient to cover the obligation of the lookback. Moreover, if when the expiry time is reached the remaining unsold calls are exercised, any further revenue is an additional surplus. □

**Remark 2.6** This upper bound, and the associated strategy apply in complete generality. No assumptions have been made about the underlying model or the dynamics of the price process.

The next step is to show that in some sense $L^+$ is the smallest fortune with which it is always possible to finance a super-replicating strategy. This will follow if we can find a market in which the rational price for call options are given by $C(k)$ and the rational price for a lookback is $L^+$.

Suppose that the call prices have arisen as the expected payoff under a martingale measure $\mathbb{Q}$ in a complete market with zero interest rates; that is we can represent the call prices $C(k) = \mathbb{E}^{\mathbb{Q}}(P_T - k)^+$. When we take a (right)-derivative with respect to $k$ we find that

$$C'(k+) = -\mathbb{Q}(P_T > k). \tag{2}$$

Thus from call prices it is possible to infer the $\mathbb{Q}$-distribution of $P_T$; and indeed $P_T$ has the law $\mu$ which we defined above. Moreover we know that under $\mathbb{Q}$ the price process $P_t$ is a martingale, and the lookback is a security whose payoff corresponds to the maximum of this martingale.

The problem of describing the possible laws of the maximum of a martingale whose terminal distribution is specified has been a subject of research in the mathematics literature for many years. A key idea is the concept of stochastic ordering of probability measures; namely that $\nu \preceq \rho$ if and only if the corresponding distribution functions $F_\nu$ and $F_\rho$ satisfy $F_\nu(x) \geq F_\rho(x)$ for all values of $x$. The relation $\preceq$ is a partial ordering. Write $\nu \vee \rho$ for the probability measure with distribution function $F_{\nu\vee\rho}(x) \equiv F_\nu(x) \wedge F_\rho(x)$. Then:

**Theorem 2.7 (Blackwell and Dubins [2], Kertz and Rösler [13])**
*Suppose $M$ is a martingale with $M_0 = 1$. Suppose $M_T$ has law $\nu$ (where $\nu$ has unit mean), and denote by $\rho$ the law of the maximum value attained by $M$ in $[0,T]$. Then*

$$\nu \vee \delta_1 \preceq \rho \preceq \nu^*$$

*where $\delta_1$ is the unit mass at 1, and $\nu^*$ is the Hardy-Littlewood transform of $\nu$. Moreover these bounds are attained.*

Kertz and Rösler [13] prove a converse to Theorem 2.7, namely that if $\nu \vee \delta_1 \preceq \rho \preceq \nu^*$ then there is a martingale $M$ with $M_0 = 1$ and terminal distribution $\nu$ whose maximum process has law $\rho$.

**Corollary 2.8** *There is a martingale price process whose terminal value has law $\mu$ and maximum has law $\mu^*$. In this model call prices are given by $C(k)$, and the price of a lookback option is $L^+$.*

**Proof of Lower Bound.** Since

$$S_T \geq (P_T - 1)^+ + 1, \tag{3}$$

the payoff from the lookback option is sure to dominate the payoff from a call with unit strike by at least 1. Consequently you can risklessly offer to pay $L^- \equiv C(1) + 1$ for the lookback.

Note that $C(1) + 1$ is the expected value of a random variable with law $(\mu \vee \delta_1)$. Thus, by Theorem 2.7, $C(1) + 1$ is the lowest price consistent with the law $\mu$ and the martingale assumption. In particular, consider the price process which consists of a single jump at time $T/2$ say; for this martingale there is equality in (3). □

**Remark 2.9** This lower bound is trivial. If investors are prepared to assume that the underlying price process is a continuous martingale then a higher bound may be feasible. In particular, given a martingale with specified terminal distribution, it is possible to identify a lower bound (with respect to stochastic ordering of probability measures) for the law of the maximum of that martingale. Thus it is possible to calculate a price which acts as a lower bound in the sense that in every complete market (in which the underlying asset price is continuous) the fair price of the lookback option exceeds the lower bound. However, since there is no associated super-replicating strategy, the existence of the bound does not guarantee that arbitrage profits can be made if the observed market price is lower than the theoretical bound.

## 3 Non-zero Interest Rates.

To date we have assumed that interest rates are zero. More precisely, in calculating the upper bound we assumed that the value of cash holdings does not decrease over time. Thus, provided that interest rates are non-negative, the super-replicating strategy outlined in the previous section will continue to have a terminal value which dominates the payoff from a lookback. Again we have a bound on the price of a lookback option if there are to be no arbitrage opportunities.

If there is a bond market on which investors can trade then it is possible to refine this upper bound. In this section we calculate the prices (and the associated strategies) at which you, the sceptical, risk-averse investor are prepared to trade the lookback option. These prices depend on the prices of calls and zero-coupon bonds.

Define $\beta_T$ to be the current (time 0) price of a pure discount bond which makes a unit payment at time $T$. We assume that that zero-coupon bonds with maturity $T$ are available for purchase in arbitrary quantities at time 0.

If interest rates are arbitrary, then it is possible for there to be a period of large positive interest rates followed by negative interest rates in a fashion which is consistent with $\beta_T$. If such scenarios are not excluded then there is no universal upper bound on the price of a lookback. Hence we make the reasonable assumption that interest rates are non-negative at all times. This assumption is crucial in ensuring that assumption AII–ii remains valid after the switch to non-zero interest rates.

**Assumption 3.1 (Non-negative Interest Rates)**

AIII–i Interest rates are non-negative. In particular $\beta_T \leq 1$.

Given the call price function $C(k)$ we can define a measure $\mu$ via

$$\beta_T^{-1} C(k) = \int_{(0,\infty)} (y-k)^+ \mu(dy).$$

Since $(P_T - \epsilon)^+ + \epsilon \geq P_T$, and since a payoff of $\epsilon$ at time $T$ can be purchased for $\epsilon \beta_T$ at time 0, we have that $\mu((0,\infty)) = -\beta_T^{-1} C'(0+) \leq 1$. Adding mass at 0 as necessary, $\mu$ is a probability measure with mean $\beta_T^{-1}$ and support contained in $[0,\infty)$. Later $\mu$ will turn out to be the law of $P_T$ under the $T$-forward measure.

Associated with $\mu$ is its Hardy-Littlewood transform $\mu^*$ and barycentre function $b_\mu$. As in the previous section the pair $(\mu, b_\mu)$ can be used to define a price and a trading strategy which, under the assumptions AI–i-ii, AII–i-ii and AIII–i, will be an upper bound on the price of a lookback and a super-replicating strategy respectively. Moreover, this upper bound is universal in the sense that in every arbitrage-free market which is consistent with the call prices $C(k)$ the price of a lookback is not greater than the universal upper bound.

**Theorem 3.2** *Under the assumptions AI–i-ii, AII–i-ii and AIII–i the universal upper bound for the price of a lookback is $L^+ \equiv \beta_T \int x \mu^*(dx)$. Specifically:*

- *it is possible to purchase a portfolio of call options at time 0, costing $L^+$, and to trade this portfolio to generate at least $S_T$ at time $T$.*
- *there is a market for which the (unique) price of the lookback option is $L^+$.*

*Under the assumption AIII–i the universal lower bound for the price of a lookback is $L^- \equiv C(1) + \beta_T$. Specifically:*

- *it is possible to sell a portfolio of call options at time 0, of value $L^-$, and to trade this portfolio in such a way that the revenue from the lookback, $S_T$, is sure to exceed the obligations which arise from the portfolio of calls at time $T$.*

- *there is a market for which the (unique) price of the lookback option is $L^-$.*

**Proof of Upper Bound: Super-replicating Strategy.** $C$, $\beta_T$ and $\mu$ are related by

$$\mu((k,\infty)) = -\beta_T^{-1} C'(k) \tag{4}$$

where $C'$ denotes the right derivative. The barycentre function is again best described pictorially: $b_\mu(k)$ is the $x$-coordinate of the point where the tangent to $C$ at $k$ crosses the $x$-axis, recall Figure 1. Thus

$$-\beta_T^{-1} C'(b_\mu^{-1}(s)) = \beta_T^{-1} \frac{C(b_\mu^{-1}(s))}{s - b_\mu^{-1}(s)} = \mu^*((s,\infty)). \tag{5}$$

Note that $b_\mu(0-) = \int x\mu(dx) = \beta_T^{-1}$ so that if interest rates are non-zero then $b_\mu(0) > 1$.

The aim is to show that it is possible to finance a super-replicating strategy with initial fortune $L^+$, so that $L^+$ is a universal upper bound on the price of a lookback.

Our strategy is motivated by the zero interest-rate case. As before purchase $ds/(s-k)$ calls with strike $k \equiv b_\mu^{-1}(s)$. The cost of this initial portfolio is $L^+ - 1$ (recall (1)), which leaves one unit to be invested in bonds, giving a return of $\beta_T^{-1}$ at time $T$.

Again the dynamic component of the strategy consists of selling those calls with strike $b_\mu^{-1}(s)$ as the maximum rises from $s - ds$ to $s$. Note that since $b_\mu(0-) > 1$ no calls are sold until the maximum first reaches $\beta_T^{-1}$. The choice of initial portfolio and the assumptions AII–ii and AIII–i ensure that the proceeds generated from the sales of calls are at least $(S_T - \beta_T^{-1})^+$. If the return from the bonds are now added then the total revenues from the strategy are at least $(S_T - \beta_T^{-1})^+ + \beta_T^{-1} \geq S_T$. □

We now wish to show that $L^+$ is the smallest fortune with which it is always possible to finance a super-replicating strategy. Suppose that call prices can be represented as the discounted expected payoff of the option under the $T$-forward measure $\mathbb{P}^T$, so that

$$C(k) \equiv \beta_T \mathbb{E}^T (P_T - k)^+. \tag{6}$$

By taking derivatives it is possible to show that the $\mathbb{P}^T$-distribution of $P_T$ is $\mu$.

**Proposition 3.3**
*Suppose $M$ is a submartingale with $M_0 = 1$. Suppose $M_T$ has law $\nu$, and denote by $\rho$ the law of the maximum value attained by $M$ in $[0,T]$. Then*

$$\nu \vee \delta_1 \preceq \rho \preceq \nu^*$$

*where $\delta_1$ is the unit mass at 1, and $\nu^*$ is the Hardy-Littlewood transform of $\nu$. Moreover these bounds are attained.*

The proof of the upper bound in Proposition 3.3 follows exactly as in the proof of the upper bound in Theorem 2.7 given in Blackwell and Dubins citeB+D, except that the use of a submartingale means that some of the equalities become inequalities. The upper bound is attained by, for example, the submartingale which rises monotonically from 1 to $\int x\nu(dx)$ over the time-period $[0, \frac{1}{2}]$ and which satisfies the martingale property, with appropriate maximal conditions on the law of the maximum, for $(\frac{1}{2} \le t \le 1)$. The proof of the lower bound in Proposition 3.3 is direct and straight-forward.

**Corollary 3.4** *There is a submartingale price process whose terminal value has law $\mu$ and maximum has law $\mu^*$. In this model call prices are given by $C(k)$ and the price of a lookback option is $L^+$.*

**Proof of Lower Bound.** Since

$$S_T \ge (P_T - 1)^+ + 1 \tag{7}$$

the payoff from the call dominates the combined payoff from a call with unit strike and a zero-coupon bond maturing at $T$. Moreover, for the submartingale consisting of a single jump, there is equality in (7). □

## 4 Knock-In Call Options and Forward-Start Lookbacks

The purpose of this section is to find further bounds on the prices of derivative securities. These bounds depend on the market prices of call options and represent the price at which you, the sceptical, risk-averse options trader are prepared to trade. At these prices there is a super-replicating strategy for the derivative security which does not depend on knowledge of the dynamics of the asset price. Arbitrage considerations prevent derivatives from trading outside these price bounds.

One simple generalisation of §2 is to consider a respecification of the payoff of the lookback. For example a *lookback put* allows the purchaser to sell a unit of asset at time $T$ for the highest price attained during the interval $[0, T]$, and has payoff

$$S_T - P_T \equiv \sup_{0 \le t \le T} (P_t) - P_T.$$

The super-replication price of such a security is exactly one less than the price calculated in §2 and the super-replicating strategy is identical except for the additional sale of one unit of the underlying asset at time 0.

Equally, as well as working with the maximum price attained, we can consider derivative securities whose payoff is contingent on the value of the minimum. Bounds follow by considering the possible maxima of the process $(-P)_t$.

Recall that for a given maturity $T$ we assume that there is a family $C(k)$ of prices of traded call options with strike $k$. Either interest rates are zero, or they are non-negative and there is a bond with maturity $T$ whose current price is $\beta_T$. Associated with the function $\beta_T^{-1}C(k)$ is a measure $\mu$ whose barycentre function $b_\mu$ satisfies

$$-C'(k)(b_\mu(k)-k)=C(k),$$

at least at continuity points of $C'$ and $b_\mu$. Moreover we can interpret $\mu$ as the law of $P_T$ under the $T$-forward measure $\mathbb{P}^T$; whence $C(k)=\beta_T\mathbb{E}^T(P_T-k)^+$.

### 4.1 Knock-In Call Options

Consider a security which pays

$$(P_T-k)^+I_{(S_T\geq\kappa)}.$$

The security is called a *knock-in call* option because it is a call option which pays off if and only if it has been "knocked-in" by the underlying asset price rising above some threshold, or barrier, at an intermediate time.

**Theorem 4.1** *The smallest universal upper bound on the price of a knock-in call is*

$$\begin{array}{ll} C(k) & \text{if } b_\mu(k)\geq\kappa \\ \dfrac{\kappa-k}{\kappa-b_\mu^{-1}(\kappa)}C(b_\mu^{-1}(\kappa)) & \text{otherwise} \end{array} \tag{8}$$

**Proof.** Suppose $(M_t)_{0\leq t\leq T}$ is a submartingale with $M_0=1$ and such that $M_T$ has law $\mu$. Let $S_t^M$ denote the maximum process of $M$. Trivially

$$(M_T-k)^+I_{[S_T^M\geq\kappa]}\leq(M_T-k)^+$$

and the upper bound $C(k)$ on the price of a knock-in call follows easily.

Moreover, by Proposition 3.3,

$$\mathbb{P}(S_T^M\geq\kappa)\leq\mu^*([\kappa,\infty))=\mu([b_\mu^{-1}(\kappa),\infty))=-C'(b_\mu^{-1}(\kappa))\beta_T.$$

Then, using the fact that $\mathbb{P}(S_T^M\geq\kappa)\leq\mathbb{P}(M_T\geq b_\mu^{-1}(\kappa))$, and the simple property of a positive random variable $X$ that $\mathbb{E}(XI_A)\leq\mathbb{E}(XI_{X\geq b})$ for all sets $A$ with $\mathbb{P}(A)\leq\mathbb{P}(X\leq b)$, we get that

$$\mathbb{E}((M_T-k)^+I_{[S_T^M\geq\kappa]})\leq\mathbb{E}((M_T-k)^+I_{[M_T\geq b_\mu^{-1}(\kappa)]}). \tag{9}$$

An inspection of the proof of the Skorokhod Embedding Theorem in Azéma and Yor [1] or Rogers [16], (see also Kertz and Rösler [13]), shows that there is an identification of the sets $(S_T^M\geq\kappa)$ and $(M_T\geq b_\mu^{-1}(\kappa))$ for certain martingales so that there can be equality in (9). Then, for $b_\mu(k)<\kappa$,

$$\mathbb{E}((M_T-k)^+I_{[M_T\geq b_\mu^{-1}(\kappa)]})=\mathbb{E}((M_T-b_\mu^{-1}(\kappa))I_{[M_T\geq b_\mu^{-1}(\kappa)]})+(b_\mu^{-1}(\kappa)-k)\mathbb{P}(M_T\geq b_\mu^{-1}(\kappa)).$$

Under the assumption that the prices of options can be expressed as a discounted expected payoff under the $T$-forward measure then an application of the above result to the $\mathbb{P}^T$-submartingale $P$ yields

$$\beta_T \mathbb{E}^T\left((P_T - k)^+ I_{[P_T \geq b_\mu^{-1}(\kappa)]}\right) = C(b_\mu^{-1}(\kappa)) + (b_\mu^{-1}(\kappa) - k)(-C'(b_\mu^{-1}(\kappa))) \equiv \frac{\kappa - k}{\kappa - b_\mu^{-1}(\kappa)} C(b_\mu^{-1}(\kappa)),$$

this last equality following from (5). Thus the lowest price for which there could be an universal replicating strategy is given by (8).

If $b_\mu(k) \geq \kappa$ then the associated super-replicating strategy is extremely simple and consists of purchasing a call with strike $k$ and holding it to maturity. If $b_\mu(k) < \kappa$ then the first step of the strategy is to purchase $(\kappa - k)/(\kappa - b_\mu^{-1}(\kappa))$ calls with strike $b_\mu^{-1}(\kappa)$. The cost of this initial portfolio is

$$\frac{\kappa - k}{\kappa - b_\mu^{-1}(\kappa)} C(b_\mu^{-1}(\kappa))$$

so that it can be financed using the revenue from the sale of the knock-in call.

One of these calls is held to maturity. The remaining $(b_\mu^{-1}(\kappa) - k)/(\kappa - b_\mu^{-1}(\kappa))$ calls are sold the instant that the underlying asset price first reaches $\kappa$. By assumption this sale raises at least $(b_\mu^{-1}(\kappa) - k)$.

If the price of the underlying asset never reaches the threshold $\kappa$, then the payoff of the knock-in call is zero whereas the super-replicating strategy raises $(P_T - b_\mu^{-1}(\kappa))^+$. Conversely, if $P_t$ reaches the barrier $\kappa$, then the wealth

$$(P_T - b_\mu^{-1}(\kappa))^+ + (b_\mu^{-1}(\kappa) - k)$$

raised by the super-replicating strategy, always dominates the obligation $(P_T - k)^+$ arising from the sale of the knock-in call. $\square$

## 4.2 Forward-Start Lookback Options

Consider a security on sale at time 0 which at time $T_2$ pays the maximum value attained by the underlying security in the period $[T_1, T_2]$. Thus the payoff is given by

$$F_{T_1,T_2} = \sup_{T_1 \leq t \leq T_2} P_t$$

The purpose of this section is to place an upper bound on the price of a *forward-start lookback* option with payoff $F_{T_1,T_2}$ relative to the prices of calls with maturities $T_1$ and $T_2$. For simplicity we assume that interest rates are zero. See the end of this section for some remarks on the general case.

Denote the call prices with expiries $T_1$, $T_2$ by $C_1(k)$ and $C_2(k)$ respectively. These call price functions must satisfy a consistency condition.

**Assumption 4.2**

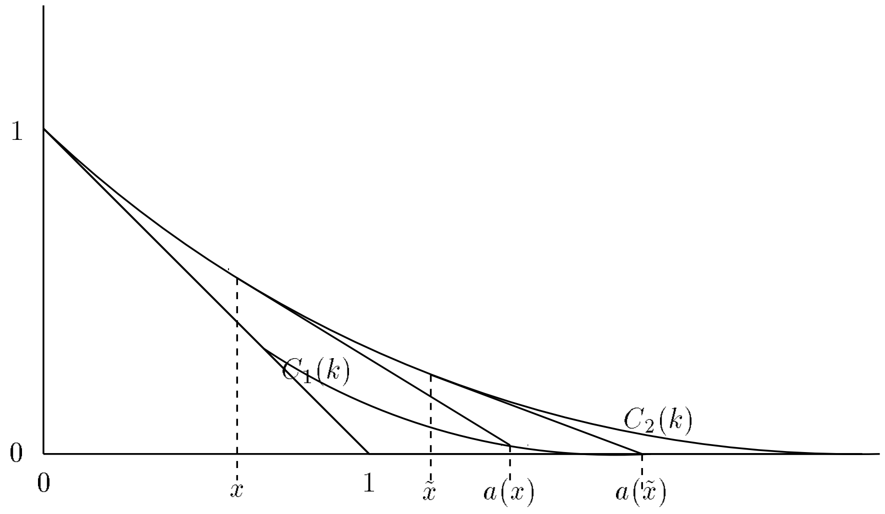


Figure 2: The function $a$.

*AII–iii The prices of call options with a given strike increase with time to maturity:*

$$C_1(k) \leq C_2(k) \qquad \forall k \tag{10}$$

To understand (10) in financial terms note that the payoff $(P_{T_2} - k)I_{[P_{T_2} > k]}$ from the option with maturity $T_2$ dominates $(P_{T_2} - k)I_{[P_{T_1} > k]}$. But this second payoff can be replicated by buying a call with strike $k$ and maturity $T_1$, and, if and only if this call finishes in the money, buying a unit of the underlying asset at time $T_1$ and holding it until $T_2$. Thus (10) is a very natural condition. Moreover (10) is a necessary condition for call prices to be consistent with martingale pricing. This is a corollary of the following result:

**Lemma 4.3 (Rost [19])** *Suppose $\nu$ and $\rho$ are probability measures on $\mathbb{R}$ with unit mean. Then there is a martingale $M$ with initial law $\nu$ and terminal law $\rho$ if and only if*

$$\mathbb{E}(X_\nu - k)^+ \leq \mathbb{E}(X_\rho - k)^+ \qquad \forall k$$

*where $X_\nu$ and $X_\rho$ are random variables with the laws $\nu$ and $\rho$ respectively.*

The first step in finding bounds on the forward-start lookback is to define a function $a$ which is an analogue of the barycentre function. $a(k)$ is the $x$-coordinate of the point where the tangent to the curve $C_2(.)$ at the point $k$, taken in the direction of increasing $k$, intersects the curve $C_1(.)$ We assume that $C_2$ is a smooth function so that there is no ambiguity in choice of tangent, and so that $a$ is a continuous function. See Hobson [11] for the general case. Then,

$$C_2(k) + (a(k) - k)C_2'(k) = C_1(a(k)). \tag{11}$$

Let $\mu_1$ and $\mu_2$ denote the probability measures associated with $C_1$ and $C_2$ respectively, in accordance with Lemma 2.3. If $X$ is a random variable with law $\mu_2$ then let $\mu^*_{1,2}$ be the law of $a(X)$.

**Proposition 4.4** *The smallest universal upper bound on the price of the forward-start lookback call is* $L^F \equiv \int x\mu^*_{1,2}(dx)$.

**Proof.** The first step is to show that there is a super-replicating strategy which can be financed by an initial investment $L^F$.

Consider the following strategy: given initial fortune $L^F$ purchase an initial portfolio consisting of calls with various strikes and maturities $T_1$ and $T_2$; at time $T_1$ exercise those calls with maturity $T_1$; during the period $[T_1, T_2)$ sell some of the calls with maturity $T_2$ according to a dynamic hedging strategy; finally at time $T_2$ exercise the remaining calls which are in the money. It remains to specify the initial portfolio and the dynamic hedging strategy. We wish to choose these items in such a way that we super-replicate the lookback option.

For initial portfolio take $\theta_1(dk)$ call options with strike $k$ and maturity $T_1$ and $\theta_2(dk)$ call options with strike $k$ and maturity $T_2$ where

$$\theta_1(dk) = \frac{-dk}{k - a^{-1}(k)} \qquad \text{and} \qquad \theta_2(dk) = \frac{a(dk)}{a(k) - k}$$

The fact that $\theta_1 < 0$ implies that the initial strategy is to sell calls with maturity $T_1$ rather than purchase them. The cost of this initial portfolio is

$$\begin{aligned}
\int \theta_2(dk)C_2(k) + \int \theta_1(dk)C_1(k) &= \int \frac{a(dk)C_2(k)}{a(k) - k} - \int \frac{a(dk)C_1(a(k))}{a(k) - k} \\
&= \int a(dk)(-C_2'(k)) \\
&= L^F - a(0)
\end{aligned}$$

leaving $a(0)$ units to be held as cash.

At time $T_1$ the maturing calls create an obligation of

$$-\int \theta_1(dk)(P_{T_1} - k)^+ = \int_{k \le P_{T_1}} \frac{P_{T_1} - k}{k - a^{-1}(k)} dk.$$

At time $T_1$ we also sell those calls with maturity $T_2$ and strikes below $a^{-1}(P_{T_1})$. By assumption AII-ii each unit sold with strike $k$ realises at least $(P_{T_1} - k)$. Then the new cash holdings are at least

$$\begin{aligned}
&a(0) + \int \theta_1(dk)(P_{T_1} - k)^+ + \int_{k \le a^{-1}(P_{T_1})} \theta_2(dk)(P_{T_1} - k) \\
&\qquad = a(0) - \int_{k \le a^{-1}(P_{T_1})} \frac{a(dk)(P_{T_1} - a(k))}{a(k) - k} + \int_{k \le a^{-1}(P_{T_1})} \frac{a(dk)(P_{T_1} - k)}{a(k) - k} \\
&\qquad = P_{T_1}
\end{aligned}$$

The dynamic hedging component of the super-replicating strategy is to sell some of the remaining calls as the maximum over the period $[T_1, T_2]$ rises. In particular, when the maximum reaches (or exceeds) $s$ those calls with strike $a^{-1}(s)$ are sold. If $s = a(k)$ then there are $\theta_2(dk) = a(dk)/(a(k) - k) = ds/(s - a^{-1}(s))$ such calls. By assumption the unit price for these calls is at least $s - a^{-1}(s)$, so that the cash generated as the maximum rises from $P_{T_1}$ to $F$ is at least

$$\int_{P_{T_1}}^{F} ds = F - P_{T_1}.$$

Adding this to the cash holdings at time $T_1$ yields at least $F$, the current maximum. Pursuing this strategy to time $T_2$ guarantees that sufficient funds will be raised to cover the obligation arising from the sale of the lookback option. Thus the strategy is a super-replicating strategy. Moreover there is the possibility of excess profits either from selling calls for more than the intrinsic value at intermediate times, or from exercising in the money calls at maturity $T_2$.

It remains to show that $L^F$ is the smallest initial fortune which can be used to finance a super-replicating strategy. Suppose that call prices have arisen in a complete market as the expected payoff of the option under an equivalent martingale measure $\mathbb{Q}$. Then $\mu_1$ and $\mu_2$ are the implied laws of $P_{T_1}$ and $P_{T_2}$ respectively under $\mathbb{Q}$. Moreover since $P$ is a $\mathbb{Q}$-martingale, the tightness of the upper bound $L^+$ is a corollary of the following Theorem.

**Theorem 4.5 (Hobson [11])** *Let $(M_t)_{0 \le t \le 1}$ be a martingale with initial law $\mu_1$ and terminal law $\mu_2$. Denote the maximum of $M$ by $S^M$; then the law $\nu$ of $S^M$ must satisfy*

$$(\mu_1 \vee \mu_2) \preceq \nu \preceq \mu^*_{1,2}.$$

*Moreover there is a martingale for which this upper bound is attained.*

□

To complete this section we now consider the pricing of the forward-start lookback option when interest rates are non-zero. Assume that the call price functions $(C_i(k))_{i=1,2}$ are known, and that the price of a bond with maturity $T_i$ is given by $\beta_i$. By the non-negativity assumption on interest rates $1 \equiv \beta_0 \ge \beta_1 \ge \beta_2$. For $i = 1, 2$ the call price functions $C_i$ and discount factors $\beta_i$ are related by

$$-C'_i(0-) = \beta_i.$$

Define the modified-strike call price function $\tilde{C}$ by $\tilde{C}_i(k) \equiv C_i(k/\beta_i)$, and assume that

$$\tilde{C}_1(k) \le \tilde{C}_2(k) \qquad \forall k. \tag{12}$$

Let $\tilde{\mu}_i$ be the measure associated with $\tilde{C}_i$. Let $\tilde{a}$ be the generalised barycentre function satisfying (11) for the call price functions $\tilde{C}_i$ and denote by $\tilde{\mu}^*_{1,2}$ the measure arising from Theorem 4.5 given the initial and terminal laws $\tilde{\mu}_1$ and $\tilde{\mu}_2$. Finally, set $\tilde{L}^F \equiv \int x \tilde{\mu}^*_{1,2}(dx)$.

It is clear that there is a simple adaptation of the trading strategy outlined in the proof of Proposition 4.4 which is based on the entities $\tilde{C}_i, \tilde{\mu}_i, \tilde{\mu}^*_{1,2}$ and $\tilde{a}$. For an initial fortune $\tilde{L}^F$, this strategy will super-replicate the forward-start lookback call, provided that a call with strike $k$ and expiry $T_2$ can be sold at time $T_1$ for at least $(P_{T_1} - k\beta_2/\beta_1)$. This will be true if, for example, interest rates are deterministic. The details of this super-replicating strategy are left as an exercise to the interested reader.

If interest rates are independent of the stock-price process, and if call prices have arisen as the discounted expected payoff under an equivalent martingale measure $\mathbb{Q}$, then $\beta_i^{-1} \equiv \mathbb{E}^{\mathbb{Q}}(P_{T_i})$ and

$$C_i(k) = \beta_i \mathbb{E}^{\mathbb{Q}}(P_{T_i} - k)^+.$$

Moreover the discounted price process $(\tilde{P}_i) = (\beta_i P_{T_i})_{i=0,1,2}$ is a $\mathbb{Q}$-martingale, with the property that the $\mathbb{Q}$-law of $\tilde{P}_i$ is $\tilde{\mu}_i$. Condition (12) is then an immediate consequence of Lemma 4.3.

We can take $\tilde{P}$ to be a martingale with time parameter $t \in [0, T_2]$ and then

$$F_{T_1,T_2} \equiv \sup_{T_1 \le t \le T_2} (P_t) \le \beta_2^{-1} \sup_{T_1 \le t \le T_2} (\tilde{P}_t).$$

so that in this model the discounted expected payoff from the lookback option satisfies

$$\beta_2 \mathbb{E}^{\mathbb{Q}}(F_{T_1,T_2}) \le \mathbb{E}^{\mathbb{Q}}(\sup_{T_1 \le t \le T_2} (\tilde{P}_t)) \le \tilde{L}^F. \tag{13}$$

Moreover, for a particular choice of martingale it is possible to have equality simultaneously in both parts of (13) so that the universal upper bound cannot be smaller than $\tilde{L}^F$.

# 5 Summary

In a Black-Scholes world, where the asset prices perform a geometric Brownian motion and the volatility is a known constant, there are formulæ for the prices of both call options and for many other exotic derivatives. For example the price of a lookback option was given in Goldman, Sosin and Gatto [9]. The book by Hull [12] is a general reference to options and derivatives, and contains a discussion of many of the securities we have considered here, including their Black-Scholes prices. If options trade at any other prices then there are arbitrage profits to be made *provided the underlying model is correct with probability one.*

In practice the underlying model is not known with certainty. Even if the asset price is assumed to follow an exponential Brownian motion then the model parameter volatility is often estimated from the price of a traded call option. However this implied volatility is rarely independent of the call chosen. This means that when exotic options are priced there is confusion over which volatility to use, and moreover the exotic pricing formula is derived from a necessarily invalid model. One potential solution to this problem is to assume that the instantaneous volatility is not fixed and known, but instead is merely constrained to lie in an interval. This is the approach taken by Lyons [15] to price barrier options. Another idea is to find a model for the dynamics of the underlying asset price which is consistent with the call options price data. This is the approach of Dupire [6]. Here we make fewer assumptions so that there may be many models which are consistent with observed call prices. To place an upper bound on the price of an exotic option we choose the model within which the price is maximised. This choice is justified because this is the lowest initial fortune with which it is possible to finance a super-replicating strategy. Moreover this strategy super-replicates whether or not the model choice is correct.

The securities on which this article concentrates are exotic options whose payoff is contingent on the final value of the maximum, and the bounds we find depend on the traded prices of call options. If the exotic ever trades at a price which is outside these bounds then there are arbitrage profits to be made, *independently of the true underlying model*. Moreover there is a simple strategy for capturing these profits.

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