Stochastic volatility models, correlation and the q-optimal measure

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The aim of this paper is to study the minimal entropy and variance-optimal martingale measures for stochastic volatility models. In particular, for a diffusion model where the asset price and volatility are correlated, we show that problem of determining the q-optimal measure can be reduced to finding a solution to a representation equation. The minimal entropy measure and variance-optimal measure are seen as the special cases $q = 1$ and $q = 2$ respectively.

In the case where the volatility is an autonomous diffusion we give a stochastic representation for the solution of this equation. If the correlation ρ between the traded asset and the autonomous volatility satisfies $\rho^2 < 1/q$, and provided certain smoothness and boundedness conditions on the parameters are satisfied, then the q-optimal measure exists. If $\rho^2 \geq 1/q$ then the q-optimal measure may cease to exist beyond a certain time horizon. As an example we calculate the q-optimal measure explicitly for the Heston model.

Keywords: Martingale measures, variance-optimal measure, minimal entropy measure, stochastic volatility, correlation, Heston model.

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1 Introduction and main results.

In this article we consider stochastic volatility models for asset prices. The typical stochastic volatility model is incomplete and there are no unique preference-independent option prices. Instead there are various approaches to derivative claim pricing. One approach, motivated by the analysis of complete markets, is to postulate that the prices of options should be given as the expectation under an equivalent martingale measure. This approach has the advantage that it leads to a linear pricing scheme which is also arbitrage free. However, since the market is incomplete, there are many such martingale measures, and we need to specify a criterion by which to justify a particular choice.

One choice of measure is the variance-optimal measure. Under certain regularity conditions this is the mean-variance option pricing measure introduced in the martingale case by Föllmer and Sondermann (1986), and extended to the general case by Duffie and Richardson (1991), Schweizer (1991) and Gouriéroux et al (1998) amongst others. The use of the variance-optimal measure in stochastic volatility models has been extensively investigated by Laurent and Pham (1999), Biagini et al (2000) and Heath et al (2001) and these studies provide a starting point for our investigations. The variance-optimal measure has links with quadratic utility functions, and has the distinct advantage that it is convenient for calculations (due to the quadratic structure) but has the disadvantage that it may cease to be defined beyond a certain time horizon. Furthermore, in models with jumps it may lead to a signed measure, see Schweizer (1996), although this problem will not concern us here.

A second choice of pricing measure is the minimal entropy measure, see Frittelli (2000) and Rouge and El Karoui (2000). There are links between pricing under the minimal entropy measure and pricing under an exponential (constant absolute risk aversion) utility function. This link is via the dual problem, see Delbaen et al (2002). Moreover, the minimal relative entropy measure exists in many circumstances when the variance-optimal measure is not defined, see for example Rheinländer (2002), and indeed some of the results below.

The connection between these two theories is provided by the q-optimal measure, Delbaen et al (1997). The q-optimal measure minimises the q^{th} moment of the Radon-Nikodym derivative of the pricing measure with respect to the original real-world measure. When $q = 2$ we have the variance-optimal measure and under certain technical conditions (see for example Grandits and Rheinländer (2002)) the minimal entropy measure arises in the limit $q \downarrow 1$. Our aim is to study the q-optimal measure for stochastic volatility models with correlation.

For the purposes of this section, and in order to state the main results, we define the model as follows. Let S be the traded asset (we work in a world with zero interest rates, so that S is really a discounted price) and let Y be the stochastic volatility. In principle both S and Y could be vector valued quantities, but in this paper we will only consider the univariate case. Then S and Y solve

$$\frac{dS_t}{S_t} = Y_t(\lambda_t dt + dB_t) \qquad\qquad dY_t = \alpha_t dt + \beta_t dW_t \tag{1}$$

subject to initial conditions, where λ_t, α_t and β_t are previsible processes, and B, W are correlated Brownian motions with correlation co-efficient ρ_t taking values in $(-1,1)$. We write $\overline{\rho}_t = \sqrt{1-\rho_t^2}$ so that W is given by $dW_t = \rho_t dB_t + \overline{\rho}_t dZ_t$ for a Brownian motion Z independent of B. The form of the SDE for the traded asset, and in particular the fact that it is driven by a Brownian motion and the drift can be written as $\lambda_t Y_t S_t$, means that this model satisfies the 'structure condition' (Pham et al (1998)). It will turn out that the increasing process $K_t := \int_0^t \lambda_t^2 dt$ is a key quantity of interest. K_t is often called the 'mean-variance tradeoff process'.

Theorem 1.1 *Suppose there are previsible processes $\eta_t = \eta_t^{(q)}, \xi_t = \xi_t^{(q)}$ and a finite constant $c = c^{(q)}$ such that*

$$\frac{q}{2}K_T = M_T - \frac{(q-1)}{2}[M]_T + L_T + \frac{1}{2}[L]_T + c \tag{2}$$

where

$$M_t = \int_0^t \eta_u(dB_u + q\lambda_u du), \qquad\qquad L_t = \int_0^t \xi_u dZ_u.$$

Then, provided certain regularity assumptions are satisfied (see later sections for more details), the q-optimal martingale measure $\mathbb{Q}^{(q)}$ is given by

$$\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}} = \mathcal{E}(-\lambda\cdot B - \xi\cdot Z)_T \tag{3}$$

and then

$$\mathbb{E}\left[\left(\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right)^q\right] = e^{(q-1)c}; \tag{4}$$

moreover the q-optimal measure is of the form

$$\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}} = C\exp\left(\int_0^T \chi_t dS_t - \frac{(q-1)}{2}\int_0^T \chi_t^2 d[S]_t\right) \tag{5}$$

for the process $\chi_t = (\eta_t - \lambda_t)/(S_t Y_t)$ and constant $C = e^c$. The expressions for (3), (4) *and* (5) *hold for the relative entropy measure $\mathbb{Q}^{(1)}$ on taking $q = 1$, save that in that case the expression in* (4) *becomes*

$$\mathbb{E}\left[\frac{d\mathbb{Q}^{(1)}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}^{(1)}}{d\mathbb{P}}\right] = c.$$

Remark 1.2 We will call (2) the fundamental representation equation. As Rheinländer (2002) remarks, this equation corresponds to a non-trivial martingale representation theorem because of the presence of the square bracket processes $[M]$ and $[L]$.

Remark 1.3 Many of the conclusions of Theorem 1.1 are not new, except perhaps in the case $q \neq 1, 2$. However the cases $q = 1$ and $q = 2$ are usually treated separately, and in different papers, and the unifying role of the fundamental equation (2) which is key for all $q \geq 1$ has not previously been recognised. The representation equation (2) is a generalisation of Equation (2.2) in Biagini et al (2000) in the variance-optimal case ($q = 2$) and Equations (4.1) in Rheinländer (2002) and (3.31) in Mania et al (2003) for the minimal entropy problem ($q = 1$). Further, (5) extends the results of Delbaen and Schachermeyer (1996) and Gouriéroux et al (1998) in the variance-optimal case, and Grandits and Rheinländer (2002) in the q-optimal case. Again one of the advantages of considering the q-optimal measure is that the minimal relative entropy measure arises as a special case.

It remains to identify the parameters η and ξ which help define the q-optimal measure, and the main contribution of this paper is to do exactly this for stochastic volatility models with correlation. To date most explicit formulae have been derived in the zero correlation case, see Laurent and Pham (1999) and Biagini et al (2000). The only exception to this of which the author is aware is the result in Rheinländer (2002) for the Stein-Stein model in the minimal entropy case. (Laurent and Pham (1999) have some results in the variance-optimal case with correlation. They are able to reformulate the problem in terms of a new stochastic control problem $\overline{\mathcal{D}}$, but they are only able to solve this reformulated problem when the correlation is zero.)

Suppose that the model is Markov, and that the model parameters are functions of the volatility Y_t and time alone. Then the volatility process is an autonomous diffusion, and the dynamics become

$$(6) \qquad \frac{dS_t}{S_t} = Y_t(\lambda(Y_t, t)dt + dB_t) \qquad\qquad dY_t = \alpha(Y_t, t)dt + \beta(Y_t, t)dW_t$$

Theorem 1.4 *Suppose that ρ is constant and let $R = 1 - q\rho^2$.*

Then the process ξ involved in the definition of the q-optimal measure is given by

$$\xi_t \equiv \xi(Y_t, t) = \overline{\rho} f'(Y_t, t)\beta(Y_t, t)$$

where, for $R \neq 0$,

$$(7) \qquad f(y, t) = -\frac{1}{R} \log \hat{\mathbb{E}} \left[\exp\left(-\frac{q}{2} R \int_t^T \lambda(Y_u, u)^2 du \right) \middle| Y_t = y \right],$$

and for $R = 0$,

$$f(y, t) = \hat{\mathbb{E}} \left[\left(\frac{q}{2} \int_t^T \lambda(Y_u, u)^2 du \right) \middle| Y_t = y \right].$$

Here the measure $\hat{\mathbb{P}}$ is such that the dynamics of Y are modified to become

$$(8) \qquad dY_t = (\alpha(Y_t, t) - q\rho\beta(Y_t, t)\lambda(Y_t, t))\, dt + \beta(Y_t, t))d\hat{W}_t$$

for a $\hat{\mathbb{P}}$ Brownian motion $\hat{W}$.

Furthermore the constant c is given by $c = f(Y_0, 0)$.

Remark 1.5 It is clear from Theorem 1.4 that the case $\rho = 0$ is very much a special case (more so than $q = 2$) because when correlation is zero the dynamics of Y used to calculate the function f, and indirectly the Girsanov change of drift on the Brownian motion Z, are the real-world dynamics. In contrast, when the correlation is non-zero, the dynamics in (8) are modified versions of the real-world dynamics in (6).

Remark 1.6 In the special case $q = 2$ and $\rho = 0$, so that $R = 1$, the formula in (7) and its relationship with ξ_t and the variance-optimal measure are given in Biagini et al (2000, Proposition 2.5) and Laurent and Pham (1999, Theorem 6.1). According to Heath et al (2001, p396): 'It would be highly desirable to remove these restrictions and allow in particular nonzero correlation, but ... this problem awaits further research.' Theorems 1.1 and 1.4 combine to give the variance-optimal measure when ρ is nonzero.

In many ways this paper is based on the paper by Rheinländer (2002) on the minimal entropy measure in a stochastic volatility model. In his paper Rheinländer shows how the fundamental representation equation (2), with $q = 1$, can be used to characterise the martingale measure with minimal entropy with respect to the original real world measure. Further, he outlines a method for solving this equation which can be used for the Stein-Stein stochastic volatility model.

This paper extends Rheinländer (2002) in two significant ways. Firstly, we show how for all values of $q > 1$, the equation (2) can be used to characterise the q-optimal measure, with the entropy measure arising when $q = 1$. This highlights the fact that the minimal entropy measure can be thought of as the limit of the q-optimal measures. (The fact that we can characterise the q-optimal measures using similar equations facilitates comparisons between the various measures. For an interesting application of such comparisons see the recent papers by Henderson (2002) and Henderson et al (2003) which compare option prices in a stochastic volatility model under the various q-optimal measures.) Secondly, in the setting of Theorem 1.4, we also provide a more general method for solving the fundamental representation equation (2), which allows us to give the form of the q-optimal measure for a general stochastic volatility model. Broadly speaking, for a diffusion model, there is a solution to (2) if and only if there exists a martingale measure which has a density with respect to $\mathbb{P}$ which has finite q^{th} moment.

The remainder of this paper is structured as follows. The next section describes the model in more detail and introduces the concepts of the variance-optimal, minimal entropy and q-optimal martingale measures. In Section 3 we show how a solution of the representation equation (2) leads to an expression for the q-optimal measure. The subject of Section 4 is the solution of (2) for a general stochastic volatility model. Using the Feynman-Kac formula we give a stochastic representation for this solution. In the final section we discuss the Heston volatility model in detail and derive explicit forms for various quantities of interest. We are then able to check some of the integrability assumptions which form part of the full statements of the main theorems.

2 The model and the q-optimal measure.

We suppose we have a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions and generated by a pair of Brownian motions B and Z . $\mathbb{P}$ is the real world measure and $T \in (0, \infty)$ is a termination date for the economy. In this model we have a single traded asset S although of course generalisations to the multi-dimensional problem are possible and indeed standard in the literature. We however, will concentrate later on specific solutions to one-dimensional problems. By default the interest rate has been set to zero, so we work with discounted price processes. Clearly in the case of deterministic interest rates this assumption is made without any loss of generality, but if interest rates are stochastic then there is some debate as to what is the appropriate functional to minimise, see the discussion in Gouriéroux et al (1998).

The dynamics of the asset price process are given by the pair of stochastic differential equations (1) which we assume to have sufficient regularity properties to guarantee the existence of a pathwise unique non-explosive strong solution. In the case of a diffusion process a Lipschitz condition is sufficient, (Rogers and Williams (1987, Theorems 11.2, 12.1)).

In general the information set available to an agent is $\mathcal{F}^S_t = \sigma(\{S_u; u \le t\})$. However Y_t^2 is clearly measurable with respect to $\mathcal{F}^S_t$. If further Y_t is measurable with respect to $\mathcal{F}^S_t$, for example if Y is necessarily non-negative, then B_t, W_t and Z_t are measurable with respect to $\mathcal{F}^S_t$ and $\mathcal{F}^S_t \equiv \mathcal{F}_t$. Henceforth we will make this assumption, although it rules out some common cases from the literature. For further remarks on this issue, with special reference to the Stein-Stein volatility model, see Rheinländer (2002).

In general stochastic volatility models are incomplete, and there are derivative assets which cannot be replicated via a hedging strategy based on the stock alone. In turn this means that there will exist several probability measures under which the traded asset process becomes a (local) martingale. (The absence of arbitrage typically guarantees the existence of at least one such measure). Our goal now is to follow the analysis in Frey (1997) and to characterise the family of equivalent local martingale measures.

Let V be a local martingale with $V_0 = 1$. Then V_tS_t is a positive local martingale if and only if

$$V_t = \mathcal{E}(-\lambda \cdot B - \psi \cdot Z)_t \equiv \mathcal{E}(-\lambda \cdot B)_t \mathcal{E}(-\psi \cdot Z)_t$$

for some process ψ. Here $\mathcal{E}$ denotes the Doléans exponential and the expression factorises since $[B, Z] = 0$, see Protter (1990, p79). V is itself a non-negative local martingale and hence supermartingale. Hence if $\mathbb{E}(V_T) = 1$ then V is a true martingale.

For such a process V, we identify V_T with the density of a (local) martingale measure $\mathbb{Q}$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = V_T = \exp\left(\int_0^T \left[-\lambda_t dB_t - \frac{1}{2}\lambda_t^2 dt - \psi_t dZ_t - \frac{1}{2}\psi_t^2 dt\right]\right) \tag{9}$$

If both $\int_0^T \lambda_t^2 dt < \infty$ and $\int_0^T \psi_t^2 dt < \infty$, $\mathbb{P}$-almost surely then V_T is strictly positive and if V is a true martingale then $\mathbb{Q}$ is equivalent to $\mathbb{P}$. The space of equivalent martingale measures is parameterised by the process ψ which governs the change of drift of Z.

Under $\mathbb{Q}$, both $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are Brownian motions where

$$dB_t^{\mathbb{Q}} = dB_t + \lambda_t dt \qquad\qquad dZ_t^{\mathbb{Q}} = dZ_t + \psi_t dt$$

and S and Y solve the equations $dS_t = S_t Y_t dB_t^{\mathbb{Q}}$ and

$$dY_t = (\alpha_t - \beta_t\rho_t\lambda_t - \beta_t\overline{\rho}_t\psi_t)dt + \beta_t(\rho_t dB_t^{\mathbb{Q}} + \overline{\rho}_t dZ_t^{\mathbb{Q}}). \tag{10}$$

In particular S is a $\mathbb{Q}$-local martingale, but not in general a martingale. For many volatility models Sin (1996) investigates whether S is indeed a true martingale.

In principle any martingale measure $\mathbb{Q}$ could be used to define arbitrage-free prices for derivatives and provides a possible consistent extension from complete to incomplete markets. One such choice of pricing measure is the minimal martingale measure. Under the minimal martingale measure traded assets become (local) martingales but the drifts of all processes orthogonal to the traded assets are left unchanged. This corresponds to the choice $\psi \equiv 0$, and can be summarised as assuming that non-hedgeable risk is not priced. But other processes ψ are possible and each choice of ψ corresponds to a potential pricing measure.

In order to fix on a particular choice of martingale measure one strategy is to choose $\mathbb{Q}$ to possess some additional optimality property. In particular we can choose $\mathbb{Q}$ so that it is 'as close as possible to $\mathbb{P}$'. Thus, for $q > 1$ we can define[1]

$$H_q(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}[(V_T)^q] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise} \end{cases}$$

and the q-optimal measure is the measure $\mathbb{Q}^{(q)}$ which minimises $H_q(\mathbb{P},\mathbb{Q})$. When $q = 1$ we can define the relative entropy

$$H_1(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}[V_T \ln(V_T)] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ \infty & \text{otherwise} \end{cases}$$

and the minimal entropy measure $\mathbb{Q}^{(1)}$ as the local martingale measure which minimises $H_1(\mathbb{P},\mathbb{Q})$. Of course it may be that there is no martingale measure $\mathbb{Q}$ for which H_q is finite in which case the q-optimal measure is not well defined.

As far as the literature is concerned the two most popular choices are $q = 2$ and $q = 1$. When $q = 2$ we have the variance-optimal martingale measure. Under certain (non-trivial) technical conditions the expected value of a contingent claim under the variance-optimal measure is precisely the 'approximation' price of Schweizer (1996). (In that case we can identify an optimal hedge for a derivative as well as an option price.) Conversely, the

[1] In fact these definitions can be modified to cover all real q, see Henderson et al (2003). Many of our conclusions and results have extensions to general q.

utility indifference price of a contingent claim under exponential utility is related via the dual problem to the solution of a minimal entropy problem, see Delbaen et al (2002), and in the limit as the co-efficient of risk aversion tends to zero, the utility indifference price tends to the expected value under the minimal entropy martingale measure. With these links between martingale measures and pricing in mind, the focus of this article will be on finding the q-optimal measure. The tasks of justifying its use for derivative pricing, of determining an appropriate hedge and of making a choice between the various q-optimal measures will not concern us at all.

3 Sufficiency of the Fundamental Representation Equation

Recall the fundamental representation equation (2):

$$\frac{q}{2}K_T = M_T - \frac{(q-1)}{2}[M]_T + L_T + \frac{1}{2}[L]_T + c \tag{11}$$

Here

$$M_t = \int_0^t \eta_u(dB_u + q\lambda_u du), \qquad L_t = \int_0^t \xi_u dZ_u$$

and $c = c(Y_0)$ is a constant which is measurable at time 0. Note that (11) is an identity of random variables and not of processes. The goal of this section is to show that if we can find a solution to (11) which holds $\mathbb{P}$ almost surely, then we can use that solution to construct the q-optimal measure.

Theorem 3.1 *Fix $q \geq 1$. Suppose that we can find a solution to* (11) *for which K_T, $[M]_T$, $[L]_T$ and c are finite. Define the local martingale $V^{(q)}$ via*

$$V_t^{(q)} = \mathcal{E}\left(-\lambda \cdot B - \xi \cdot Z\right)_t$$

and suppose that $V^{(q)}$ is a true martingale. Define $\mathbb{Q}^{(q)}$ via $(d\mathbb{Q}^{(q)}/d\mathbb{P})|_{\mathcal{F}_T} = V_T^{(q)}$. Then $\mathbb{Q}^{(q)}$ is equivalent to $\mathbb{P}$.

Suppose $q > 1$. Define the local martinagle $\tilde{V}^{(q)}$ via

$$\tilde{V}_t^{(q)} = \mathcal{E}\left(((q-1)\eta - q\lambda) \cdot B - \xi \cdot Z\right)_t$$

and suppose that $\tilde{V}_t^{(q)}$ is a true martingale. Then $H_q(\mathbb{Q}^{(q)}, \mathbb{P}) = \exp((q-1)c)$, and $\mathbb{Q}^{(q)}$ is the q-optimal measure.

Now suppose $q = 1$. If $(\eta - \lambda) \cdot B^{\mathbb{Q}^{(1)}}$ is a $\mathbb{Q}^{(1)}$-martingale and if $(\eta - \lambda) \cdot B^{\mathbb{Q}}$ is a true $\mathbb{Q}$-martingale for every martingale measure $\mathbb{Q}$ with finite entropy with respect to $\mathbb{P}$, then $H_1(\mathbb{Q}^{(1)}, \mathbb{P}) = c$ and $\mathbb{Q}^{(1)}$ is the minimal entropy measure.

Proof:
For $q \geq 1$ let $\mathbb{Q}^{(q)}$ be as given in the theorem. Since K_T and $[L]_T$ are finite, $\mathbb{P}$ almost

surely, we have $V_T^{(q)} > 0$, and since $\mathbb{E}[V_T^{(q)}] = 1$ by assumption, it follows that $\mathbb{Q}^{(q)}$ is a probability measure, equivalent to $\mathbb{P}$. Then

$$
\begin{aligned}
\ln V_T^{(q)} &= \int_0^T \left[-\lambda_t dB_t - \frac{1}{2}\lambda_t^2 dt - \xi_t dZ_t - \frac{1}{2}\xi_t^2 dt\right] \\
&= -\int_0^T \lambda_t dB_t^{\mathbb{Q}^{(q)}} + \frac{1}{2}K_T - \int_0^T \xi_t dZ_t^{\mathbb{Q}^{(q)}} + \frac{1}{2}\int_0^T \xi_t^2 dt \qquad (12)
\end{aligned}
$$

where $B^{\mathbb{Q}^{(q)}}$ and $Z^{\mathbb{Q}^{(q)}}$ are $\mathbb{Q}^{(q)}$-Brownian motions. If we also rewrite the representation equation (11) in terms of $B^{\mathbb{Q}^{(q)}}$ and $Z^{\mathbb{Q}^{(q)}}$ we get

$$
\frac{q}{2}K_T = \int_0^T \eta_t dB_t^{\mathbb{Q}^{(q)}} + (q-1)\int_0^T \left[\eta_t\lambda_t - \frac{1}{2}\eta^2\right] dt + \int_0^T \xi_t dZ_t^{\mathbb{Q}^{(q)}} - \frac{1}{2}\int_0^T \xi_t^2 dt + c.
$$

Note that since $\mathbb{Q}^{(q)}$ is equivalent to $\mathbb{P}$ this equation holds $\mathbb{Q}^{(q)}$-almost surely. Using the trivial decomposition

$$
\frac{1}{2}K_T = \frac{q}{2}K_T - \frac{(q-1)}{2}K_T
$$

we can substitute this expression into (12) to obtain

$$
\ln V_T^{(q)} = c + \int_0^T \left((\eta_t - \lambda_t)dB_t^{\mathbb{Q}^{(q)}} - \frac{(q-1)}{2}(\eta_t - \lambda_t)^2 dt\right). \qquad (13)
$$

Now suppose $q > 1$. Then

$$
\begin{aligned}
H_q(\mathbb{Q}^{(q)}, \mathbb{P}) = \mathbb{E}[(V_T^{(q)})^q] &= \mathbb{E}^{\mathbb{Q}^{(q)}}[(V_T^{(q)})^{q-1}] \\
&= e^{(q-1)c}\mathbb{E}^{\mathbb{Q}^{(q)}}\left[\mathcal{E}\left((q-1)(\eta-\lambda)\cdot B^{\mathbb{Q}^{(q)}}\right)_T\right] \\
&= e^{(q-1)c}\mathbb{E}\left[\mathcal{E}\left(((q-1)\eta - q\lambda)\cdot B - \xi\cdot Z\right)_T\right] \\
&= e^{(q-1)c}
\end{aligned}
$$

where we have used the fact that $\tilde{V}^{(q)}$ is a true martingale.

In particular the density of the martingale measure $\mathbb{Q}^{(q)}$ has a finite q^{th} moment. Hence, (Delbaen and Schachermayer (1996, Theorem 1.3), Grandits and Rheinländer (2002, Proposition 4.3)) the q-optimal measure exists and is equivalent to $\mathbb{P}$. Moreover we can write

$$
V_T^{(q)} = e^c\left\{\mathcal{E}\left((q-1)(\eta-\lambda)\cdot B^{\mathbb{Q}^{(q)}}\right)_T\right\}^{1/(q-1)}
$$

so that by Delbaen and Schachermayer (1996, Lemma 2.2) and Grandits and Rheinländer (2002, Proposition 4.2) $\mathbb{Q}^{(q)}$ is the q-optimal measure.

Now consider the case $q = 1$. Then, from (13),

$$
\begin{aligned}
H_1(\mathbb{Q}^{(1)}, \mathbb{P}) = \mathbb{E}[V_T^{(1)}\ln V_T^{(1)}] &= \mathbb{E}^{\mathbb{Q}^{(1)}}[\ln V_T^{(1)}] \\
&= c + \mathbb{E}^{\mathbb{Q}^{(1)}}\int_0^T (\eta_t - \lambda_t)dB_t^{\mathbb{Q}^{(1)}} \\
&= c.
\end{aligned}
$$

Hence, $\mathbb{Q}^{(1)}$ has finite entropy, and we conclude that the minimal entropy measure exists and is equivalent to $\mathbb{P}$. Further

$$V_T^{(1)} = e^c \exp\left(\int_0^T (\eta - \lambda) dB^{\mathbb{Q}^{(1)}}\right)$$

so that by Frittelli (2000, Theorems 2.2, 2.3) and Grandits and Rheinländer (2002, Theorems 3.1, 3.2), $\mathbb{Q}^{(1)}$ is the martingale measure of minimal entropy. The assumption that $\mathbb{E}^{\mathbb{Q}}[\int_0^T (\eta_t - \lambda_t) dB_t^{\mathbb{Q}}] = 0$ for all martingale measures with finite entropy is used to check the final condition of Theorem 2.3 in Frittelli (2000).

Remark 3.2 Given that we have a solution to (11) we will describe the measure $\mathbb{Q}^{(q)}$ given by $V_T^{(q)} = \mathcal{E}(-\lambda \cdot B - \xi \cdot Z)_T$ as the candidate q-optimal measure. $\mathbb{Q}^{(q)}$ is the true q-optimal measure provided the remaining assumptions of Theorem 3.1 are satisfied.

By construction, for any measure $\mathbb{Q}$ which makes the price process into a martingale we must have $dB_t^{\mathbb{Q}} = dB_t + \lambda_t dt = dS_t/(S_t Y_t)$. As a corollary we can use the representation (13) to write the expression for the q-optimal measure in terms of the asset price process S.

Corollary 3.3 *Assume the conditions of Theorem* 3.1 *are satisfied. The q-optimal measure is of the form*

$$\left.\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = C \exp\left(\int_0^T \chi_t dS_t - \frac{(q-1)}{2} \int_0^T \chi_t^2 d[S]_t\right)$$

Here the constant C equals e^c and the process χ_t is given by $(\eta_t - \lambda_t)/(S_t Y_t)$.

In the proof of Theorem 3.1 we quoted results from Delbaen and Schachermayer (1996), Grandits and Rheinländer (2002) and Frittelli (2000) to show that $\mathbb{Q}^{(q)}$ is indeed the q-optimal martingale measure (or the minimal entropy martingale measure if $q = 1$). However, modulo certain technical assumptions, we can sketch a proof of the optimality of $\mathbb{Q}^{(q)}$ directly. We begin with a simple lemma.

Lemma 3.4 *For any $\alpha > 0$, adapted integrand θ_t, measure $\mathbb{Q}$, and $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$,*

$$\mathbb{E}^{\mathbb{Q}}\left[\left\{\mathcal{E}\left(\theta \cdot W^{\mathbb{Q}}\right)_T\right\}^{-\alpha}\right] \geq 1$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \theta_t dW_t^{\mathbb{Q}} + \frac{1}{2}\int_0^T \theta_t^2 dt\right] \geq 0.$$

Proof:
Note that if θ has suitable integrability properties then the result is immediately true. More generally, let $N_t = \mathcal{E}(-\theta \cdot W^{\mathbb{Q}})_t$. Then N_t is a non-negative local $\mathbb{Q}$-martingale, and hence a supermartingale for which $\mathbb{E}^{\mathbb{Q}}(N_T) \leq 1$.

Now take $f : \mathbb{R}_+ \to \mathbb{R}$ decreasing and concave. By Jensen's inequality and the monotonicity we have

$$\mathbb{E}^{\mathbb{Q}}[f(N_T)] \geq f(\mathbb{E}^{\mathbb{Q}}[N_T]) \geq f(1).$$

The results follow if we take $f(x) = x^{-\alpha}$ and $f(x) = -\ln x$.

Let $\mathbb{Q}$ be any equivalent local martingale measure with Radon-Nikodym derivative V_T with respect $\mathbb{P}$ where $V_t = \mathcal{E}(-\lambda \cdot B - \psi \cdot Z)_t$. Then, as in (12)

$$\begin{aligned} \ln V_T &= \int_0^T \left[-\lambda_t dB_t - \frac{1}{2}\lambda_t^2 dt - \psi_t dZ_t - \frac{1}{2}\psi_t^2 dt \right] \\ &= -\int_0^T \lambda_t dB_t^{\mathbb{Q}} + \frac{1}{2}K_T - \int_0^T \psi_t dZ_t^{\mathbb{Q}} + \frac{1}{2}\int_0^T \psi_t^2 dt \end{aligned} \tag{14}$$

where $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are $\mathbb{Q}$-Brownian motions. Rewriting (11) in terms of $B^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$, and decomposing $K_T/2$ into two parts as before we obtain

$$\begin{aligned} \ln V_T &= c + \int_0^T \left((\eta_t - \lambda_t) dB_t^{\mathbb{Q}} - \frac{(q-1)}{2}(\eta_t - \lambda_t)^2 dt \right) \\ &\quad + \int_0^T \left((\xi_t - \psi_t) dZ_t^{\mathbb{Q}} + \frac{1}{2}(\xi_t - \psi_t)^2 dt \right). \end{aligned} \tag{15}$$

Suppose $q > 1$. Then

$$\begin{aligned} H_q(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}^{\mathbb{Q}}[(V_T)^{q-1}] \\ &= e^{(q-1)c}\mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}\left((q-1)(\eta - \lambda) \cdot B^{\mathbb{Q}} \right)_T \left\{ \mathcal{E}\left(-(\xi - \psi) \cdot Z^{\mathbb{Q}} \right)_T \right\}^{-(q-1)} \right] \end{aligned}$$

If $\mathbb{Q}$ is a martingale measure, then we can think about $\mathcal{E}(((q-1)(\eta - \lambda)) \cdot B^{\mathbb{Q}})_T$ as a change of measure acting on $B^{\mathbb{Q}}$ alone. In particular $Z^{\mathbb{Q}}$ is still a Brownian motion after this change of measure, and from Lemma 3.4 we deduce $H_q(\mathbb{Q}, \mathbb{P}) \geq e^{(q-1)c} = H_q(\mathbb{Q}^{(q)}, \mathbb{P})$.

Conversely, if $q = 1$, then for any measure with finite entropy with respect to $\mathbb{P}$, (15) yields

$$\begin{aligned} H_1(\mathbb{Q}, \mathbb{P}) &= \mathbb{E}^{\mathbb{Q}}[\ln V_T] \\ &= c + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\eta_t - \lambda_t) dB_t^{\mathbb{Q}} \right] + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\xi_t - \psi_t) dZ_t^{\mathbb{Q}} + \frac{1}{2}\int_0^T (\xi_t - \psi_t)^2 dt \right] \end{aligned}$$

By assumption the middle term in this expression is zero, and the final term is non-negative and equal to zero when $\xi = \psi$. Hence, for any martingale measure $\mathbb{Q}$, $H_1(\mathbb{Q}, \mathbb{P}) \geq c = H_1(\mathbb{Q}^{(1)}, \mathbb{P})$, and we have demonstrated the optimality of our candidate measure.

The key conclusion from Theorem 3.1 is that the fundamental representation equation can be used to define a candidate q-optimal measure, and that measure is indeed optimal if certain regularity conditions are met. In the remainder of this section we discuss a set of sufficient conditions under which the regularity conditions can be seen to hold.

When $q = 1$ we need a result that guarantees that a family of local martingales are simultaneously true martingales. The following result adapted from Delbaen et al (2002) and Rheinländer (2002) provides a sufficient condition.

Proposition 3.5 *Suppose $\int_0^T(\eta_s-\lambda_s)^2ds$ has an exponential moment under $\mathbb{P}$. Let $\mathbb{Q}$ be any equivalent martingale measure of the form (9) with finite entropy with respect to $\mathbb{P}$, and let $B^{\mathbb{Q}}$ be a $\mathbb{Q}$-Brownian motion. Then $(\eta-\lambda)\cdot B^{\mathbb{Q}}$ is a $\mathbb{Q}$-martingale.*

Proof: For $x>0$ we have $\ln x \le x-1$. Hence, for a positive random variable X and a change of measure V_T

$$E^{\mathbb{Q}}[\ln X] = \mathbb{E}[V_T \ln X] \le \mathbb{E}[V_T(X-1)] = \mathbb{E}[V_T X]-1.$$

Now, for $\alpha>0$ let $X=e^{\alpha U}/V_T$. Then

$$\alpha\mathbb{E}^{\mathbb{Q}}[U] - \mathbb{E}^{\mathbb{Q}}[\ln V_T] \le \mathbb{E}[e^{\alpha U}]-1.$$

In particular, if $\mathbb{E}[e^{\alpha U}]<\infty$, then $\mathbb{E}^{\mathbb{Q}}[U]<\infty$ for any $\mathbb{Q}$ with finite entropy with respect to $\mathbb{P}$.

Let $\mathbb{Q}$ be a measure with finite relative entropy. Clearly $(\eta-\lambda)\cdot B^{\mathbb{Q}}$ is a local $\mathbb{Q}$-martingale. Further, if we set $U=\int_0^T(\eta_s-\lambda_s)^2ds$ then by assumption for some suitable choice of α we have $\mathbb{E}[e^{\alpha U}]<\infty$, and then we deduce that $\mathbb{E}^{\mathbb{Q}}[U]<\infty$. Hence $(\eta-\lambda)\cdot B^{\mathbb{Q}}$ is a true martingale.

Now we discuss conditions under which the local martingales $V^{(q)}$ and, when $q>1$, $\tilde{V}^{(q)}$ are true martingales. The following Proposition is adapted from Hobson and Rogers (1998).

Suppose Y_t is an autonomous diffusion. In particular, to be consistent with (6) and our correlated stochastic volatility model, suppose Y_t solves $Y_0=y$ and

$$dY_t = \alpha(Y_t,t)dt + \beta(Y_t,t)(\rho(Y_t,t)dB_t + \overline{\rho}(Y_t,t)dZ_t),$$

where B and Z are independent Brownian motions. Let θ and ϕ be Markov functions of Y_t, which are continuous where finite, and are such that $\int_0^T[\theta(Y_t,t)^2+\phi(Y_t,t)^2]dt$ is finite, $\mathbb{P}$ almost surely. Let V solve $V_0=1$ and

$$\frac{dV_t}{V_t} = \theta(Y_t,t)dB_t + \phi(Y_t,t)dZ_t \tag{16}$$

so that $V_t=\mathcal{E}(\theta\cdot B+\phi\cdot Z)_t$. Now let $\tilde{Y}_t$ solve $\tilde{Y}_0=y$ and

$$\begin{aligned} d\tilde{Y}_t &= \{\alpha(\tilde{Y}_t,t)+\rho(\tilde{Y}_t,t)\beta(\tilde{Y}_t,t)\theta(\tilde{Y}_t,t)+\overline{\rho}(\tilde{Y}_t,t)\beta(\tilde{Y}_t,t)\phi(\tilde{Y}_t,t)\}dt \\ &\qquad +\beta(\tilde{Y}_t,t)(\rho(\tilde{Y}_t,t)dB_t+\overline{\rho}(\tilde{Y}_t,t)dZ_t). \end{aligned}$$

Suppose that the co-efficients of this stochastic differential equation are such that it has a pathwise unique strong solution.

Proposition 3.6 *Define $\mathcal{A}_t \equiv \mathcal{A}_t^{\theta,\phi} = \{y : |\theta(y,t)|+|\phi(y,t)|=\infty\}$. Suppose that $\mathbb{P}(\tilde{Y}_t\in\mathcal{A}_t,$ for some $t)=0$. Then V is a true martingale.*

Proof.
Since V given by (16) is a non-negative local martingale it is sufficient to prove that $\mathbb{E}[V_T]=1$. Let $\tau_n = \inf\{t : \int_0^t [\theta(Y_u,u)^2 + \phi(Y_u,u)^2]du \geq n\} \wedge T$, and define $\tilde{\tau}_n = \inf\{t : \int_0^t [\theta(\tilde{Y}_u,u)^2 + \phi(\tilde{Y}_u,u)^2]du \geq n\} \wedge T$. Let $\mathbb{Q}^{<n>}$ be the measure with density $V^{<n>}$ with respect to $\mathbb{P}$ given by $V_t^{<n>} = V_{t\wedge\tau_n}$. For $t \leq \tau_n$, the law of Y under $\mathbb{Q}^{<n>}$ is the same as the law of $\tilde{Y}$ under $\mathbb{P}$, and moreover,

$$\mathbb{Q}^{<n>}[\tau_n < T] = \mathbb{P}[\tilde{\tau}_n < T],$$

and this term tends to zero by assumption.

By the Novikov criterion, $V^{<n>}$ is a true martingale. It follows that

$$\begin{aligned} 1 = \mathbb{E}[V_T^{<n>}] &= \mathbb{E}[V_T; \tau_n = T] + \mathbb{E}[V_T^{<n>}; \tau_n < T] \\ &= \mathbb{E}[V_T; \tau_n = T] + \mathbb{Q}^{<n>}[\tau_n < T] \end{aligned}$$

Hence $1 \geq \mathbb{E}[V_T] \geq \mathbb{E}[V_T; \tau_n = T] \to 1$ and V is a true martingale.

In order to apply the proposition and to decide whether $V_t^{(q)}$ is a martingale, it is sufficient to consider the diffusion whose dynamics are given by (10) with $\psi_t = \xi_t^{(q)}$, and to determine whether this diffusion Y reaches the set $\mathcal{A}_t^{\lambda,\xi^{(q)}}$.

4 An autonomous diffusion model for volatility

In the previous section we saw how the representation equation (2) is key to determining the q-optimal measure. The purpose of this section is to show how to solve (2), beginning with a couple of special cases before moving on to the Markovian volatility models which are the main subject of this paper.

If $\lambda_t = \lambda(t)$ then the left-hand-side of (2) is deterministic and there is a simple solution involving the choice $\eta \equiv \xi = 0$ and

$$c = \frac{q}{2}\int_0^T \lambda(t)^2 dt.$$

In this case the q-optimal measure, the minimal entropy measure and the minimal martingale measure (amongst others) all coincide.

The second special case is the set of *almost complete* models of Pham et al (1998) and Laurent and Pham (1999). Suppose λ_t is adapted to the filtration generated by B alone. Then for each $q \geq 1$ there is a solution to (2) with $\xi \equiv 0$, and hence the candidate q-optimal measure is the same for each q, and equal to the minimal martingale measure. To see that there is such a solution consider first the case $q=1$. If we take $\xi \equiv 0$ then (2) becomes

$$\frac{1}{2}K_T = \int_0^T \eta_t dB_t^{\mathbb{P}^{(1)}} + c$$

where $dB_t^{\mathbb{P}^{(1)}} = dB_t + \lambda_t dt$. But under suitable integrability conditions this equation has a solution by the Martingale Representation Theorem for the Brownian motion $B^{\mathbb{P}^{(1)}}$. In particular $c = \mathbb{E}^{\mathbb{P}^{(1)}}[K_T/2]$. Now suppose $q > 1$, and let $\mathbb{P}^{(q)}$ be a measure under which $B^{\mathbb{P}^{(q)}}$ given by $dB_t^{\mathbb{P}^{(q)}} = dB_t + q\lambda_t dt$ is a Brownian motion. The equation we want to solve is

$$\exp\left(\frac{q(q-1)}{2}K_T\right) = e^{(q-1)c}\mathcal{E}\left((q-1)\eta \cdot B^{\mathbb{P}^{(q)}}\right)_T$$

Again, provided $\mathbb{E}^{\mathbb{P}^{(q)}}[\exp(q(q-1)K_T/2)] < \infty$ the Martingale Representation Theorem guarantees that we have a solution.

Now we consider the case where the mean-variance trade-off process K_t depends on the volatility process Y. In order to solve (2) it is necessary to make some simplifying assumptions. In particular we assume that the stochastic dynamics are Markov, and moreover that the parameters of the model depend on Y alone and not on the asset price S. In this case the model becomes

$$\frac{dS_t}{S_t} = Y_t(\lambda(Y_t,t)dt + dB_t), \qquad\qquad dY_t = \alpha(Y_t,t)dt + \beta(Y_t,t)dW_t. \tag{17}$$

where B and W are correlated Brownian motions with correlation co-efficient $\rho(Y_t,t)$. We assume $|\rho_t| < 1$. The volatility Y is now an autonomous diffusion.

Suppose temporarily that the Brownian motions B and W are uncorrelated. Then the dynamics for Y depend on the Brownian motion $Z = W$ alone, and it follows that we can take $M_t \equiv 0$ and (2) simplifies to

$$\frac{q}{2}K_T = L_T + \frac{1}{2}[L]_T + c. \tag{18}$$

It is easy to identify the constant c in this case also. Multiplying by -1, exponentiating and taking expectations yields

$$c = -\ln \mathbb{E}\left[\exp\left(-\frac{q}{2}K_T\right)\right] \tag{19}$$

Further we can identify ξ by representing $\exp(-qK_T/2)$ as the constant c plus a stochastic integral with respect to the Brownian motion Z. In the case of variance-optimal measures this analysis is carried out in Biagini et al (2000). Our goals are to deduce the form of ξ and η in the general case (with correlation) and to show that (19) is a special case of a more general formula.

Rearranging (2) we get

$$\frac{q}{2}\int_0^T \lambda_t^2 dt = \int_0^T \eta_t dB_t + \int_0^T \xi_t dZ_t + \int_0^T \left[q\eta_t\lambda_t - \frac{(q-1)}{2}\eta_t^2 + \frac{1}{2}\xi_t^2\right] dt + c \tag{20}$$

Y is an autonomous diffusion and, by assumption, the left-hand-side of this expression is measurable with respect to the filtration generated by W alone. Conversely, on the right-hand-side there are stochastic integrals with respect to both B and Z. Our best hope of finding a solution is to write the sum of the integrals against B and Z as a single integral against W, ie to choose $\eta = \rho\theta$ and $\xi = \overline{\rho}\theta$ for some process θ_t. The equation becomes

$$\frac{q}{2}\int_0^T \lambda_t^2 dt = \int_0^T \theta_t(\rho_t dB_t + \overline{\rho}_t dZ_t) + \int_0^T \left(q\rho_t\theta_t\lambda_t - \frac{(q-1)}{2}\rho_t^2\theta_t^2 + \frac{1}{2}\overline{\rho}^2\theta^2\right) dt + c$$

and finally, if we emphasise the dependence of certain parameters on Y_t,

$$\frac{1}{2}\int_0^T \left[q\lambda(Y_t,t)^2 - 2q\rho(Y_t,t)\theta_t\lambda(Y_t,t) + (q-1)\rho(Y_t,t)^2\theta_t^2 - \overline{\rho}(Y_t,t)^2\theta_t^2\right] dt = \int_0^T \theta_t dW_t + c. \tag{21}$$

We expect this equation to have a solution of the form $\theta_t = \theta(Y_t,t)$.

Consider a function $f = f(y,t) \in C^{2,1}$ such that $f(y,T) = 0$. Then, by Itô's formula,

$$\begin{aligned}
-f(Y_0,0) &= \int_0^T df(Y_t,t) \\
&= \int_0^T f'\beta(Y_t,t)dW_t + \int_0^T f'\alpha(Y_t,t)dt + \int_0^T \frac{1}{2}f''\beta(Y_t,t)^2 dt + \int_0^T \dot{f}dt
\end{aligned}$$

Rearranging and adding c to both sides:

$$\begin{aligned}
&\int_0^T f'(Y_t,t)\beta(Y_t,t)dW_t + c = \\
&\qquad c - f(0,Y_0) - \int_0^T \left(f'(Y_t,t)\alpha(Y_t,t) + \frac{1}{2}f''(Y_t,t)\beta(Y_t,t)^2 dt + \dot{f}(Y_t,t)\right) dt
\end{aligned}$$

If we choose $\theta_t = f'(Y_t,t)\beta(Y_t,t)$ then we have an equation of the form (21) provided

$$\begin{aligned}
&\frac{1}{2}\int_0^T dt \left(q\lambda_t^2 - 2q\rho_t\lambda_t f'\beta_t - (\overline{\rho}_t^2 - (q-1)\rho_t^2)f'^2\beta_t^2\right) \\
&= c - f(Y_0,0) - \int_0^T \left(f'\alpha_t + \frac{1}{2}f''\beta_t^2 + \dot{f}\right) dt.
\end{aligned}$$

This holds if

$$\begin{aligned}
&c(Y_0) - f(Y_0,0) = \\
&\qquad \int_0^T \left[\frac{q}{2}\lambda_t^2 - q\rho_t f'\beta_t\lambda_t - \frac{1}{2}R_t(q,\rho)f'^2\beta_t^2 + f'\alpha_t + \frac{1}{2}f''\beta_t^2 + \dot{f}\right] dt
\end{aligned} \tag{22}$$

where

$$R_t = R(q,\rho_t) = \overline{\rho}_t^2 - (q-1)\rho_t^2 = (1 - q\rho_t^2).$$

The left-hand-side of (22) is independent of the process Y, (given the starting value Y_0) so if we can choose f so that the term in square brackets depends only on t then we are done.

Hence we have a solution provided we can solve

$$\frac{q}{2}\lambda_t^2 - q\rho_t f'\beta_t\lambda_t - \frac{1}{2}R_t f'^2\beta_t^2 + f'\alpha_t + \frac{1}{2}f''\beta_t^2 + \dot{f} = H(t) \tag{23}$$

for some function H of time alone, and then $\theta_t \equiv f'(Y_t,t)\beta(Y_t,t)$. Moreover, since f is only determined up to the addition of a function of time we may and shall take $H \equiv 0$. In terms of our original variables this means that

$$c \equiv c(y_0) = f(y_0,0) \qquad \eta_t = \rho f'(Y_t,t)\beta(Y_t,t) \qquad \xi_t = \overline{\rho} f'(Y_t,t)\beta(Y_t,t). \tag{24}$$

Note that the change of drift parameters induced by the switch to the q-optimal measure $\mathbb{Q}^{(q)}$ are Markov functions of the autonomous volatility coefficient Y_t.

Remark 4.1 (i). Recall that in the definition of (2) we assumed that the constant c was finite. This is equivalent to the existence of a finite solution f to (23) with $H \equiv 0$.
(ii). Note that if $\rho = 0$, we find that $\eta \equiv 0$. In particular (2) simplifies to (18) in this case.

In order to complete the process of solving (2) and hence to construct the q-optimal measure it remains to solve the non-linear partial differential equation (23). This non-linearity is controlled by the parameter R_t. Henceforth we assume that the correlation is constant in both time and space so that $\rho_t = \rho$ with $|\rho| < 1$. Further, in order to ensure the existence of a unique solution with a stochastic representation, we need to assume that the co-efficients satisfy suitable regularity conditions. Sufficient conditions are that λ, α and β are smooth bounded functions and β is bounded below. See Karatzas and Shreve (1988, Remark 5.7.8) or the remarks before Theorem 6.1 in Laurent and Pham (1999) for a discussion of these conditions and how they may be relaxed.

We first solve (23) (with $H = 0$) when $R = 0$. In this special case (23) is linear and by the Feynman-Kac formula (Karatzas and Shreve (1988, Theorem 5.7.6)) the solution f takes the form

$$f(y,t) = \hat{\mathbb{E}}\left[\left(\frac{q}{2}\int_t^T \lambda(Y_u,u)^2 du\right)\middle| Y_t = y\right],$$

where under $\hat{\mathbb{E}}$ the volatility process Y has modified dynamics

$$dY_t = (\alpha(Y_t,t) - q\rho\lambda(Y_t,t)\beta(Y_t,t))\,dt + \beta(Y_t,t)d\hat{W} \tag{25}$$

for a $\hat{\mathbb{P}}$-Brownian motion $\hat{W}$. Note that f is necessarily positive, but not necessarily finite.

Now we consider the solution of (23) when $R \neq 0$. Consider the transformation $g = e^{-Rf}$, so that $g(y,T) = 1$. Then (23) becomes

$$\frac{q}{2}R(q,\rho)\lambda(y,t)^2 g = -q\rho\beta(y,t)\lambda(y,t)g' + \alpha(y,t)g' + \frac{1}{2}g''\beta(y,t)^2 + \dot{g}(y,t). \tag{26}$$

By the Feynman-Kac formula this has solution

$$g(t,y) = \hat{\mathbb{E}}\left[\exp\left(-\frac{q}{2}R(q,\rho)\int_t^T \lambda(Y_u,u)^2 du\right)\middle| Y_t = y\right]. \tag{27}$$

Here the dynamics of Y are as in (25). It follows that

$$f(t,y) = -\frac{1}{R}\log\hat{\mathbb{E}}\left[\exp\left(-\frac{q}{2}R(q,\rho)\int_t^T \lambda(Y_u,u)^2 du\right)\middle| Y_t = y\right].$$

Remark 4.2 (i). Note that if $\rho = 0$ then the measure $\hat{\mathbb{P}}$ introduced above is just the original measure $\mathbb{P}$. Hence f takes a particularly transparent form in this case. Conversely if $q = 1$ then the dynamics in (25) are the stochastic dynamics of Y_t under the minimal martingale measure.
(ii). If $R > 0$ then $0 < g \le 1$ and f is positive and finite. Conversely, if $R < 0$ then certainly $g > 1$, and depending on the dynamics of Y under $\hat{\mathbb{P}}$ we may have that g explodes in finite time. Combining this observation with the results in the $R = 0$ case we find that if $R \le 0$ then $0 < f \le \infty$ and additional conditions are necessary to ensure that c is finite or that the change of drift parameter ψ exists.
(iii). The condition $R > 0$ is equivalent to $\rho^2 < 1/q$. For example, if $\rho = 0$ or $q = 1$ then we have $R > 0$ and the candidate q-optimal measure is defined over all time-horizons. Further, for any given model we can find $q > 1$ such that $R > 0$ and the candidate q-optimal measure is defined for all time.

5 An Example: The Heston Stochastic Volatility Model

In this section we investigate the form of the q-optimal martingale measure for a stochastic volatility model of type proposed by Heston (1993). In particular we find the solution of the fundamental representation equation. This allows us to write down the candidate q-optimal measure, and the modified dynamics of the volatility process under this measure. Finally we check that we do indeed have the q-optimal and minimal entropy measures.

In the Heston model the squared volatility Σ_t is given by a squared Bessel process:

$$\frac{dS_t}{S_t} = \mu_0(\Sigma_t)dt + \sqrt{\Sigma_t}dB, \qquad d\Sigma_t = 2\kappa(m - \Sigma_t)dt + 2\beta_0\sqrt{\Sigma_t}dW. \tag{28}$$

where the Brownian motions B and W are imperfectly correlated with constant correlation coefficient ρ. We will consider a particular version of this model where $\mu_0(\Sigma_t) = \mu\Sigma_t$. If we write $\Sigma = Y^2$ and $\overline{m} = m - \beta_0^2/2\kappa$ then the dynamics become

$$\frac{dS_t}{S_t} = \mu Y_t^2 dt + Y_t dB, \qquad dY_t = \kappa\left(\frac{\overline{m}}{Y_t} - Y_t\right)dt + \beta_0 dW. \tag{29}$$

We assume that β_0 and $\overline{m}$ are strictly positive parameters and that κ is non-negative. The condition on $\overline{m}$ guarantees that a strictly positive solution for Σ exists in (28). In terms of our original formulation this model is equivalent to the choice $\beta(y,t) = \beta_0$, $\alpha(y,t) = \kappa((\overline{m}/y) - y)$ and $\lambda(y,t) = \mu y$.

Note that we have specified the dynamics under the real world probability measure $\mathbb{P}$, and in particular we have specified the drift of the traded asset under $\mathbb{P}$. Often authors only specify the market price of volatility risk (or equivalently the change of drift ψ on Z), which is enough to determine the pricing measure $\mathbb{Q}$ since we know that the traded asset is a (local) martingale under $\mathbb{Q}$. We must specify the complete $\mathbb{P}$-dynamics since we want to choose the martingale measure closest to $\mathbb{P}$ in the appropriate metric.

We begin by finding the solution to (2). The equation we have to solve is, see (23),

$$q\mu^2y^2 - 2q\rho\beta_0\mu y f' - R(q,\rho)\beta_0^2 f'^2 + 2\kappa\left(\frac{\overline{m}}{y} - y\right) f' + \beta_0^2 f'' + 2\dot{f} = 0.$$

We look for a solution of the form $f(y,t) = y^2F(T-t)/2 + G(T-t)$ with $F(0) = 0 = G(0)$. Then

$$\left[q\mu^2 - 2q\rho\beta_0\mu F - R(q,\rho)\beta_0^2F^2 - 2\kappa F - \dot{F}\right] y^2 + 2\kappa\overline{m}F + \beta_0^2 F - 2\dot{G} = 0. \tag{30}$$

Equating co-efficients of y we find

$$\dot{G} = \left(\kappa\overline{m} + \frac{1}{2}\beta_0^2\right) F = \kappa m F$$

where F solves

$$\dot{F} = q\mu^2 - 2(\kappa + q\rho\beta_0\mu)F - R(q,\rho)\beta_0^2F^2.$$

Proposition 5.1 *Let the constants A, B, D be given by*

$$A^2 = |R(q,\rho)|\beta_0^2 \qquad B = \frac{\kappa + q\rho\beta_0\mu}{\beta_0^2|R(q,\rho)|} \qquad D = q\mu^2 + \frac{(\kappa + q\rho\beta_0\mu)^2}{\beta_0^2 R(q,\rho)} \tag{31}$$

and let $C = +\sqrt{D}$ if $D > 0$ and $C_- = +\sqrt{-D}$ if $D < 0$.

For the Heston model the function F used in the construction of the q-optimal measure (observe that $\theta(y,t) = yF(T-t)$) is given by:
Case 1: $R(q,\rho) = 0$; if $\kappa + q\rho\beta_0\mu_2 = 0$ then $F(\tau) = q\mu^2\tau$, otherwise

$$F(\tau) = \frac{q\mu^2}{2(\kappa + q\rho\beta_0\mu)}\left(1 - \exp\left(-2(\kappa + q\rho\beta_0\mu_2)\tau\right)\right).$$

Case 2: $R(q,\rho) > 0$;

$$F(\tau) = \frac{C}{A}\tanh\left(AC\tau + \tanh^{-1}\left(\frac{AB}{C}\right)\right) - B. \tag{32}$$

Case 3: $R(q,\rho) < 0$; if $D > 0$ we have

$$F(\tau) = \frac{C}{A}\tan\left(AC\tau - \tan^{-1}\left(\frac{AB}{C}\right)\right) + B. \tag{33}$$

Conversely, if $D < 0$, we have

$$F(\tau) = \frac{(A^2B^2 - C_-^2)\left(e^{2AC_-\tau} - 1\right)}{2AC_- + A(AB + C_-)\left(e^{2AC_-\tau} - 1\right)} \tag{34}$$

Finally, if $D = 0$ then

$$F(\tau) = \frac{B^2A^2\tau}{1 + BA^2\tau}.$$

Proof:
If $R(q,\rho)=0$ then F solves

$$\dot{F} = q\mu^2 - 2(\kappa + q\rho\beta_0\mu)F$$

and the result follows easily.

If $R(q,\rho)>0$ then it follows that F solves

$$\dot{F} = D - [A(F+B)]^2$$

which has the solution given in (32). Note that $A^2B^2 < C^2 = D$ in this case. Conversely if $R(q,\rho)<0$ then F solves

$$\dot{F} = [A(F-B)]^2 + D$$

which has the solutions given, depending on the sign of D. Note that if $D<0$ then $A^2B^2 > C_-^2$.

Remark 5.2 (i). It is interesting to consider when the non-negative increasing function F has a solution which is defined for all time, and when the solution explodes. From Proposition 5.1 we find that F has a solution which explodes to infinity in finite time if and only if $R(q,\rho)<0$ and either $D>0$ or $B<0$.
(ii). Recall that $\dot{G} = \kappa m F$. Since κ, m and F are all non-negative, so is G. Further G is finite on $[0,\tau)$ if and only if F is finite over that same interval, and if F and G explode at all then they explode at the same instant. It follows that if F is bounded for all time then $c = G(T) + F(T)Y_0^2/2$ is finite. However, if F explodes in finite time, then so does c and there is no martingale measure with density in $L^q(\mathbb{P})$. Given the first remark above a simple sufficient condition for the existence of a candidate martingale measure with finite q^{th} moment is $R \geq 0$.
(iii). The change of drift of Z under the q-optimal measure is governed by

$$\xi_t^{(q)} = \overline{\rho}\beta_0 F(T-t)Y_t$$

and the q-optimal measure $\mathbb{Q}^{(q)}$, if it exists, is of the form

$$\frac{d\mathbb{Q}^{(q)}}{d\mathbb{P}} = \mathcal{E}\left(-\mu Y\cdot B - \xi^{(q)}\cdot Z\right)_T.$$

Here F depends on q. The dynamics of Y under $\mathbb{Q}^{(q)}$ are

$$dY_t = \kappa\left(\frac{\overline{m}}{Y_t} - Y_t\right)dt - \rho\beta_0\mu Y_t dt - \overline{\rho}^2\beta_0^2 F(T-t)Y_t dt + \beta_0(\rho dB^{\mathbb{Q}^{(q)}} + \overline{\rho} dZ^{\mathbb{Q}^{(q)}})$$

where $B^{\mathbb{Q}^{(q)}}$ and $Z^{\mathbb{Q}^{(q)}}$ are independent $\mathbb{Q}^{(q)}$-Brownian motions. Note that although the real-world dynamics of the model under $\mathbb{P}$ are time-homogeneous, the dynamics under the q-optimal measure are time-inhomogeneous.
(iv). Note that the function F does not depend on the value of the parameter m which

determines the mean reversion level. As a corollary the form of q-optimal measure is also independent of the value of m, although since the dynamics of Y do depend on m, the constant c and equivalently the q^{th} moment of the density of the q-optimal measure do depend on the value of m.

(v). The constants A, B, C which implicitly depend on q are continuous in q and we deduce that F which also implicitly depends on q converges uniformly over finite time horizons as $q \downarrow 1$ to the function F_1 from the minimal entropy problem.

(vi). Henderson (2002) (see also Henderson et al (2003)) investigates the dependence of the q-optimal measure on q to deduce comparison theorems for option prices under different martingale measures.

Example 5.3 (Minimal Entropy) Suppose $q = 1$. Then $R > 0$ and the candidate q-optimal measure is well defined over all time horizons. In this case the function F takes the form

$$F(\tau) = \frac{\Delta}{\beta_0^2 \overline{\rho}^2} \tanh\left(\Delta\tau + \tanh^{-1}\left(\frac{(\kappa + \rho\beta_0\mu)}{\Delta}\right)\right) - \frac{(\kappa + \rho\beta_0\mu)}{\beta_0^2\overline{\rho}^2}$$

where $\Delta^2 = \kappa^2 + 2\rho\kappa\beta_0\mu + \beta_0^2\mu^2$. Then

$$\frac{d\mathbb{Q}^{(1)}}{d\mathbb{P}} = \mathcal{E}\left(-\mu Y \cdot B - \xi^{(1)} \cdot Z\right) \tag{35}$$

where $\xi_t^{(1)} = \overline{\rho}\beta_0 F(T-t) Y_t$.

Example 5.4 (Zero Correlation) Suppose $\rho = 0$. Then $R = 1$ and the function F is given by

$$F(\tau) = \frac{\Gamma}{\beta_0^2} \tanh\left(\Gamma\tau + \tanh^{-1}\left(\frac{\kappa}{\Gamma}\right)\right) - \frac{\kappa}{\beta_0^2}.$$

where $\Gamma^2 = (\kappa^2 + q\mu^2\beta_0^2)$. When $q = 2$ and we are interested in the variance-optimal measure then up to a change of labels we recover the expression in Heath et al (2001). In particular if we write parameters from Heath et al (2001) with a check-mark then $2\beta_0 \equiv \hat{\Sigma}$, $2\kappa = \hat{\kappa}$, $2\Gamma = \hat{\Gamma}$, $\mu = \hat{\gamma}$, $F = 2\hat{\gamma}^2\hat{\alpha}$.

Example 5.5 (Stein and Stein) If we take $\overline{m} = 0$ then the model is a variant of the Stein and Stein (1991) model. If we take $q = 1$ and consider the minimal entropy measure then we can recover the result of Rheinländer (2002) who shows that $\xi(y,t) = y(c_1 \tanh(c_2(T-t) + c_3) + c_4)$ but does not identify the constants.

To complete the analysis of the Heston model it remains to check that the conditions of Theorem 3.1 are satisfied. We have $\lambda_t = \mu Y_t$, $\eta_t = \rho\beta_0 F(T-t) Y_t$ and $\xi_t = \overline{\rho}\beta_0 F(T-t)Y_t$.

Consider the processes Y_t, $\hat{Y}_t$ and $\tilde{Y}_t$, each with initial value y and dynamics

$$dY_t \;=\; \beta_0(\rho dB + \overline{\rho} dZ) + \kappa\left(\frac{\overline{m}}{Y_t} - Y_t\right) dt$$

$$
\begin{aligned}
d\hat{Y}_t &= \beta_0(\rho dB + \overline{\rho} dZ) + \left(\kappa\left(\frac{\overline{m}}{\hat{Y}_t} - \hat{Y}_t\right) - \beta_0\rho\mu\hat{Y}_t - \overline{\rho}^2\beta_0^2 F(T-t)\hat{Y}_t\right) dt \\
d\tilde{Y}_t &= \beta_0(\rho dB + \overline{\rho} dZ) + \left(\kappa\left(\frac{\overline{m}}{\tilde{Y}_t} - \tilde{Y}_t\right) - q\beta_0\rho\mu\tilde{Y}_t - R(q,\rho)\beta_0^2 F(T-t)\tilde{Y}_t\right) dt
\end{aligned}
$$

Suppose T is such that $F(T) < \infty$. Then there exists a solution to (2) with $c < \infty$. Further, it is clear from the above dynamics that each of Y_t, $\hat{Y}_t$ and $\tilde{Y}_t$ is non-explosive to either 0 or ∞. It follows from Proposition 3.6 that $V^{(q)}$ and $\tilde{V}^{(q)}$ are true martingales and hence that for $q > 1$, the candidate q-optimal measure is the q-optimal measure.

For the minimal entropy case it remains to check that $(\eta - \lambda) \cdot B^{\mathbb{Q}}$ is a true $\mathbb{Q}$ martingale both for $\mathbb{Q} \equiv \mathbb{Q}^{(1)}$ and more generally for each martingale measure $\mathbb{Q}$ with finite entropy with respect to $\mathbb{P}$. Note first that

$$
\int_0^T (\eta_t - \lambda_t)^2 dt = \int_0^T (\rho\beta_0 F(T-t) - \mu)^2 Y_t^2 dt \leq K \int_0^T Y_t^2 dt
$$

for some finite constant K. To show that $(\eta - \lambda) \cdot B^{\mathbb{Q}^{(1)}}$ is a $\mathbb{Q}^{(1)}$ martingale it is sufficient to observe that $\mathbb{E}^{\mathbb{Q}^{(1)}} \int_0^T Y_t^2 dt < \infty$, or equivalently that the continuous function $J(t) = \mathbb{E}^{\mathbb{Q}^{(1)}} Y_t^2$ is finite for each t, and this follows by a stochastic comparison theorem.

Finally we show that $(\eta - \lambda) \cdot B^{\mathbb{Q}}$ is a true $\mathbb{Q}$ martingale for each martingale measure $\mathbb{Q}$ with finite entropy with respect to $\mathbb{P}$. By Proposition 3.5 it is sufficient to show that $\int_0^T (\eta_t - \lambda_t)^2 dt$ has an exponential moment under $\mathbb{P}$. If we define $\tilde{g}(y,t)$ via

$$
\tilde{g}(y,t) = \mathbb{E}\left[\exp\left(\alpha K \int_t^T Y_s^2 ds \,\middle|\, Y_t = y\right)\right],
$$

then we can solve for $\tilde{g}$ exactly as in Section 4 of this paper. We find

$$
\tilde{g}(y,t) = \exp\left(\frac{y^2}{2}\tilde{F}(T-t) + \tilde{G}(T-t)\right)
$$

where $\tilde{F}$ solves (33) for some suitable values of the constants A, B and C. Finally we can choose α sufficiently small so that $\tilde{F}$ does not explode by time T and $g(y,0)$ is finite. Hence $\int_0^T (\eta_t - \lambda_t)^2 dt$ has an exponential moment under $\mathbb{P}$ and $\mathbb{Q}^{(1)}$ as given by (35) is the minimal entropy measure.

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