The Range of Traded Option Prices

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Abstract

Suppose we are given a set of prices of European call options over a finite range of strike prices and exercise times, written on a financial asset with deterministic dividends which is traded in a frictionless market with no interest rate volatility. We ask: when is there an arbitrage opportunity? We give conditions for the prices to be consistent with an arbitrage-free model (in which case the model can be realised on a finite probability space). We also give conditions for there to exist an arbitrage opportunity which can be locked in at time zero. There is also a third boundary case in which prices are recognisably misspecified, but the ability to take advantage of an arbitrage opportunity depends upon knowledge of the null sets of the model.

Key words: Option pricing, implied distributions, no-arbitrage conditions.

1 Introduction

Given a finite matrix of prices for traded European options on a financial asset, is there an arbitrage opportunity? We answer this question by giving necessary and sufficient conditions for the existence of a price model in which the given prices are expressed as discounted expectations under a martingale measure. By standard theorems, there is no arbitrage under these conditions. It turns out that if a model can be constructed at all, it can be done on a finite probability space. When these conditions fail there is an an arbitrage. This may be a model-independent arbitrage, or may take the form of what we term a weak arbitrage opportunity, in which case it is necessary to know which events are null in order to choose the strategy that generates the arbitrage.

There has been extensive study of the relationship between option prices and price distributions ever since the original work of Breeden and Litzenberger (1978). In particular, Rubinstein (1994), Dupire (1994), Derman and Kani (1998) and others have constructed 'implied trees',

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which are discrete models calibrated to given option prices. Generally, however – but see the discussion in subsequent paragraphs – the calibration is to a smooth volatility surface obtained by interpolating the raw market data, and the arbitrage question is not directly considered, although of course the ability to construct an implied tree is a sufficient condition for absence of arbitrage. Arbitrage is discussed by Rebonato (1999) – see §6.6 – but no precise conditions are given.

The starting point for the present paper was the short note by Carr and Madan (2004). These authors consider a countably infinite set of strike prices K_i and assume that interest rates and dividends are zero. They obtain essentially the result of Corollary 4.1 below. We must however allow for non-zero interest rates and dividends, because otherwise the results do not cover the obvious applications to equity index options and FX options. When we do so, we find that the intertemporal conditions involve all the options jointly, as opposed to the simpler condition (non-negativity of calendar spread values) stated in Corollary 4.1.

The Carr and Madan (2004) note was predated by the work of Laurent and Leisen (2000). In this paper the authors take a finite set of options and option prices and discuss when it is possible to construct a model which is consistent with those prices. In the terminology of the next section they are interested in determining whether the option prices are consistent with absence of arbitrage. The main result of Laurent and Leisen (2000) is similar in spirit to the first part of our main result (Theorem 4.2 below), but we provide a much more direct proof based on the Sherman-Stein-Blackwell Theorem. We are also much more careful in determining the conditions on the prices of the options with the largest strikes which are necessary for no arbitrage.

Two recent preprints by Cousot (2004) and Buehler (2004) also attack the problem of determining when quoted options prices are consistent with some martingale model. These papers use Kellerer's Theorem (Kellerer (1972)) rather than the Sherman-Stein-Blackwell Theorem. Again the focus is on determining when quoted options prices are consistent with some model, rather than whether it is possible to construct an arbitrage strategy. Cousot (2004) takes care to exclude the situation which arises in the notion of a weak arbitrage (Definition 2.3) whereas Buehler (2004) excludes this case with an ad-hoc assumption. However, Buehler (2004) makes the important observation that the model which all these papers essentially construct is the model which leads to the highest option prices for European options with convex payoffs.

Since the problem is in general in an incomplete market setting, in the cases where there is absence of arbitrage there are typically many models which are consistent with the quoted option prices. Laurent and Leisen (2000), Cousot (2004) and Buehler (2004) each make some attempt to describe how to choose a model from this class of consistent models. In the first two papers the idea is to choose the model which minimises entropy relative to a prior model. In contrast Buehler (2004) suggests choosing a model which minimises the variance of price movements. In contrast we do not discuss model fitting. Instead, our focus is on the arbitrage/no arbitrage distinction and the notion of a weak arbitrage. Where possible we exhibit explicit static, model-independent strategies which realise the arbitrage.

The basic problem of determining when prices are consistent with absence of arbitrage is a feasability isssue in a linear semi-infinite programming setting. The problem is semi-infinite because the space of measures over which the search takes place is infinite dimensional, see Hettlich and Kortanek (1993) for an overview and d'Aspremont and Ghaoui (2005) for an application to option pricing. However, in our case the structure of the problem is such that we can bypass the general theory, and construct an explicit model on a finite probability space.

The remainder of the paper is structured as follows. The problem is formulated in precise terms in Section 2. In Section 3 we consider option prices specified only at a single exercise time. There are three possibilities, identified in Theorem 3.1. Either there is a model-independent arbitrage, or the prices are recognisably mis-specified but an arbitrage cannot be realized without more information, or an arbitrage-free model can be constructed in a simple and explicit way. The general case is tackled in Section 4, giving the main result of the paper, Theorem 4.2. As mentioned above, the result depends on the Sherman-Stein-Blackwell theorem (Theorem 4.1 below), a classical result in mathematical statistics concerned with the existence of martingales with given marginal distributions. For completeness we give, in an appendix, a self-contained proof of this result.

2 Problem formulation

Let $\{S_t\}_{t\in\mathcal{T}}$ be the price of a financial asset, where $\mathcal{T}$ denotes the set of times at which the asset can be traded. The present time is $0 \in \mathcal{T}$. Suppose that a finite number of European call options, denoted $C_{i,j}$, are written on this asset, where the (i,j)th option has maturity time $T_j \in \mathcal{T}$ and strike $K_{i,j}$, with $T_1 < T_2 \ldots < T_m$ and $K_{1,j} < K_{2,j} \ldots < K_{n(j),j}$.

We make the following standing assumptions about the financial market and the asset on which the options are written.

1. The market is frictionless: assets can be traded in arbitrary amounts, short or long, without transactions costs, and the interest rate for borrowing and lending is the same.

2. There is no interest rate volatility. We denote by $D(t)$ the market discount factor for time t, i.e. the price at time 0 of a zero-coupon bond maturing at t. Under the stated assumption, the value of this bond at time $s < t$ is just $D(t)/D(s)$. Finally, we define $D_j = D(T_j), j = 1, \ldots, m$.

3. Either the asset does not pay dividends or else the dividend yield is deterministic. In this case there is a uniquely specified model-free forward price $F(t)$ for delivery of the asset at t. More specifically, let $\Gamma(t)$ be the number of shares which will be owned by time t if dividend income is re-invested in shares. Then $F(t) = S_0/D(t)\Gamma(t)$. For example, if the asset has a constant dividend yield q then $\Gamma(t) = e^{qt}$ and $F(t) = e^{-qt}S_0/D(t)$. However, the exact structure of the dividend payments is not important.

Since we assume no interest rate volatility and deterministic dividends, the forward price at time s for delivery at a later time t is a known (at time 0) multiple $F(t)/F(s)$ of the time s-spot price of the asset. Let $F_j = F(T_j), j = 1, \ldots, m$.

Option $C_{i,j}$ has exercise value $H_{i,j} = [S_{T_j} - K_i^j]^+$ and quoted price $p_{i,j}$ at time 0. Based on these quoted prices, we want to decide if there is an arbitrage opportunity. To answer this question we have to distinguish between *model-independent* and *model-dependent* arbitrage.

Definition 2.1 *There is a* model-independent arbitrage *if we can form a semi-static portfolio in the underlying asset and the options such that the initial portfolio value is strictly negative but all subsequent cash flows are non-negative.*

Here we describe a portfolio as semi-static if it involves a fixed position in the traded options taken at time zero, and if the position in the underlying asset can only be modified at a finite number of trading times $t \in \mathcal{T}$. After time-0 the strategy is kept self-financing with balancing transactions in the risk-free bond. The fact that the initial portfolio value is strictly negative means that we receive a positive cashflow for entering the position.

As a simple example of a model-independent arbitrage, suppose $p_{1,1} > S_0$. Then we could sell the option and buy the asset. We make an immediate profit of $p_{1,1} - S_0$ and the value on exercise at T_1 is $S_{T_1} \wedge K_1 \geq 0$. A more interesting example is the 'butterfly spread', which is the contract $\lambda C_{i-1,j} - (\lambda + \gamma)C_{i,j} + \gamma C_{i+1,j}$ with $\lambda = 1/(K_i - K_{i-1})$ and $\gamma = 1/(K_{i+1} - K_i)$. Its price is $q = \lambda p_{i-1,j} - (\lambda + \gamma)p_{i,j} + \gamma_{i+1} p_{i+1,j}$ and the exercise value, shown in Figure 1, is non-negative. There is therefore a model-independent arbitrage unless $q \geq 0$.

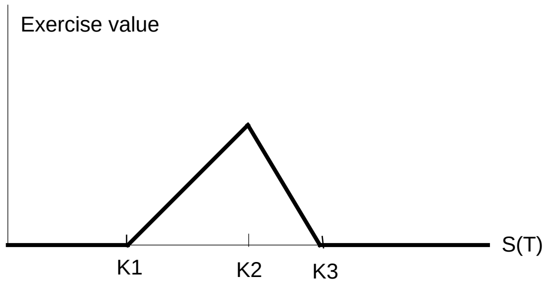

Figure 1: Exercise value of butterfly spread option.

Now we turn to the definition of a model-dependent arbitrage. Absence of arbitrage is normally defined in relation to a specific model of market prices. A *model* $\mathbb{M}$ is a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathcal{T}}, P)$ with $\mathcal{F}_0 = \{\Omega, \emptyset\}$ augmented with all null sets of $\mathcal{F}_{T_m}$, carrying an adapted process $(S_t)_{t \in \mathcal{T}}$ with S_0 equal to the given asset price at time 0. Recall that $\mathcal{T}$ is the given set of trading times, the same for every model. If also there is an equivalent martingale measure (EMM), i.e. a measure $Q \approx P$ under which the process $M_t = S_t/F(t)$ is a martingale, where $F(t)$ is the model-independent forward price introduced above, then $\mathbb{M}$ is arbitrage-free. Given a model $\mathbb{M}$, the option prices $p_{i,j}$ are *arbitrage-free prices* if we can construct price processes $Y_t^{i,j}$ on $(\Omega, \mathcal{F}_{T_m}, P)$ for the corresponding options $C_{i,j}$ such that $\mathbb{M}$ is still arbitrage-free if we include the $Y_t^{i,j}$ as extra traded assets. The requirements for $Y_t^{i,j}$ are (i) $Y_t^{i,j} \geq 0$ for $t \in \mathcal{T}$, a.s., (ii) $Y_0^{i,j} = p_{i,j}$ and (iii) $Y_{T_j}^{i,j} = H_{i,j}$ a.s. It is shown by the argument of Theorem 5.30 of Föllmer and Schied (2004) that the set of arbitrage-free prices is in 1-1 correspondence with the set of discounted expectations $\{\tilde{p}_{i,j} = D_j \mathbb{E}_{\tilde{Q}}[H_{i,j}] : \tilde{Q} \in \mathcal{P}_{\mathbb{M}}\}$, where $\mathcal{P}_{\mathbb{M}}$ denotes the set of EMMs $\tilde{Q}$ such that $\mathbb{E}_{\tilde{Q}}[H_{i,j}] < \infty$.

In our case, we are not presented with a model *a priori.* The appropriate concept is as follows:

Definition 2.2 *The set of option prices $p_{i,j}$ is* consistent with absence of arbitrage *if there is a model $\mathbb{M}$ and $Q \in \mathcal{P}_{\mathbb{M}}$ such that $p_{i,j} = D_j \mathbb{E}_Q[H_{i,j}]$ for $i = 1, \ldots, n(j)$ and $j = 1, \ldots, m$.*

The aim of this paper is to characterise the situations when either the traded call prices yield a *model-independent arbitrage*, or they are *consistent with the absence of arbitrage.* The original hope was that any set of prices would fall in to one of these two settings. In fact there is a third case, which arises when the prices in the market are recognisably mis-specified, in that there is no model in which the prices are expressed as discounted expectations, but there is no model-independent arbitrage.

Definition 2.3 *There is a* weak arbitrage opportunity *if there is no model-independent arbitrage, but, given the null sets of the model, there is a semi-static portfolio such that the initial portfolio value is non-positive, but all subsequent cashflows are non-negative, and the probability of a positive cashflow is non-zero.*

From the remarks before Definition 2.2 it is clear that absence of arbitrage implies that there can neither be a weak nor a model-independent arbitrage, and therefore, since we define the weak and model-independent arbitrages to be mutually exclusive phenomena, a given set of option prices is consistent with at most one of these situations.

The differences between a weak arbitrage opportunity and a model-independent arbitrage are firstly that we do not insist that the initial portfolio value is strictly negative, and secondly that for a weak arbitrage the choice of portfolio is allowed to depend on the null sets of the model. Suppose for example that $p_{i+1,j} = p_{i,j} > 0$ for some i, j. Obviously, this pair of prices cannot arise as discounted expectations in any model. In a model where $\mathbb{P}[S_{T_j} > K_i^j] > 0$ an arbitrage opportunity is realized by (at zero cost) buying the spread option $C_{i,j} - C_{i+1,j}$, while if $\mathbb{P}[S_{T_j} > K_{i,j}] = 0$ an arbitrage is realized by selling $C_{i,j}$ (or $C_{i+1,j}$). It will never be exercised. Thus a weak arbitrage opportunity is a situation where we know there must be an arbitrage but we cannot tell, without further information, what strategy will realize it. In fact it will turn out that in describing the strategy which generates the weak arbitrage it is always sufficient to know whether $\mathbb{P}[S_{T_j} > K_{i,j}]$ is positive for some sufficiently large strike, rather than knowing all the null sets of the model.

Recall that, in a model $\mathbb{M}$, the price process is expressed as $S_t = F(t)M_t$ where M_t is a Q-martingale. If we define

$$r_{i,j} = \frac{p_{i,j}}{D_j F_j}, \qquad k_{i,j} = \frac{K_{i,j}}{F_j}, \tag{2.1}$$

the prices can be expressed in normalised form as

$$r_{i,j} = \frac{1}{F_j}\mathbb{E}[S_{T_j} - K_{i,j}]^+ = \mathbb{E}[M_{T_j} - k_{i,j}]^+. \tag{2.2}$$

We will use this notation throughout the paper, generally suppressing the suffix Q, as in (2.2). We extend the notation to include the zero-strike option $K_{0,j} = 0$ which has value $p_{0,j} = D_j F_j$, the normalised values being $k_{0,j} = 0, r_{0,j} = 1$.

A further item of notation is as follows. Let $\mathcal{S} = \{(x_i, y_i),\ i = 0, 1, \ldots, n\}$ be a set of pairs of real numbers with x_i increasing and $y_i \geq 0$. We will call $f : [x_0, \infty) \to \mathbb{R}$ the *support function* of $\mathcal{S}$ if f is the largest decreasing convex function such that $f(x_i) \leq y_i$, $i = 0, 1, \ldots, n$. Note that the support function of $\mathcal{S}$ passes through each of the points (x_i, y_i) if and only if the linear interpolation of the points $\{(x_i, y_i),\ i = 0, 1, \ldots, n\}$ is identical to the support function on $[x_0, x_n]$.

We assume that all option prices $p_{i,j}$ are non-negative else there is a trivial arbitrage.

3 Single exercise time

Our first result concerns the sequence of prices $p_{1,1}, \ldots, p_{n(1),1}$ for options maturing at the first exercise time T_1, as shown in Figure 2, where the linear interpolation of these prices is also shown. For notational convenience we drop the time-subscript 1, and write $D_1 = D$, $H_{i,1} = H_i$, etc., and $n(1) = n$. Let $n_0 = \inf\{i : r_i = 0\}$ be the index of the first call with zero price, and set $n_0 = \infty$ if $r_i > 0$ for all i.

Theorem 3.1 *Let $\mathcal{R}(\cdot)$ denote the support function of the set $\{(k_0, r_0), \ldots, (k_n, r_n)\}$, where r_i, k_i are defined by (2.1). The prices p_i are consistent with absence of arbitrage if and only if $\mathcal{R}$ is a strictly decreasing function on $[0, k_{n_0 \wedge n}]$ such that $(d\mathcal{R}/dk)|_{k=0+} \geq -1$ and $\mathcal{R}(k_i) = r_i$.*

If r_n is positive, $(d\mathcal{R}/dk)|_{k=0+} \geq -1$ and $\mathcal{R}(k_i) = r_i$, but $\mathcal{R}$ is not strictly decreasing on $[0, k_n]$, then there is a weak arbitrage opportunity.

Otherwise there is a model-independent arbitrage.

Proof: The theorem has an equivalent statement in terms of the un-normalised quantities. In describing the strategies which yield model independent and weak arbitrages we will revert to un-normalised quantities, whereas for constructing the model which shows absence of arbitrage we will use the normalised variables. Either part can be rewritten in terms of the other coordinate system.

For $i = 1, \ldots, n$ define

$$\alpha_i = \frac{1}{(k_i - k_{i-1})} \qquad\qquad \beta_i = (r_{i-1} - r_i)\alpha_i,$$

and for $i = 1, \ldots, n-1$ set

(3.1) $$B_i = \alpha_{i+1}H_{i+1} - (\alpha_{i+1} + \alpha_i)H_i + \alpha_i H_{i-1}$$

together with $B_0 = 1 - \alpha_1(H_0 - H_1)$, and $B_n = \alpha_n(H_{n-1} - H_n)$. By construction $B_i = 1$ if $S_T = K_i$, and $B_i = 0$ outside (K_{i-1}, K_{i+1}), so that B_i plays the role of an Arrow-Debreu security. Similarly define

(3.2) $$q_i = \alpha_{i+1}r_{i+1} - (\alpha_{i+1} + \alpha_i)r_i + \alpha_i r_{i-1} = \beta_i - \beta_{i+1}$$

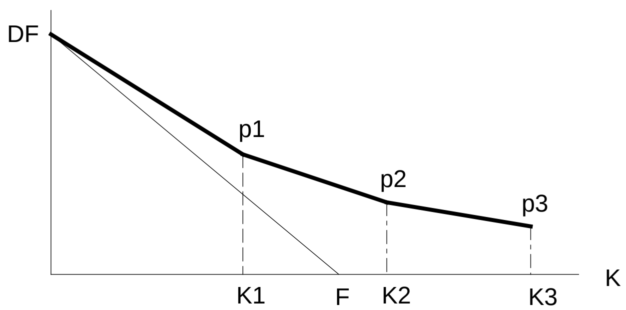

Figure 2: Option prices at a single maturity time. These prices are consistent with absence of arbitrage.

together with $q_0 = 1 - \alpha_1(r_0 - r_1) = 1 - \beta_1$ and $q_n = \alpha_n(r_{n-1} - r_n) = \beta_n$. It is immediate that $\sum_{i=0}^n q_i = 1$.

The condition $(d\mathcal{R}/dk)|_{k=0} \geq -1$ is equivalent to $q_0 \geq 0$; and for $1 \leq i \leq n$, $\mathcal{R}(k_i) = r_i$ is equivalent to $q_i \geq 0$. This second equivalence follows from the fact that the coefficients β_i are (minus) the slopes of the linear interpolation created from $\{(k_i, r_i), i = 0, 1, \dots, n\}$ and if this linear interpolation is to agree with the support function then these slopes must be non-decreasing. Note also that $\mathcal{R}$ is strictly decreasing on $[0, k_{n_0 \wedge n}]$ if and only if $r_i = 0$ for some i or $q_n > 0$.

There are three cases to consider:

1. $q_i < 0$ for some i;
2. $q_i \geq 0$, $q_n = 0$ and $r_n > 0$;
3. $q_i \geq 0$ and either $q_n > 0$ or $r_n = 0$.

Our aim is to show that these three cases correspond to model-independent arbitrage, weak arbitrage and absence of arbitrage respectively.

We begin by considering the third of these cases, which by the above remarks corresponds exactly to the conditions given is the statement of the theorem as necessary and sufficient for absence of arbitrage. If $r_n > 0$, then fix $\theta \in (0,1)$, and using the fact that $q_n > 0$ define $\tilde{q}_n = \theta q_n$, $\tilde{q}_{n+1} = (1-\theta)q_n$, $\alpha_{n+1} = \tilde{q}_{n+1}/r_n$, $k_{n+1} = k_n + \alpha_{n+1}^{-1}$ and $r_{n+1} = 0$. Consider the augmented set of call prices $\{(k_i, r_i), 0 \leq i \leq n+1\}$. (Effectively we are assuming that there is an extra fictitious call with strike k_{n+1} and price zero. The key point is that we choose k_{n+1} sufficiently large so that the support functions for the sets $\{(k_i, r_i), 0 \leq i \leq n\}$ and $\{(k_i, r_i), 0 \leq i \leq n+1\}$ agree on $[0, k_n]$.) If we now define $\hat{q}_i$ relative to this augmented set of calls via the formula (3.2) then we find that for $i \leq n-1$, $q_i = \hat{q}_i$ together with $\hat{q}_n = \tilde{q}_n$ and $\hat{q}_{n+1} = \tilde{q}_{n+1}$. All the $\hat{q}_i$ are non-negative by construction. In this way we reduce the case with $r_n > 0$ to the case where the call with the largest strike has a price of zero.

Hence, without loss of generality we may assume $r_n = 0$. (Note that in most practical applications all options will trade for positive prices, so we have shown how the typical situation may be reduced to an atypical one.) Define the measure μ by

$$\mu = \sum_{i=0}^n q_i \delta_{k_i}$$

where δ_x denotes the Dirac measure at x. Our price model $\mathbb{M}$ is now defined as follows. The probability space is $\Omega = \{\omega_0, \dots, \omega_n\}$ with $Q(\{\omega_i\}) = q_i$. The normalised price process is $M_t(\omega_i) = \mathbf{1}_{(t<T)} + k_i \mathbf{1}_{t=T}$ for $t \in \mathcal{T}$. For this normalised price process $\mathbb{E}[M_T - k_n]^+ = r_n = 0$ and

$$\mathbb{E}[M_T - k_{i_0}]^+ - \mathbb{E}[M_T - k_{i_0+1}]^+ = \sum_{i=i_0+1}^n q_i(k_{i_0+1} - k_{i_0}) = \beta_{i_0}(k_{i_0+1} - k_{i_0}) = r_{i_0} - r_{i_0+1}.$$

Thus we have shown that $\mathbb{M}$ satisfies the properties specified in Definition 2.2.

Now consider the first case above, so that $q_k < 0$ for some k. If $q_0 < 0$ then $1 - r_1 > k_1$, and this condition is equivalent to $p_1 < D(F - K_1)$. In this case a model-independent arbitrage can be realised by buying the call with strike K_1, selling DF/S_0 units of the underlying and investing

DK_1 in the bond. If $q_i < 0$ for some $i \in \{1, \ldots, n-1\}$ then there is a model-independent arbitrage to be realised from buying the butterfly-spread as described after Definition 2.1, or equivalently from buying the portfolio B_i in (3.1). Finally, if $q_n < 0$ there is a model-independent arbitrage from buying the call with strike K_{n-1} and selling the call with strike K_n.

Finally consider the situation where $q_k \geq 0$, $q_n = 0$ and $r_n > 0$. In this case there is a weak arbitrage, exactly as described after Definition 2.3. All that remains is to show that there is no model-independent arbitrage in this situation. The idea is to show that if there were a model-independent arbitrage in this case, then after a small perturbation of the prices there would still be a model-independent arbitrage even though the prices would be consistent with absence of arbitrage; a contradiction.

So suppose there is a model-independent arbitrage, i.e. there exists a portfolio $C_\alpha = \sum_{i=-1}^n \alpha_i C_i$ (where α_{-1} is the coefficient of the cash component C_{-1} for which $H_{-1} = 1$, $r_{-1} = 1$) such that $H_\alpha = \sum_{i=-1}^n \alpha_i H_i \geq 0$, and $r_\alpha = \sum_{i=-1}^n \alpha_i r_i < 0$. Define

$$\epsilon = - \min_{\alpha : \sum_{i=-1}^n |\alpha_i| = 1} r_\alpha,$$

which is a measure of the largest certain gain which can be realised for a portfolio of a given size, and let α^* be the minimiser.

Set $i_0 = \max\{i : r_i = 1 - k_i\}$, and define

$$\delta = \min \left\{ \frac{\epsilon}{2} \; ; \; r_n \frac{\sum_{i \geq i_0} (k_i - k_{i_0})}{(k_n - k_{i_0})} \; ; \; (r_{i_0+1} + k_{i_0+1} - 1) \frac{\sum_{i \geq i_0} (k_i - k_{i_0})}{(k_{i_0+1} - k_{i_0})} \right\}.$$

Now define $\tilde{r}_i = r_i$ for $i \leq i_0$, and for $i > i_0$

$$\tilde{r}_i = r_i - \delta \frac{(k_i - k_{i_0})}{\sum_{i \geq i_0} (k_i - k_{i_0})}.$$

The second and third elements in the minimum of the definition of δ are exactly sufficient to ensure that $\tilde{r}_n \geq 0$ and $\tilde{r}_{i_0+1} \geq 1 - k_{i_0+1}$. Given that the modified prices $\tilde{r}_i$ are a piecewise linear transformation of the original normalised prices r_i, it is clear that they now satisfy the conditions of the first paragraph of Theorem 3.1 which are sufficient for absence of arbitrage.

With respect to the modified prices the portfolio C_α is associated with the same cashflow at time T, and with an initial cashflow of

$$\begin{aligned} \tilde{r}_{\alpha^*} = \sum_{i=-1}^n \alpha_i^* \tilde{r}_i &= \sum_{i=-1}^n \alpha_i^* r_i - \sum_{i_0+1}^n \alpha_i^* (r_i - \tilde{r}_i) \\ &\leq -\epsilon + \delta \sum_{i_0+1}^n |\alpha_i^*| \frac{(k_i - k_{i_0})}{\sum_{i \geq i_0} (k_i - k_{i_0})} \\ &\leq -\epsilon + (\epsilon/2) \sum_{i_0+1}^n |\alpha_i^*| \leq -\epsilon/2 \end{aligned}$$

But this contradicts the fact that the modified prices $\tilde{r}_i$ are consistent with absence of arbitrage. Hence there is no model-independent arbitrage. ■

Remark: A by-product of this proof is that if an arbitrage-free price system can be realized at all, it can be done on a finite probability space with $n + 2$ points.

4 The General Case

We now revert to the general case as described in Section 2 and to the normalised parameters (2.1). A necessary condition for absence of arbitrage is that the prices at each exercise time T_j satisfy the conditions of Theorem 3.1. The intertemporal conditions are however a more subtle matter, since we have to construct a joint distribution for $S_0, S_{T_1}, \ldots, S_{T_m}$ rather than merely a one-dimensional distribution. The key ingredient is the following theorem, due to Sherman (1951), Stein (1951) and Blackwell (1953). Some background to this result, and a proof, are given for the reader's convenience in Appendix A.

Let $X = (a_1, \ldots, a_n)$ be a finite set of points in $\mathbb{R}^N$ and let m be a probability mass distribution on $X \times X$. If Z_1, Z_2 denote the projections onto the two factors, then the 'process' (Z_1, Z_2) is a martingale if $E[Z_2|Z_1] = Z_1$, i.e. $\sum_j q_{ij} a_j = a_i$, $i = 1, \ldots, n$ where q_{ij} is the transition matrix $q_{ij} = m_{ij} / \sum_k m_{ik}$. We call this a martingale transition matrix.

Theorem 4.1 (Sherman-Stein-Blackwell) *If μ and ν are any two probability measures on a finite set $X = (a_1, \ldots, a_n)$ in $\mathbb{R}^N$ such that*[1] *$\mu\phi \geq \nu\phi$ for every continuous concave function defined on the convex hull of X, then there is a martingale transition matrix Q such that $\mu Q = \nu$.*

To formulate the main result let $\hat{\mathcal{R}}_j$ be the support function defined on $[0, \infty)$ of

$$\hat{\mathcal{S}}_j = \{(k_{i,l}, r_{i,l}) : 0 \leq i \leq n(l), j \leq l \leq m\}.$$

Note that the support function is eventually constant, and let $c^j = \min\{r_{i,l} : 0 \leq i \leq n(l) + 1, j \leq l \leq m\}$ be the large-k limit of $\hat{\mathcal{R}}_j$. For j such that $c^j > 0$ let $\hat{k}^j$ be the strike at which the function first becomes constant, and let $\hat{\beta}^j$ be the slope of $\hat{\mathcal{R}}_j$ just to the left of this point. For j such that $c^j = 0$, let $\hat{k}^j_{n_0}$ be the strike at which $\hat{\mathcal{R}}_j$ first becomes zero.

Theorem 4.2 *The prices $\{p_{i,j}, i = 1, \ldots, n(j), j = 1, \ldots, m\}$ are consistent with absence of arbitrage if and only if, for each j, $\hat{\mathcal{R}}_j$ is a strictly decreasing function on $[0, \hat{k}^j_{n_0} \wedge k_{n(j),j}]$ such that $(d\hat{\mathcal{R}}_j/dk)|_{k=0+} \geq -1$ and $\hat{\mathcal{R}}(k_{i,j}) = r_{i,j}$ for $i = 1, \ldots, n(j)$.*

If the support functions $\hat{\mathcal{R}}_j$ satisfy $(d\hat{\mathcal{R}}_j/dk)|_{k=0+} \geq -1$ and $\hat{\mathcal{R}}(k_{i,j}) = r_{i,j}$ but for some j with $r_{n(j),j}$ positive $\hat{\mathcal{R}}$ is not strictly decreasing on $[0, k_{n(j),j}]$, then there is a weak arbitrage opportunity.

Otherwise there is a model-independent arbitrage.

PROOF:

Suppose the conditions stated for absence of arbitrage hold. In the cases for which $c^j > 0$ then, as in the proof of Theorem 3.1, we want to introduce a fictitious extra strike for which the call price is zero. Fix $k^\infty > \max_{j:c^j>0}(\hat{k}^j + c_j|\hat{\beta}^j|)$ and define strikes $k_{n(j)+1,j} = k^\infty$ for which the associated call prices are $r_{n(j)+1,j} = 0$. It follows that the support functions generated from the points $\hat{\mathcal{S}}_j$ and

$$\bar{\mathcal{S}}_j = \{(k_{i,l}, r_{i,l}) : 0 \leq i \leq n(l) + 1, j \leq l \leq m\}$$

agree on $[0, k_{n(j),j}]$. Let $\bar{\mathcal{R}}_j$ be the support function of $\bar{\mathcal{S}}_j$.

Let $\mathcal{K} = \bigcup\{k_{i,j} : 0 \leq i \leq n(j) + 1, 1 \leq j \leq m\}$ and set $\mathcal{K}_j = \{k_{i,l} : (k_{i,l}, r_{i,l}) \in \bar{\mathcal{S}}_j$ and $r_{i,l} = \bar{\mathcal{R}}_j(k_{i,l})\}$. In the case where there the prices are consistent with no arbitrage we will produce a

[1]Notation: real-valued functions on X are written as column vectors, and measures as row vectors.

model in which the normalised asset price takes values in $\mathcal{K}$ only. Let $\tilde{\mu}_j$ be the distribution on $\mathcal{K}_j$ given by the construction in the proof of Theorem 3.1 but based on the prices $\{(k_{i,l}, r_{i,l}) : k_{i,l} \in \mathcal{K}_j\}$, and let μ_j be the distribution on $\mathcal{K}$ defined by $\mu_j(\{k\}) = \tilde{\mu}_j(\{k\})$ for $k \in \mathcal{K}_j$ and $\mu_j(\{k\}) = 0$ for $k \in \mathcal{K}\backslash\mathcal{K}_j$. The measure μ_j will be the marginal distribution of M_{T_j}.

It is sufficient to show that there is a model such that $\mathbb{E}[M_{T_j} - k_{i,j}]^+ = r_{i,j}$. We construct a martingale M on a finite probability space, with state-space $\mathcal{K}$ and indexed by the times T_j which is consistent with the prices implied by $\bar{\mathcal{R}}_j$, and which therefore is consistent with the prices $r_{i,j}$. We proceed by induction; by the arguments of Section 2 there is no problem in constructing a process for which the time-T_1 calls are given by $\bar{\mathcal{R}}_1$. Suppose we have constructed a martingale process which is consistent with the observed prices for $j \leq J$. Let $\phi : [0, k^\infty]$ be a continuous convex function and let $\psi : [0, k^\infty]$ be the piecewise linear interpolant of the points $\{(k, \phi(k)) : k \in \mathcal{K}_{J+1}\}$. Then $\psi = \phi$ a.e.$(d\mu_{J+1})$ and $\mu_{J+1}\psi = \mu_{J+1}\phi$. Now ψ can be expressed as a positive linear combination of put and call payoffs with strikes in $\mathcal{K}_{J+1}$. By the assumptions of the Theorem $\bar{\mathcal{R}}_J \leq \bar{\mathcal{R}}_{J+1}$, implying that $\mu_J\psi \leq \mu_{J+1}\psi$. But $\psi \geq \phi$, so that $\mu_J\psi \geq \mu_J\phi$; hence $\mu_J\phi \leq \mu_{J+1}\phi$. Thus the conditions of the Sherman-Stein-Blackwell Theorem are met, and we can find a martingale transition matrix for the period $[T_J, T_{J+1}]$. In this way we construct a martingale $M_{T_1}, \ldots, M_{T_m}$ such that the marginal distribution of M_{T_j} is μ_j, ensuring the correct option prices. As in Section 3 we extend the definition of M_t to all $t \in \mathcal{T}$ by setting $M_t = M_{T_j}$ for $t \in \mathcal{T} \cap [T_j, T_{j+1}[$.

Now consider the conditions for a model-independent arbitrage. As in the proof of Theorem 3.1 if for some j we have $(d\hat{\mathcal{R}}_j/dk)|_{k=0+} < -1$ then there is an arbitrage involving a call and the underlying asset. Suppose instead that the condition $\hat{\mathcal{R}}_j(k_{i,j}) = r_{i,j}$ fails. Then, either there exists $j, k_{i,j}$ with $k_{i,j} > \hat{k}^j$ and $r_{i,j} > c^j$ or there exists j, $k_{i,j}$ together with $l_1, l_2 \geq j$ and $k_{i_1,l_1} \leq k_{i,j} \leq k_{i_2,l_2}$ such that $\theta r_{i_1,l_1} + (1-\theta) r_{i_2,l_2} < r_{i,j}$, or equivalently

$$\theta \frac{D_j F_j}{D_{l_1} F_{l_1}} p_{i_1,l_1} + (1-\theta) \frac{D_j F_j}{D_{l_2} F_{l_2}} p_{i_2,l_2} < p_{i,j},$$

where $\theta = (k_{i_2,l_2} - k_{i,j})/(k_{i_2,l_2} - k_{i_1,l_1})$. The aim in the second case is to combine the elements of butterfly and calendar spreads to produce an arbitrage. A similar, but simpler argument involving two calls and a potential forward transaction can be used in the first case.

Consider the strategy of selling the call with maturity T_j and strike $K_{i,j}$ and buying a quantity $\theta D_j F_j / D_{l_1} F_{l_1}$ units of the call with maturity T_{l_1}, strike K_{i_1,l_1} and $(1-\theta) D_j F_j / D_{l_2} F_{l_2}$ units of the call with maturity T_{l_2}, strike K_{i_2,l_2}. By the above inequality this transaction earns a small initial profit. We want to show that, notwithstanding this initial profit, the portfolio of calls can be hedged to give a non-negative payoff, and hence that there is an arbitrage opportunity.

If the call with maturity T_j expires out-of-the-money, then the portfolio is guaranteed a non-negative terminal value.

Now consider the case where the call with maturity T_j expires in-the-money. We consider the case $T_{l_2} \geq T_{l_1}$ the other case being similar. In addition to the options above, and conditional on $S_{T_j} > K_{i,j}$, at time T_j sell $D_j F_j / D_{l_2} F_{l_2}$ units of the forward with maturity T_{l_2}. Under the assumptions listed in the introduction this transaction occurs at a unit price of $S_{T_j} F_{l_2} / F_j$. To close out this position, buy back $\theta D_j F_j / D_{l_2} F_{l_2}$ units of the forward at T_{l_1} and the remaining $(1-\theta) D_j F_j / D_{l_2} F_{l_2}$ units at T_{l_2}. Once the stock options have been exercised (for their cash equivalents) the portfolio at T_{l_2} contains no stock, and the cash value of the strategy is

$$
\begin{aligned}
&\theta\frac{D_jF_j}{D_{l_1}F_{l_1}}(S_{T_{l_1}}-K_{i_1,l_1})^+\frac{D_{l_1}}{D_{l_2}}+(1-\theta)\frac{D_jF_j}{D_{l_2}F_{l_2}}(S_{T_{l_2}}-K_{i_2,l_2})^+\\
&\quad -(S_{T_j}-K_{i,j})\frac{D_j}{D_{l_2}}+\frac{D_jF_j}{D_{l_2}F_{l_2}}\frac{S_{T_j}F_{l_2}}{F_j}-\theta\frac{D_jF_j}{D_{l_2}F_{l_2}}\frac{S_{T_{l_1}}F_{l_2}}{F_{l_1}}-(1-\theta)\frac{D_jF_j}{D_{l_2}F_{l_2}}S_{T_{l_2}}\\
\geq\;& \theta\left[(S_{T_{l_1}}-K_{i_1,l_1})\frac{D_jF_j}{D_{l_2}F_{l_1}}-S_{T_{l_1}}\frac{D_jF_j}{D_{l_2}F_{l_1}}\right]\\
&\quad +(1-\theta)\left[(S_{T_{l_2}}-K_{i_2,l_2})\frac{D_jF_j}{D_{l_2}F_{l_2}}-\frac{D_jF_j}{D_{l_2}F_{l_2}}S_{T_{l_2}}\right]+K_i^j\frac{D_j}{D_{l_2}}\\
=\;& \frac{F_jD_j}{D_{l_2}}\left[k_{i,j}-\theta k_{i_1,l_1}-(1-\theta)k_{i_2,l_2}\right]\\
=\;& 0
\end{aligned}
$$

Hence, whatever the value of S at T_j we have an arbitrage opportunity.

Finally we consider the case when the prices are not consistent with any model, but neither is there a model-independent arbitrage. Suppose the conditions stated for a weak arbitrage opportunity hold. Then for some $j \leq j'$, and for some i' we have $r_{n(j),j} = r_{i',j'} > 0$ with $k_{i',j'} \leq k_{n(j),j}$. If $\mathbb{P}[M_{T_j} > k_{n(j),j}] = 0$ then there is an arbitrage from selling $C_{n(j),j}$. If $\mathbb{P}[M_{T_j} > k_{n(j),j}] > 0$ then the strategy of buying $1/D_{j'}F_{j'}$ units of the call $C_{i',j'}$ and selling $1/D_jF_j$ units of $C_{n(j),j}$ and possibly entering forward transaction at time T_j makes a profit with positive probability at zero initial cost. Furthermore, exactly as in the single-time case, it is clear that there cannot be a model-independent arbitrage since an arbitrarily small perturbation of the option prices is sufficient to take us into the case where prices are consistent with absence of arbitrage.

■

There is a special case in which the result can be stated in simpler form. This is essentially the case considered by Carr and Madan (2004).

Corollary 4.1 *Suppose that, for $j = 2, \dots, m$, $n(j) = n(1)$ and*

$$k_{i,j} = k_{i,1}, \tag{4.1}$$

or, equivalently, $K_{i,j} = K_{i,1}F_j/F_1$. Then the observed prices are consistent with an arbitrage free model if and only if (i) the prices at each exercise time T_j satisfy the conditions of Theorem 3.1 and (ii) all calendar spreads $C_{i,j+1} - (D_{j+1}F_{j+1}/D_jF_j)C_{i,j}$ have non-negative value.

Proof: The stated condition is equivalent to $r_{i,j+1} \geq r_{i,j}$. It is clear that under condition (4.1) the condition given in Theorem 4.2 for consistency with absence of arbitrage is satisfied if and only if $r_{i,l} \geq r_{i,j}$ for all $l > j$. This in turn is equivalent to the stated calendar spread condition. ■

The following simple corollary converts a geometric condition involving convex hulls into an alternative algebraic condition.

Corollary 4.2 *The prices $p_{i,j}$ are consistent with no arbitrage if and only if*
- *$k_{i,j} = k_{i',j'}$ and $j < j'$ implies $r_{i,j} \leq r_{i',j'}$, and*
- *for all $1 \leq i \leq n$, $1 \leq j \leq m$,*

$$\inf_{j'\geq j}\ \inf_{\{i':k_{i',j'}<k_{i,j}\}}\frac{r_{i,j}-r_{i',j'}}{k_{i,j}-k_{i',j'}} \leq \inf_{j'\geq j}\ \inf_{\{i':k_{i',j'}>k_{i,j}\}}\frac{r_{i',j'}-r_{i,j}}{k_{i',j'}-k_{i,j}}.$$

Given the relationship $k_{i,j} = K_{i,j}/F_j$ this can be rewritten in terms of the original quantities.

Corollary 4.3 *The prices $p_{i,j}$ are consistent with no arbitrage if and only if*
- $K_{i,j}F_j = K_{i',j'}F_{j'}$ *and* $j < j'$ *implies* $p_{i,j} \leq p_{i',j'}$, *and*
- *for all* $1 \leq i \leq n$, $1 \leq j \leq m$,

$$\inf_{j' \geq j} \inf_{\{i' : K_{i',j'}F_{j'} < K_{i,j}F_j\}} \frac{p_{i,j} - p_{i',j'}}{K_{i,j}F_j - K_{i',j'}F_{j'}} \leq \inf_{j' \geq j} \inf_{\{i' : K_{i',j'}F_{j'} > K_{i,j}F_j\}} \frac{p_{i',j'} - p_{i,j}}{K_{i',j'}F_{j'} - K_{i,j}F_j}.$$

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A Appendix: The Sherman-Stein-Blackwell theorem

Theorem 4.1 has an illustrious history, starting in 1929 with a special case for $N = 1$ (i.e. points on a line) due to Hardy et al. (1929), which also appears in *Inequalities* (Hardy et al., 1934). It was generalised to $N > 1$ around 1950 by Sherman (1951)[2] Stein (1951)and Blackwell (1953). A generalisation to probability measures in $\mathbb{R}^N$ was carried out by Strassen (1965), and there is related work by Kellerer (1972). See also Theorem 2.58 of Föllmer and Schied (2004). The proof given below is Strassen's proof restricted to the finite case.

Proof of Theorem 4.1: Let Π be the set of probability measure on $X \times X$, so that Π is a simplex in $\mathbb{R}^{n^2}$, and let Λ be a closed convex subset of Π. A measure $m \in \Pi$ has marginals m^1, m^2 given, in obvious notation, by

$$m_i^1 = \sum_j m_{ij}, \quad m_j^2 = \sum_i m_{ij}.$$

Let $\hat{m}$ be the $2n$-vector (m^1, m^2) and denote $\hat{M} = \{\hat{m} : m \in \Lambda\}$, a closed convex set. If $(\mu, \nu) \notin \hat{M}$ then there is a separating hyperplane, i.e. a vector (ϕ, ψ) and a number K such that $\mu\phi + \nu\psi > K$ while $\sum_{ij}(\phi_i + \psi_j)m_{ij} \leq K$ for all $m \in \Lambda$. Thus $(\mu, \nu) \in \hat{M}$ if and only if for all vectors (ϕ, ψ)

$$\mu\phi + \nu\psi \leq \sup_{m\in\Lambda} \sum_{i,j} (\phi_i + \psi_j)m_{ij}. \tag{A.1}$$

Now let Λ be the set of martingale measures, i.e.

$$\Lambda = \left\{ m \in \Pi : \sum_j (a_j - a_i)m_{ij} = 0, \ i = 1, \dots, n \right\}. \tag{A.2}$$

Take arbitrary vectors ϕ, ψ (thought of as functions on X) and let ψ^0 be the smallest concave function majorising ψ. Then

$$\begin{aligned} \mu\phi + \nu\psi &\leq \mu\phi + \nu\psi^0 \\ &\leq \mu(\phi + \psi^0) \\ &\leq \max_i(\phi_i + \psi_i^0) \\ &= \phi_{i^*} + \psi_{i^*}^0, \end{aligned} \tag{A.3}$$

[2]Sherman's paper was communicated to the journal by John von Neumann.

where the second inequality uses the condition stated in the theorem. Next, for $t \in R^N$, define

$$\hat{\psi}(t) = \sup\{\eta\psi : \eta \text{ is a probability measure with expectation } \sum \eta_i a_i = t\}.$$

Then $\hat{\psi}$ is a concave function and $\psi_i^1 \equiv \hat{\psi}(a_i) \geq \psi_i$, so that in particular $\psi^1 \geq \psi^0$, and given $\epsilon > 0$ there is a probability measure η_i^ϵ such that $\eta_i^\epsilon \psi \geq \psi_i^1 - \epsilon$. Define $m^* = \delta_{i^*} \times \eta_{i^*}^\epsilon$. Then $m^* \in \Lambda$ and

$$\phi_{i^*} + \psi_{i^*}^0 \;\leq\; \phi_{i^*} + \eta_{i^*}^\epsilon \psi + \epsilon = \sum_{i,j} (\phi_i + \psi_j) m_{ij}^* + \epsilon.$$

In view of (A.3) this shows that condition (A.1) is satisfied and hence that $(\mu, \nu) \in \Lambda$. ■