RECOVERING A TIME-HOMOGENEOUS STOCK PRICE PROCESS FROM PERPETUAL OPTION PRICES

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Abstract. It is well-known how to determine the price of perpetual American options if the underlying stock price is a time-homogeneous diffusion. In the present paper we consider the inverse problem, i.e. given prices of perpetual American options for different strikes we show how to construct a time-homogeneous model for the stock price which reproduces the given option prices.

1. Introduction

In the classical Black-Scholes model, there is a one-to-one correspondence between the price of an option and the volatility of the underlying stock. If the volatility σ is assumed to be given, for example by estimation from historical data, then the arbitrage free option price can be calculated using the Black-Scholes formula. Conversely, if an option price is given, then the implied volatility can be obtained as the unique σ that would produce this option price if inserted in the Black-Scholes formula. It has been well documented that if the implied volatility is inferred from real market data for option prices with the same maturity date but with different strike prices, then typically a non-constant implied volatility is obtained. Since the implied volatility often resembles a smile if plotted against the strike price, this phenomenon is referred to as the smile effect. The smile effect is one indication that the Black-Scholes assumption of normally distributed log-returns is too simplistic.

A wealth of different stock price models has been proposed in order to overcome the shortcomings of the standard Black-Scholes model, of which the most popular are jump models and stochastic volatility models. Given a model, option prices can be determined as risk neutral expectations. However, models are typically governed by a small number of parameters and only in exceptional circumstances can they be calibrated to perfectly fit the full range of options data.

Instead, there is a growing literature which tries to reverse the procedure and uses option prices to make inferences about the underlying price process. At one extreme models exist which take a price surface as the initial value of a Markov process on a space of functions. In this way the Heath-Jarrow-Morton [6] interest rate models can be made to perfectly fit an initial term structure. Such ideas inspired Dupire [5] to introduce the local volatility model which calibrates perfectly to an initial volatility surface. For a local

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volatility model, Dupire derived the PDE

$$P_T(T,K) + rKP_K(T,K) = \frac{1}{2}\sigma^2(T,K)K^2P_{KK}(T,K)$$

where $P(T,K)$ is the European put option price, T is time to maturity and K is the strike price. Solving for the (unknown) local volatility $\sigma(T,K)$ gives a formula for the time inhomogeneous local volatility in terms of derivatives of the observed European put option prices.

The local volatility model gives the unique martingale diffusion which is consistent with observed call prices. (Alternative, non-diffusion models also exist, see for example, Madan and Yor [9].) The recent literature (eg Schweizer and Wissel [13]) has included attempts to extend the theory to allow for a stochastic local volatility surface. However, it relies on the knowledge of a double continuum of option prices, which are smooth. In contrast, Hobson [7] builds models which are consistent with a continuum of strikes, but at a single maturity, in which case there is no uniqueness.

In the current article we present a method to recover a time-homogeneous local volatility function from perpetual American option prices. More precisely, we assume that perpetual put option prices are observed for all different values of the strike price, and we derive a time-homogeneous stock price process for which theoretical option prices coincide with the observed ones.

No-arbitrage enforces some fundamental convexity and monotonicity conditions on the put prices, and if these fail then no model can support the observed prices. If the observed put prices are smooth then we can use the theory of differential equations to determine a diffusion process for which the theoretical perpetual put prices agree with the observed prices, and our key contribution in this case is to give an expression for the diffusion coefficient of the underlying model in terms of the put prices. (It turns out that this expression uniquely determines the volatility co-efficient at price levels below the current stock price, but the volatility function is undetermined above the current stock price level, except through a single integral condition.) The key idea is to construct a dual function to the perpetual put price, and then the diffusion co-efficient can be found easily by taking derivatives of this dual.

The second contribution of this paper is to give time-homogeneous models which are consistent with a given set of perpetual put prices, even when those put prices are not twice differentiable or not strictly convex (in the continuation region where it is not optimal to exercise immediately). Again the key is the dual function, coupled with a change of scale and a time-change. We give a construction of a time-homogeneous process consistent with put prices, which we assume to satisfy the no-arbitrage conditions, but otherwise has no regularity properties.

One should perhaps note that in reality put prices are only given in the market for a discrete set of strike prices. Therefore, as a first step one needs to interpolate between the strikes. If a stock price is modeled as the solution to a stochastic differential equation with a continuous volatility function, then the perpetual put price exhibits certain regularity properties with respect to the strike price. Therefore, if one aims at recovering a continuous volatility, then one has to use an interpolation method that produces option prices exhibiting this regularity. On the other hand, if a linear spline method is used, then a continuous volatility cannot be recovered. This is one of the motivations for searching for price processes which are consistent with a general perpetual put price function (which is convex but may be neither strictly convex, nor smooth).

Whilst preparing this manuscript we came across a preprint by Alfonsi and Jourdain [2]. One of the aims of [2], as in this article, is to construct a time-homogeneous process which is consistent with observed put prices. However the method is different, and considerably less direct. Alfonsi and Jourdain [2] build on their previous work [1] to construct a parallel model such that the put price function in the original model (expressed as a function of strike) becomes a call price function expressed as a function of the initial value of the stock. They then solve the perpetual pricing problem for this parallel model, and subject to solving a differential equation for the optimal exercise boundaries in this model, give an analytic formula for the volatility coefficient. In contrast, the approach in this paper is much simpler, and unlike the method of Alfonsi and Jourdain extends to the irregular case.

2. The Forward Problem

Assume that the stock price process X is modeled under the pricing measure as the solution to the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma(X_t)X_t\,dW_t, \qquad X_0 = x_0.$$

Here the interest rate r is a positive constant, the level-dependent volatility $\sigma : (0,\infty) \to (0,\infty)$ is a given continuous function, and W is a standard Brownian motion. We assume that the stock pays no dividends, and we let zero be an absorbing barrier for X. If the current stock price is x_0, then the price of a perpetual put option with strike price $K > 0$ is

$$\hat{P}(K) = \sup_\tau \mathbb{E}^{x_0}\left[e^{-r\tau}(K - X_\tau)^+\right], \tag{1}$$

where the supremum is taken over random times τ that are stopping times with respect to the filtration generated by W. From the boundedness, monotonicity and convexity of the payoff we have:

Proposition 2.1. *The function $\hat{P} : [0,\infty) \to [0,\infty)$ satisfies*

(i) $(K - x_0)^+ \le \hat{P}(K) \le K$ *for all* K.

(ii) $\hat{P}$ *is non-decreasing and convex on* $[0,\infty)$.

Example. If σ is constant, i.e. if X is a geometric Brownian motion, then

$$\hat{P}(K) = \begin{cases} \frac{K}{\beta+1}(\beta K/x_0(\beta+1))^\beta & \text{if } K < \hat{K} \\ K - x_0 & \text{if } K \ge \hat{K}, \end{cases} \tag{2}$$

where $\beta = 2r/\sigma^2$ and $\hat{K} = x_0(\beta+1)/\beta$.

Intimately connected with the solution of the optimal stopping problem (1) is the ordinary differential equation

$$\frac{1}{2}\sigma(x)^2x^2u_{xx} + rxu_x - ru = 0. \tag{3}$$

This equation has two linearly independent positive solutions which are uniquely determined (up to multiplication with positive constants) if one requires one of them to be increasing and the other decreasing, see for example Borodin and Salminen [4, p18]. We denote the increasing solution by $\hat\psi$ and the decreasing one by $\hat\varphi$. In the current setting, $\hat\psi$ and $\hat\varphi$ are given by

$$\hat\psi(x) = x$$

and

$$\hat\varphi(x) = x\int_x^\infty \frac{1}{y^2}\exp\left\{-\int_c^y \frac{2r}{z\sigma(z)^2}\,dz\right\}dy \tag{4}$$

for some arbitrary constant c. For simplicity, and without loss of generality, we choose c so that $\hat\varphi(x_0) = 1$. It is well-known that with $H_z = \inf\{t \geq 0 : X_t = z\}$,

$$\mathbb{E}^x\left[e^{-rH_z}\right] = \begin{cases} \hat\varphi(x)/\hat\varphi(z) & \text{if } z < x \\ \hat\psi(x)/\hat\psi(z) & \text{if } z > x. \end{cases} \tag{5}$$

(This result is easy to check by considering $e^{-rt}\hat\varphi(X_t)$ and $e^{-rt}\hat\psi(X_t)$ which, since they involve solutions to (3), are local martingales.)

Given the assumed time-homogeneity of the process X, it is natural to consider stopping times in (1) that are hitting times. Define

$$\begin{aligned} \tilde P(K) &:= \sup_{z:z\leq x_0\wedge K} \mathbb{E}^{x_0}\left[e^{-rH_z}(K - X_{H_z})^+\right] \\ &= \sup_{z:z\leq x_0\wedge K} (K-z)\mathbb{E}^{x_0}\left[e^{-rH_z}\right] \\ &= \sup_{z:z\leq x_0\wedge K} \frac{K-z}{\hat\varphi(z)}, \end{aligned}$$

where the last equality follows from (5). By differentiating (4) we find

$$\hat\varphi'(x_0) = \exp\left\{-\int_c^{x_0}\frac{2r}{z\sigma(z)^2}\,dz\right\}\int_{x_0}^\infty \frac{1}{y^2}\left(\exp\left\{-\int_{x_0}^y \frac{2r}{z\sigma(z)^2}\,dz\right\}-1\right)dy, \tag{6}$$

so $\hat\varphi'(x_0) < 0$. Define

$$\hat K = x_0 - 1/\hat\varphi'(x_0).$$

Then, for $K \leq \hat K$ we have

$$\tilde P(K) = \sup_{z:z\leq K}\frac{K-z}{\hat\varphi(z)} = \sup_z \frac{K-z}{\hat\varphi(z)}, \tag{7}$$

whereas for $K > \hat K$ we have $\tilde P(K) = (K - x_0)$.

Clearly, $\hat P(K) \geq \tilde P(K)$, and of course, as we show below, there is equality.

Lemma 2.2. *The functions $\hat P$ and $\tilde P$ coincide, i.e.*

$$\hat P(K) = \sup_{z:z\leq x_0}\frac{K-z}{\hat\varphi(z)}. \tag{8}$$

Proof. As noted above, clearly $\hat P \geq \tilde P$ since the supremum over all stopping times is at least as large as the supremum over first hitting times.

For the reverse implication, suppose first that $K \leq \hat K$. In that case $\hat\varphi(z) \geq (K-z)^+/\tilde P(K)$ by (7). Further, $e^{-rt}\hat\varphi(X_t)$ is a non-negative local martingale and hence a supermartingale. Thus, for any stopping time τ we have

$$1 \geq \mathbb{E}^{x_0}[e^{-r\tau}\hat{\varphi}(X_\tau)] \geq \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+/\tilde{P}(K)].$$

Hence $\tilde{P}(K) \geq \sup_\tau \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+] = \hat{P}(K)$.

Finally, let $K > \hat{K}$. It follows from the first part that $\hat{P}(\hat{K}) = \hat{K} - x_0$, so Proposition 2.1 implies that $\hat{P}(K) = K - x_0 = \tilde{P}(K)$, which finishes the proof. □

Example. If σ is constant, i.e. if X is a geometric Brownian motion, then

$$\hat{\varphi}(x) = \left(\frac{x_0}{x}\right)^\beta,$$

where $\beta = 2r/\sigma^2$. Consequently, the put option price is given by

$$\hat{P}(K) = x_0^{-\beta} \sup_{z:z\leq x_0} (K - z)z^\beta.$$

Straightforward differentiation shows that the supremum is attained for

$$z = z^* := \frac{\beta K}{\beta + 1}$$

if $z^* < x_0$, and for $z = x_0$ if $z^* \geq x_0$. Consequently $\hat{P}(K)$ is given by (2).

Proposition 2.3. *There exists a $\hat{K} \in (0, \infty)$ such that $\hat{P}(K) > (K - x_0)^+$ for all $K \in (0, \hat{K})$ and $\hat{P}(K) = K - x_0$ for all $K \geq \hat{K}$.*

Proof. Since σ is continuous and positive, the function $\hat{\varphi}(x)$ given by (4) is twice continuously differentiable with a strictly positive second derivative. Therefore, for each fixed K there exists a unique $z = z(K) \leq x_0$ for which the supremum in Lemma 2.2 is attained, i.e.

$$\hat{P}(K) = \frac{K - z(K)}{\hat{\varphi}(z(K))}. \tag{9}$$

Geometrically, $z = z(K)$ is the unique value which makes the (negative) slope of the line through $(K, 0)$ and $(z, \hat{\varphi}(z))$ as large as possible. Algebraically this means that $z = z(K)$ and K satisfy $z(K) = x_0$ if $K \geq \hat{K}$ where

$$\hat{K} = x_0 - 1/\hat{\varphi}'(x_0),$$

and

$$(K - z)\hat{\varphi}'(z) + \hat{\varphi}(z) = 0 \tag{10}$$

if $K < \hat{K}$. Therefore, if $K \geq \hat{K}$, then

$$\hat{P}(K) = \frac{K - x_0}{\hat{\varphi}(x_0)} = K - x_0,$$

and if $K < \hat{K}$, then

$$\hat{P}(K) = \frac{K - z}{\hat{\varphi}(z)} > K - x_0.$$

□

Proposition 2.4. *Suppose $\sigma : (0, \infty) \to (0, \infty)$ is continuous. In addition to the properties in Propositions 2.1 and 2.3, the following statements about the function $\hat{P} : [0, \infty) \to [0, \infty)$ hold.*

(i) $\hat{P}$ *is continuously differentiable on* $(0,\infty)$*, and twice continuously differentiable on* $(0,\infty)\setminus\{\hat{K}\}$.
(ii) $\hat{P}$ *is strictly increasing on* $(0,\infty)$ *with a strictly positive second derivative on* $(0,\hat{K})$.

Proof. It follows from (10) and the implicit function theorem that $z(K)$ is continuously differentiable for $K<\hat{K}$. Therefore, differentiating (9) gives

$$\hat{P}'(K) = \frac{(1-z'(K))\hat{\varphi}(z(K))-(K-z(K))z'(K)\hat{\varphi}'(z(K))}{(\hat{\varphi}(z(K)))^2} = \frac{1}{\hat{\varphi}(z(K))}, \tag{11}$$

where the second equality follows from (10). Equation (11) shows that $\hat{P}'(\hat{K}-)=1/\hat{\varphi}(x_0)=1$, so $\hat{P}$ is C^1 also at $\hat{K}$. Moreover, since $\hat{\varphi}(z)$ is C^1 and $z(K)$ is C^1 away from $\hat{K}$, it follows that $\hat{P}(K)$ is C^2 on $(0,\infty)\setminus\{\hat{K}\}$. In fact, for $K<\hat{K}$ we have

$$\hat{P}''(K) = \frac{-z'(K)\hat{\varphi}'(z(K))}{(\hat{\varphi}(z(K)))^2} = \frac{(\hat{\varphi}'(z(K)))^2}{(K-z(K))(\hat{\varphi}(z(K)))^2\hat{\varphi}''(z(K))} > 0,$$

where the second equality follows by differentiating (10). Thus $\hat{P}$ has a strictly positive second derivative on $(0,\hat{K})$, which finishes the proof. □

Remark. Note that $\hat{P}'(0+)\geq 0$ with equality if and only if $\hat{\varphi}(0+)=\infty$.

We end this section by showing that $\hat{\varphi}$ can be recovered directly from the put option prices $\hat{P}(K)$, at least on the domain $(0,x_0]$. To do this, define the function $\varphi:(0,x_0]\to(0,\infty)$ by

$$\varphi(z)=\sup_{K:K\geq z}\frac{K-z}{\hat{P}(K)}, \tag{12}$$

where $\hat{P}$ is given by (8).

Lemma 2.5. (i) *Suppose* $f:(0,z_0]\to[1,\infty]$ *is a non-negative, decreasing convex function on* $(0,z_0]$ *with* $f(z_0)=1$ *and* $f'(z_0)<0$. *Define* $g:(0,\infty)\to[0,\infty)$ *by*

$$g(k)=\sup_{z:z\leq z_0}\frac{k-z}{f(z)}. \tag{13}$$

Then $g(k)$ *is a non-negative, non-decreasing convex function with* $(k-z_0)^+\leq g(k)\leq k$ *and* $g(k)=k-z_0$ *for* $k\geq k^*=z_0-1/f'(z_0)$.

(ii) f *and* g *are self-dual in the sense that if for* $z\leq z_0$ *we define*

$$F(z)=\sup_{k:k\geq z}\frac{k-z}{g(k)},$$

then $F\equiv f$ *on* $(0,z_0]$.

(iii) *Similarly, assume that* $g : (0, \infty) \to [0, \infty)$ *is a non-negative, non-decreasing convex function with* $(k - z_0)^+ \leq g(k) \leq k$ *and* $g(k) = k - x_0$ *for* $k \geq k^*$*, and define*

$$f(z) = \sup_{k:k \geq z} \frac{k - z}{g(k)}$$

for $z \leq z_0$. *Then* g *can be recovered by*

$$g(k) = \sup_{z:z \leq z_0} \frac{k - z}{f(z)}.$$

Proof. See Appendix A.1. □

Corollary 2.6. *The function* φ *coincides with the decreasing fundamental solution* $\hat{\varphi}$ *on* $(0, x_0]$.

3. The Inverse Problem: The Regular Case

Now consider the inverse problem. Let $P(K)$ be observed perpetual put prices for all nonnegative values of the strike K. The idea is that since $\hat{\varphi}$ satisfies the Black-Scholes equation (3), Corollary 2.6 provides a way to recover the volatility $\sigma(x)$ for $x \in (0, x_0]$ from perpetual put prices. In this section we provide the details in the case when the observed put prices are sufficiently regular. We assume that the observed put option price $P : [0, \infty) \to [0, \infty)$ satisfies the following conditions, compare Propositions 2.1, 2.3 and 2.4 above.

Hypothesis 3.1. (i) $(K - x_0)^+ \leq P(K) \leq K$ *for all* K.

(ii) *There exists a strike price* K^* *such that* $P(K) > (K - x_0)^+$ *for all* $K < K^*$ *and* $P(K) = K - x_0$ *for all* $K \geq K^*$.

(iii) P *is continuously differentiable on* $(0, \infty)$*, and twice continuously differentiable on* $(0, \infty) \setminus \{K^*\}$.

(iv) P *is strictly increasing on* $(0, \infty)$ *with a strictly positive second derivative on* $(0, K^*)$.

Motivated by Corollary 2.6, define the function $\varphi : (0, x_0] \to (0, \infty)$ by

$$\varphi(z) = \sup_{K:K \geq z} \frac{K - z}{P(K)}. \tag{14}$$

Proposition 3.2. *The function* P *can be recovered from* φ *by*

$$P(K) = \sup_{z:z \leq z_0} \frac{K - z}{\varphi(z)}.$$

Proof. This is a consequence of part (iii) of Lemma 2.5. □

Proposition 3.3. *The function* $\varphi : (0, x_0] \to (0, \infty)$ *is twice continuously differentiable with a positive second derivative, and it satisfies* $\varphi(x_0) = 1$ *and* $\varphi'(x_0) = -1/(K^* - x_0)$.

Proof. For each $z \leq x_0$ there exists a unique $K = K(z) \in (z, K^*]$ for which the supremum in (14) is attained. Geometrically, K is the unique value which minimises the slope of the line through $(z, 0)$ and $(K, P(K))$. Clearly, $K = K(z)$ satisfies the relation

$$(K - z)P'(K) = P(K). \tag{15}$$

Reasoning as in the proof of Proposition 2.4 one finds that $K(z)$ is continuously differentiable on $(0, x_0]$, with

$$\varphi'(z) = \frac{-1}{P(K(z))} \tag{16}$$

and $\varphi'(x_0) = -1/(K^* - x_0)$. Differentiating (16) with respect to z gives

$$\varphi''(z) = \frac{K'(z)P'(K(z))}{P^2(K(z))} = \frac{(P'(K(z)))^2}{(K(z)-z)P^2(K(z))P''(K(z))}, \tag{17}$$

where the second equality follows by differentiating (15). It follows that $\varphi''(z)$ is continuous and positive, which finishes the proof. □

Next, define $\sigma(x)^2$ for $x < x_0$ so that φ is a solution to the corresponding Black-Scholes equation, i.e.

$$\sigma(x)^2 = 2r\frac{\varphi(x) - x\varphi'(x)}{x^2\varphi''(x)}. \tag{18}$$

Moreover, define $\sigma(x)^2$ for $x \geq x_0$ to be the constant

$$\sigma^2 = 2r(K^* - x_0)/x_0. \tag{19}$$

Now, given this volatility function $\sigma(\cdot)$, we are in the situation of Section 2, and can thus define $\hat{\varphi}$ to be the decreasing fundamental solution to the corresponding Black-Scholes equation scaled so that $\hat{\varphi}(x_0) = 1$. Moreover, let $\hat{P}(K)$ be the corresponding perpetual put option price as given by (8).

Theorem 3.4. *Assume that Hypothesis 3.1 holds. Then the functions $\hat{P}$ and P coincide. Consequently, the volatility $\sigma(x)$ defined by* (18)-(19) *solves the inverse problem.*

Proof. First recall that any solution to (3) can be written as a linear combination of $\hat{\psi}$ and $\hat{\varphi}$. Consequently, it follows from the definition of σ that

$$\varphi(x) = C_1\hat{\varphi}(x) + C_2 x \tag{20}$$

for $x \in (0, x_0]$ and some constants C_1 and C_2. The boundary condition $\varphi(x_0) = 1$ yields

$$C_1 + C_2 x_0 = 1. \tag{21}$$

Moreover, from (19) we have

$$\begin{aligned} x_0 \int_{x_0}^{\infty} \frac{1}{y^2} \exp\left\{-\int_{x_0}^{y} \frac{2r}{z\sigma(z)^2}\,dz\right\} dy & \\ = \; x_0 \int_{x_0}^{\infty} \frac{1}{y^2} \exp\left\{-\int_{x_0}^{y} \frac{x_0}{z(K^*-x_0)}\,dz\right\} dy & \\ = \; x_0^{K^*/(K^*-x_0)} \int_{x_0}^{\infty} y^{-2-x_0/(K^*-x_0)}\,dy = (K^*-x_0)/K^*, & \end{aligned}$$

so the requirement $\hat{\varphi}(x_0) = 1$ gives the explicit expression

$$\hat{\varphi}(x) = \frac{K^*}{K^* - x_0} x \int_x^{\infty} \frac{1}{y^2} \exp\left\{-\int_{x_0}^{y} \frac{2r}{z\sigma(z)^2}\,dz\right\} dy$$

for the decreasing fundamental solution. Consequently,

$$\begin{aligned}\hat{\varphi}'(x_0) &= \frac{K^*}{K^*-x_0}\left(\int_{x_0}^{\infty}\frac{1}{y^2}\exp\left\{-\int_{x_0}^{y}\frac{2r}{z\sigma(z)^2}\,dz\right\}dy - 1/x_0\right)\\ &= \frac{K^*}{K^*-x_0}\left(\frac{K^*-x_0}{K^*x_0}-\frac{1}{x_0}\right) = -1/(K^*-x_0) = \varphi'(x_0)\end{aligned}$$

by Proposition 3.3. It therefore follows from (20) that $(1-C_1)\varphi'(x_0) = C_2$. Since $\varphi'(x_0) < 0$, (21) gives that $C_1 = 1$ and $C_2 = 0$, so $\varphi = \hat{\varphi}$ on $(0, x_0]$. Proposition 3.2 then yields

$$\hat{P}(K) = \sup_{z:z\leq x_0}\frac{K-z}{\hat{\varphi}(z)} = \sup_{z:z\leq x_0}\frac{K-z}{\varphi(z)} = P(K),$$

which finishes the proof. □

Remark. The inverse problem does not have a unique solution. Indeed, σ can be defined arbitrarily for $x > x_0$ as long as the condition

$$\int_{x_0}^{\infty}\frac{1}{y^2}\exp\left\{-\int_{x_0}^{y}\frac{2r}{z\sigma(z)^2}\,dz\right\}dy = (K^*-x_0)/(K^*x_0)$$

is satisfied, which guarantees that φ is C^1 at the point x_0.

We next show how to calculate the volatility that solves the inverse problem directly from the observed option prices $P(K)$. To do that, note that for each fixed z, the supremum in (14) is attained at some $K = K(z)$ for which

$$\varphi(z) = \frac{K-z}{P(K)}, \tag{22}$$

$$\varphi'(z) = \frac{-1}{P(K)} \tag{23}$$

and

$$\varphi''(z) = \frac{(P'(K))^2}{(K-z)P^2(K)P''(K)}, \tag{24}$$

compare (16) and (17). Since φ satisfies the Black-Scholes equation, we get

$$\sigma(z)^2z^2 = 2r\frac{\varphi(z)-z\varphi'(z)}{\varphi''} = \frac{2rKP^2(K)P''(K)}{(P'(K))^3}. \tag{25}$$

Consequently, to solve the inverse problem one first determines z by

$$z = K - \frac{P(K)}{P'(K)},$$

and then for this z one determines $\sigma(z)$ from (25).

4. The inverse problem: the irregular case

Again, suppose we are given perpetual put prices $P(K)$ and an interest rate $r > 0$. Our goal is to construct a time-homogeneous process which is consistent with the given prices. Unlike in the regular case discussed in Section 3, we now impose no regularity assumptions on the function P, beyond the necessary conditions stated in Propositions 2.1 and 2.3. For a discussion of the necessity of the condition in Proposition 2.3, see Section 9.1.

Hypothesis 4.1. (i) *For all K we have $(K - x_0)^+ \le P(K) \le K$.*
(ii) *P is non-decreasing and convex.*
(iii) *There exists $K^* \in (x_0, \infty)$ such that $P(K) = K - x_0$ for $K \ge K^*$.*

Given $P(K)$ we define $\underline{K} \in [0, x_0]$ by $\underline{K} = \sup\{K : P(K) = 0\}$. Then, necessarily, $P(K) \equiv 0$ for $K \le \underline{K}$.

Theorem 4.2. *Given $P(K)$ satisfying Hypothesis 4.1, and given $r > 0$, there exists a right-continuous (for $t > 0$), time-homogeneous Markov process X_t with $X_0 = x_0$ such that*

$$\sup_{\tau} \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+] = P(K) \qquad \forall K > 0$$

and such that $(e^{-rt}X_t)_{t\ge 0}$ is a local martingale.

Remark. Although we wish to work in the standard framework with right-continuous processes, in some circumstances we have to allow for an immediate jump. We do this by making the process right-continuous, except perhaps at $t = 0$. At $t = 0$ we allow a jump subject to the martingale condition $\mathbb{E}[X_0] = x_0$.

To exclude a completely degenerate case, henceforth we assume $P(x_0) > 0$. If $P(x_0) = 0$ then necessarily, to preclude arbitrage, $P(K) = (K - x_0)^+$ and $X_t = x_0 e^{rt}$ is consistent with the prices $P(K)$. In this case $\tau \equiv 0$ is an optimal stopping time for every K.

Given $P(K)$ satisfying Hypothesis 4.1 and such that $P(x_0) > 0$, define φ by

$$\varphi(x) = \sup_{K:K\ge x} \frac{K - x}{P(K)} \tag{26}$$

for $x \in (0, x_0]$. By Lemma 2.5, $\varphi : (0, x_0] \to [1, \infty]$ is a convex, decreasing, non-negative function with $\varphi(x_0) = 1$. Further,

$$\varphi(x_0) - \varphi(x_0 - \epsilon) \le 1 - \frac{K^* - x_0 + \epsilon}{K^* - x_0} = \frac{-\epsilon}{K^* - x_0},$$

so $\varphi'(x_0) \le -1/(K^* - x_0) < 0$. For some values of x, the supremum in (26) may be infinite since P may vanish on a non-empty interval $(0, \underline{K}]$. We define

$$\underline{x} = \inf\{x > 0 : \varphi(x) < \infty\},$$

and in the case where $\underline{x} > 0$ we see that $\varphi(x) = \infty$ for $x < \underline{x}$. In fact $\underline{x} > 0$ if and only if $\underline{K} > 0$, and then it is easy to see that these two quantities are equal.

We extend the definition of φ to (x_0, ∞) in any way which is consistent with the convexity, monotonicity and non-negativity properties and such that $\lim_{x\to\infty} \varphi(x) = 0$. It is convenient to use $\varphi(x) = (x/x_0)^{\varphi'(x_0-)x_0}$, for then $\varphi'(x)$ is continuous at x_0, and φ is twice continuously differentiable and positive on (x_0, ∞).

Given φ, define $s : (\underline{x}, \infty) \mapsto (-\infty, \infty)$ via $s'(x) = \varphi(x) - x\varphi'(x)$, with $s(x_0) = 0$. Then s is a concave, increasing function, which is continuous on $(\underline{x}, \infty)$. (It will turn out that s is the scale function, which explains the choice of label.) The function s has a well-defined inverse $g : (s(\underline{x}), s(\infty)) \to (\underline{x}, \infty)$, and if $s(\underline{x}) > -\infty$ then we extend the definition of g so that $g(y) = \underline{x}$ for

$y \leq s(\underline{x})$. Note that $g : (-\infty, s(\infty)) \to (\underline{x}, \infty)$ is a convex, increasing function with $g(0) = x_0$. Also define $f(y) = \varphi(g(y))$. Then f is decreasing and convex, with $f(0) = \varphi(x_0) = 1$.

Example. For geometric Brownian motion, provided $\beta \neq 1$ we have

$$s(x) = x_0^\beta (x^{1-\beta} - x_0^{1-\beta})(1+\beta)/(1-\beta),$$

$$g(y) = x_0 \left[1 + \frac{y(1-\beta)}{x_0(1+\beta)} \right]^{1/(1-\beta)}$$

and

$$f(y) = \left[1 + \frac{y(1-\beta)}{x_0(1+\beta)} \right]^{-\beta/(1-\beta)}.$$

If $\beta = 1$ the corresponding formulae are $s(x) = 2x_0 \ln(x/x_0)$, $g(y) = x_0 e^{y/(2x_0)}$ and $f(y) = e^{-y/(2x_0)}$.

Remark. Recall that a scale function is only determined up to a linear transformation. The choice $s(x_0) = 0$ is arbitrary but extremely convenient, as it allows us to start the process Z, defined below, at zero. The choice $s'(x) = \varphi(x) - x\varphi'(x)$ is simple, but a case could be made for the alternative normalisation $s'(x) = (\varphi(x) - x\varphi'(x))/(1 - x_0\varphi'(x_0))$ for which $s'(x_0) = 1$. Multiplying s by a constant has the effect of modifying the construction defined in the next section, but only by the introduction of a constant factor into the time-changes. It is easy to check that this leaves the final model X_t unchanged.

Our goal is to construct a time-homogeneous process which is consistent with observed put prices, and such that $e^{-rt}X_t$ is a (local) martingale. In the regular case we have seen how to construct a diffusion with these properties. Now we have to allow for more general processes, perhaps processes which jump over intervals, or perhaps processes which have 'sticky' points. One very powerful construction method for time-homogeneous, martingale diffusions is via a time-change of Brownian motion, and it is this approach which we exploit.

5. Constructing Time-homogeneous Processes as Time-changes of Brownian Motion

In this section we extend the construction in Rogers and Williams [12, Section V.47] of martingale diffusions as time-changes of Brownian motion, see also Itô and McKean [8, Section 5.1]. The difference from the classical setting is that the processes defined below may have 'sticky' points and may jump over intervals. Since diffusions by definition are continuous, the resulting processes are not diffusions, but one might think of them as 'extended diffusions', and they are 'as continuous as possible'.

Let ν be a measure on $\mathbb{R}$ and let $\mathbb{F}^B = (\mathcal{F}_u^B)_{u \geq 0}$ be a filtration supporting a Brownian motion B started at 0 with local time process L_u^z. Define Γ to be the left-continuous increasing additive functional

$$\Gamma_u = \int_{\mathbb{R}} L_u^z \nu(dz), \qquad \Gamma_0 = 0 \tag{27}$$

and let A be the right-continuous inverse to Γ, i.e.

$$A_t = \inf\{u : \Gamma_u > t\}.$$

Note that Γ is a non-decreasing process, so that A is well defined, and A_t is an $\mathbb{F}^B$-stopping time for each time t. Set $Z_0 = 0$ and for $t > 0$ set $Z_t = B_{A_t}$ and $\mathcal{F}_t = \mathcal{F}^B_{A_t}$. Note that Z is right-continuous except perhaps at $t = 0$. The process Z_t is a time-changed Brownian motion adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and subject to mild non-degeneracy conditions on ν, (see Lemma 5.1 below), the processes Z_t and $Z_t^2 - A_t$ are local martingales. Further, if $\nu(dy) = dy/\gamma(y)^2$ then $\Gamma_u = \int_0^u \gamma(B_r)^{-2} dr$ and $A_t = \int_0^t \gamma(Z_s)^2 ds$, so that Z_t is a weak solution to $dZ_t = \gamma(Z_t)dW_t$, and Z is a diffusion in natural scale. The measure ν is called the speed measure of Z, although, as pointed out by Rogers and Williams, ν is large when Z moves slowly.

The measure ν may have atoms, and it may have intervals on which it places no mass. If there is an atom at $\hat{z}$ then $d\Gamma_u/du > 0$ whenever $B_u = \hat{z}$, and then the time-changed process is 'sticky' there. Conversely, if ν places no mass in (α, β) then Γ is constant on any time-periods that B spends in this interval, and the inverse time-change A has a jump. In particular, Z_t spends no time in this interval. If $\nu(\{\tilde{z}\}) = \infty$ then $\Gamma_u = \infty$ for any u greater than the first hitting time $H^B_{\tilde{z}}$ by B of level $\tilde{z}$. Then $A_\infty \leq H^B_{\tilde{z}}$ so that if Z hits $\tilde{z}$, then $\tilde{z}$ is absorbing for Z. The other possibility is that Z tends to this level without reaching it in finite time.

Define $\overline{z}_\nu \in (0, \infty]$ and $\underline{z}_\nu \in [-\infty, 0)$ via

$$\overline{z}_\nu = \inf\{z > 0 : \nu((0, z]) = \infty\} \quad \text{and} \quad \underline{z}_\nu = \sup\{z < 0 : \nu([z, 0)) = \infty\}.$$

The cases where $\overline{z}_\nu = 0$ or $\underline{z}_\nu = 0$ correspond to the degenerate case $X_t = x_0 e^{rt}$ mentioned in the previous section, and we exclude them. The following lemma provides a guide to sufficient conditions for a time-change of Brownian motion to be a local martingale, and therefore provides insight into the constructions of local martingales via time-change that we develop in the next section.

Lemma 5.1. *Suppose that either $\overline{z}_\nu < \infty$ or ν charges (a, ∞) for each a, and further that either $\overline{z}_\nu > -\infty$ or ν charges $(-\infty, a)$ for each a. Then $Z_t = B_{A_t}$ is a local martingale.*

Proof. See Appendix A.2. □

6. Constructing the model

We now show how to choose the measure ν which gives the process we want. Define ν via

$$\nu(dy) = \frac{1}{2r}\left(\frac{g''(y)}{g(y)}\right) dy, \tag{28}$$

and let $\nu(dy) = \infty$ for $y < s(0)$ in the case when $s(0) > -\infty$. Similarly, in the case when $s(\infty) < \infty$ we set $\nu(dy) = \infty$ for $y > s(\infty)$. Where g' is absolutely continuous it follows that ν has a density with respect to Lebesgue measure, but more generally (28) can be interpreted in a distributional sense.

Now, for this ν we can use the construction of the previous section to give a process Z_t. If we then set $X_t = g(Z_t)$, then, subject to the hypotheses of

Lemma 5.1, $Z_t = s(X_t)$ is a local martingale, so that s is a scale function for X. The process X is our candidate process for which the associated put prices are given by P.

Example. For geometric Brownian motion,

$$\frac{\nu(dy)}{dy} = \frac{1}{2r}\frac{g''(y)}{g(y)} = \frac{1}{2rx_0^2}\frac{\beta}{(1+\beta)^2}\left[1+\frac{y(1-\beta)}{x_0(1+\beta)}\right]^{-2}.$$

In the case $\beta = 1$ this simplifies to

$$\frac{\nu(dy)}{dy} = \frac{1}{8x_0^2 r} = \frac{1}{C^2}$$

where $C = 2x_0\sqrt{2r}$. Then $\Gamma_u = uC^{-2}$; $A_t = tC^2$; $Z_t = B_{tC^2} \overset{\mathcal{D}}{=} C\tilde{B}_t$ for a Brownian motion $\tilde{B}$ and $X_t = x_0 e^{Z_t/2x_0} = x_0 e^{\sqrt{2r}\tilde{B}_t}$.

Recall that $\Gamma_u = \int_{\mathbb{R}} L_u^z \nu(dz)$ and let ξ be the first explosion time of Γ. Note that by construction Γ is continuous for $t < \xi$, and left continuous at $t = \xi$. Since ν is infinite outside the interval $[s(0), s(\infty)]$ we also have the expression $\xi = \inf\{u : B_u \notin [s(0), s(\infty)]\} = H^B_{s(0)} \wedge H^B_{s(\infty)}$. The inverse scale function g is concave on $(s(0), s(\infty))$, but may have a jump (from a finite to an infinite value) at $s(\infty)$. In that case we take it left-continuous at $s(\infty)$ so that we may have $\bar{g} := \lim_{z\uparrow\infty} g(s(z))$ is finite.

For $0 < u < \xi$, define $M_u = e^{-r\Gamma_u} g(B_u)$ and $N_u = e^{-r\Gamma_u} f(B_u)$.

Lemma 6.1. *$M = (M_u)_{0\le u<\xi}$ and $N = (N_u)_{0\le u<\xi}$ are $\mathbb{F}^B$-local martingales.*

Sketch of proof: Suppose that φ is twice continuously differentiable with a positive second derivative. Then g is twice continuously differentiable. For $u < \xi$, applying Itô's formula to $M_u = e^{-r\Gamma_u} g(B_u)$ gives

$$e^{r\Gamma_u} dM_u = g'(B_u)dB_u + \left[-r\frac{d\Gamma_u}{du} g(B_u) + \frac{1}{2}g''(B_u)\right] du.$$

But, by definition, $d\Gamma_u/du = g''(B_u)/(2rg(B_u))$, so M is a local martingale as required.

A similar argument can be provided for the process N. For the general case, see the Appendix. □

Since M and N are non-negative local martingales on $[0, \xi)$ they converge almost surely to finite values, which we label M_ξ and N_ξ. In particular, if $\xi = H^B_{s(0)}$, then $M_\xi = 0$. However, if $\xi = H^B_{s(\infty)}$ then there are several cases. The fact that a non-negative local martingale converges means that we cannot have both $\Gamma_\xi < \infty$ and $\bar{g} = \lim_{z\uparrow\infty} g(s(z)) = \infty$. Instead, if $\Gamma_\xi < \infty$ then $\bar{g} < \infty$ and $M_\xi = e^{-r\Gamma_\xi}\bar{g}$. If $\Gamma_\xi = \infty$, and $\bar{g} < \infty$ then $M_\xi = 0$, whereas if $\Gamma_\xi = \infty$ and $\bar{g} = \infty$ then $(M_u)_{u<\xi}$ typically has a non-trivial limit. Similar considerations apply to N.

Recall that A is the right-continuous inverse to Γ and define the time-changed processes $\tilde{M}_t = M_{A_t}$ and $\tilde{N}_t = N_{A_t}$. Note that these processes are adapted to $\mathbb{F}$ and that, at least for $t \le \Gamma_\xi$, we have $\Gamma_{A_t} = t$, $\tilde{M}_t = e^{-rt} g(Z_t) = e^{-rt} X_t$ and $\tilde{N}_t = e^{-rt} f(Z_t) = e^{-rt}\varphi(X_t)$.

If $(s(0), s(\infty)) = \mathbb{R}$ then $\xi = \infty$, $\Gamma_\xi = \infty$ and $\tilde{M}$ is defined for all t.

If $s(0) > -\infty$ then we may have $\xi = H^B_{s(0)}$. In this case either $\Gamma_\xi = \Gamma_{H^B_{s(0)}} = \infty$, whence $\tilde{M}$ is defined for all t as before, or $\Gamma_\xi < \infty$. Then $\tilde{M}_{\Gamma_\xi} = M_\xi = e^{-r\Gamma_\xi} g(B_\xi) = 0$, and we set $\tilde{M}_t = 0$ for all $t > \Gamma_\xi$. It follows that $X_t = 0$ for all $t \geq \Gamma_\xi$ and 0 is an absorbing state.

Similarly, if $s(\infty) < \infty$ then we may have $\xi = H^B_{s(\infty)}$. Then either $\Gamma_\xi = \infty$, whence $\tilde{M}$ is defined for all t, or $\Gamma_\xi < \infty$. In the latter case, if $\xi = H^B_{s(\infty)} < \infty$ and $\Gamma_\xi < \infty$ then $\tilde{M}_{\Gamma_\xi} = M_\xi = e^{-r\Gamma_\xi}\bar{g}$. We set $\tilde{M}_t = \tilde{M}_{\Gamma_\xi}$ for all $t > \Gamma_\xi$, and it follows that for $t > \Gamma_\xi$, $X_t := e^{rt}\tilde{M}_t = e^{r(t-\Gamma_\xi)}\bar{g}$. Thus, for $t > \Gamma_\xi$, X grows deterministically. An example of this situation is given in Example 8.4 below. (In fact, the case where $\bar{g} < \infty$, which depends on the behaviour of the scale function s to the right of x_0, can always be avoided by suitable choice of the extension to φ.)

We want to show how $\tilde{M}$ and X inherit properties from M. The key idea below is that, loosely speaking, a time-change of a martingale is again a martingale. Of course, to make this statement precise we need strong control on the time-change. (Without such control the resulting process can have arbitrary drift. Indeed, as Monroe [10] has shown, any semi-martingale can be constructed from Brownian motion via a time-change.) We have the following result the proof of which is given in the Appendix.

Corollary 6.2. *The process $(e^{-rt}X_t)_{t\geq 0}$ is a local martingale.*

We can perform a similar analysis on N and $\tilde{N}$ and use similar ideas to ensure that $\tilde{N}$ is defined on $\mathbb{R}_+$. The proof that $\tilde{N}$ is a local martingale mirrors that of Corollary 6.2.

Corollary 6.3. *The process $(e^{-rt}\varphi(X_t))_{t\geq 0}$ is a local martingale.*

7. Determining the put prices for the candidate process

Recall the definitions of s, g and ν via $s'(x) = \varphi(x) - x\varphi'(x)$, $g \equiv s^{-1}$ and $\nu(dy) = g''(y)/(2rg(y))dy$. Suppose that Z is constructed from ν and a Brownian motion using the time-change Γ, and construct the candidate price process via $X_t = g(Z_t)$. By Corollary 6.2, the discounted price $e^{-rt}X_t$ is a (local) martingale. To complete the proof of Theorem 4.2 we need to show that for the candidate process X_t the function

$$\hat{P}(K) := \sup_\tau \mathbb{E}[e^{-r\tau}(K - X_\tau)^+]$$

is such that $\hat{P}(K) \equiv P(K)$ for all $K \geq 0$.

Unlike the regular case the process X that we have constructed may have jumps. For this reason, for $x < x_0$ we modify the definition of the first hitting time so that $H_x = \inf\{u > 0 : X_u \leq x\}$.

Theorem 7.1. *The perpetual put prices for X are given by P.*

Proof: Fix $x \in (\underline{x}, x_0)$. Suppose first that x is such that Γ is strictly increasing whenever the Brownian motion B takes the value $s(x)$. Then $X_{H_x} = x$. More generally the same is true whenever $\nu((s(x) - \delta, s(x)]) > 0$ for every $\delta > 0$. By Corollary 6.3 we have that $(e^{-rt}\varphi(X_t))_{t\leq H_x}$ is a local martingale, and φ is bounded on $[x,\infty)$ so it follows that $e^{-r(t\wedge H_x)}\varphi(X_{t\wedge H_x})$ is a bounded martingale and $\varphi(x_0)=\mathbb{E}^{x_0}[e^{-rH_x}\varphi(x)]$. Hence,

$$\hat{P}(K)\geq \mathbb{E}^{x_0}[e^{-rH_x}(K-x)]=(K-x)\frac{\varphi(x_0)}{\varphi(x)}=\frac{K-x}{\varphi(x)}.$$

Otherwise, fix $x^-(x)=\inf\{w<x:\nu((s(w),s(x)])=0\}$ and $x^+(x)=\sup\{w>x:\nu([s(x),s(w))=0\}$. It must be the case that φ is linear on $(x^-(x),x^+(x))$ and bounded on $[x^-(x),\infty)$ and

$$\begin{aligned}\hat{P}(K) &\geq \max_{w\in\{x^-,x^+\}}\mathbb{E}^{x_0}[e^{-rH_w}(K-w)]\\ &= \max_{w\in\{x^-,x^+\}}\frac{K-w}{\varphi(w)}\geq\frac{K-x}{\varphi(x)}.\end{aligned}$$

It follows that

$$\hat{P}(K)\geq\sup_{x:x\leq x_0}\frac{K-x}{\varphi(x)}=P(K). \tag{29}$$

(Clearly, if $x<\underline{x}$ then $(K-x)/\varphi(x)=0$, so the supremum cannot be attained for such an x.)

To prove the reverse inequality, we first claim that the left derivative $D^-\varphi$ of the convex function φ satisfies

$$D^-\varphi(x_0):=\lim_{\epsilon\downarrow 0}\frac{\varphi(x_0)-\varphi(x_0-\epsilon)}{\epsilon}=\frac{-1}{K^*-x_0}. \tag{30}$$

To prove (30), first note that choosing $K=K^*$ in (26) yields $\varphi(x)\geq(K^*-x)/(K^*-x_0)$, so

$$D^-\varphi(x_0)=\lim_{x\uparrow x_0}\frac{\varphi(x_0)-\varphi(x)}{x_0-x}\leq\frac{-1}{K^*-x_0}.$$

Conversely, note that for each $\delta>0$ there exists a non-empty interval $(x_0-\epsilon,x_0)$ on which

$$\varphi(x)\leq\frac{K^*-\delta-x}{K^*-\delta-x_0}.$$

Consequently, for $x\in(x_0-\epsilon,x_0)$ we have

$$\frac{\varphi(x_0)-\varphi(x)}{x_0-x}\geq\frac{-1}{K^*-\delta-x_0}.$$

Thus $D^-\varphi(x_0)\geq -1/(K^*-x_0)$ since $\delta>0$ is arbitrary, so (30) follows.

We next claim that for each fixed $K\leq K^*$ we have

$$\varphi(x)\geq(K-x)^+/P(K) \tag{31}$$

for all x. Clearly, this holds for $x\geq K$ and for $x\leq x_0$. Similarly, if $x_0<x<K$, then it follows from (30) and the convexity of φ that

$$\varphi(x)\geq\frac{K^*-x}{K^*-x_0}\geq\frac{K-x}{K-x_0}\geq\frac{K-x}{P(K)}.$$

It follows from (31) and Corollary 6.3 that for any stopping rule τ we have

$$\mathbb{E}^{x_0}[e^{-r\tau}(K-X_\tau)^+]\leq P(K)\mathbb{E}^{x_0}[e^{-r\tau}\varphi(X_\tau)]\leq P(K)\varphi(x_0)=P(K).$$

Hence $\hat{P}(K)\leq P(K)$ for $K\leq K^*$, and in view of (29), $\hat{P}(K)=P(K)$.

For $K > K^*$ it follows from $\hat{P}(K^*) = P(K^*) = K^* - x_0$, the convexity of $\hat{P}$ and Hypothesis 4.1 that $\hat{P}(K) = K - x_0 = P(K)$, which finishes the proof. □

8. Examples

The following examples illustrate the construction of the previous sections. The list of examples is not intended to be exhaustive, but rather indicative of the types of behaviour that can arise. In each example we assume $x_0 = 1$.

8.1. **The smooth case.** We have studied the case of exponential Brownian motion throughout. It is very easy to generate other examples, for example by choosing a smooth decreasing convex function (with $\varphi(x_0) = 1$ and $\lim_{x\uparrow\infty} \varphi(x) = 0$), and defining other quantities from φ.

Example 8.1. Suppose $\varphi(x) = (x+1)/(2x^2)$. Then, from (3) we obtain

$$\sigma^2(x) = r\frac{2x+3}{x+3} \qquad x > 0$$

and, from (8)

$$P(K) = \frac{(K+9)^{3/2}(K+1)^{1/2} - (27+18K-K^2)}{4} \qquad K \leq 5/3$$

with $P(K) = (K-1)$ for $K \geq 5/3$.

8.2. **Kinks in P.** If the first derivative of P is not continuous then we find that φ is linear over an interval (α, β) say. Then s' is constant on this interval and g is linear over the interval $(s(\alpha), s(\beta))$. It follows that ν does not charge this interval, so that Γ_u is constant whenever $B_u \in (s(\alpha), s(\beta))$, and A_t has a jump. Then Z_t jumps over the interval $(s(\alpha), s(\beta))$, and X_t spends no time in (α, β).

Example 8.2. Suppose $P(K)$ satisfying Hypothesis 4.1 is given by

$$P(K) = \begin{cases} K^2/8 & 0 < K \leq 27/32 \\ 4K^3/27 & 27/32 \leq K \leq 3/2 \\ (K-1) & 3/2 \leq K. \end{cases}$$

Then P is continuous, but P' has a jump at $K = 27/32$.

Using (26) we find that

$$\varphi(x) = \begin{cases} 2x^{-1} & 0 < x \leq 27/64 \\ x^{-2} & x > 9/16. \end{cases}$$

Over the region $I = [27/64, 9/16]$ φ is given by linear interpolation. The corresponding scale function is linear on I, and in the construction of Z, ν assigns no mass to $s(I)$. The process X is a generalised diffusion with diffusion coefficient given by $\sigma(x) = \sqrt{2r}$ for $x \leq 27/64$, $\sigma(x) = \sqrt{r}$ for $x \geq 9/16$ (strictly speaking there is some freedom in the choice of σ for $x \geq x_0 \equiv 1$, but the constant value $\sqrt{r}$ is a natural choice) and $\sigma(x) = \infty$ for $x \in I$.

8.3. **Linear parts to P.** In this case the derivative of $\varphi(x)$ is discontinuous at a point γ say. Then s' is also discontinuous at this point, and g' is discontinuous at $s(\gamma)$. It follows that ν has a point mass at $s(\gamma)$, and that Γ_u includes a multiple of the local time at $s(\gamma)$.

Example 8.3. Suppose $P(K)$ satisfying Hypothesis 4.1 is given by

$$P(K) = \begin{cases} 8K^3/27 & 0 < K \leq 3/4 \\ (2K-1)/4 & 3/4 \leq K \leq 1 \\ K^2/4 & 1 \leq K \leq 2 \\ (K-1) & 2 \leq K. \end{cases}$$

Then P is convex, but is linear on the interval $[3/4, 1]$.

Then

$$\varphi(x) = \begin{cases} x^{-2}/2 & 0 < x \leq 1/2 \\ x^{-1} & x > 1/2 \end{cases}$$

where we have chosen to extend the definition of φ to $(1, \infty)$ in the natural way. Then $s(x) = 3 - 2\ln 2 - 3/2x$ for $x < 1/2$ and $s(x) = 2\ln x$ otherwise. It follows that g is everywhere convex but has a discontinuous first derivative at $z = -2\ln 2$, and that the corresponding measure ν has a positive density with respect to Lebesgue measure *and* an atom of size $r^{-1}/12$ at $-2\ln 2$. In the terminology of stochastic processes the process Z is 'sticky' at this point — for a discussion of sticky Brownian motion see Amit [3] or, for the one-sided case Warren [14].

If P is piecewise linear (for example if P is obtained by linear interpolation from a finite number of options) then φ is piecewise linear, s is piecewise linear, g is piecewise linear, and ν consists of a series of atoms. As a consequence the process Z_t is a continuous-time Markov process on a countable state-space (at least whilst $Z_t < s(x_0) \equiv 0$), in which transitions are to nearest neighbours only. Holding times in states are exponential and the jump probabilities are such that Z_t is a martingale.

In turn this means that X_t is a continuous-time Markov process on a countable set of points (at least whilst $X_t < x_0$).

Example 8.4. Suppose

$$P(K) = \begin{cases} K/3 & K \leq 1 \\ (2K-1)/3 & 1 \leq K \leq 2 \\ (K-1) & K \geq 2. \end{cases}$$

This is consistent with a situation in which only two perpetual American put options trade, with strikes 1 and 3/2 and prices 1/3 and 2/3, in which case we may assume that we have extrapolated from the traded prices to a put pricing function $P(K)$ which is consistent with the traded prices.

Consider the process X_t with state space $\{0\} \cup \{1/2\} \cup [2, \infty)$ and such that

- at $t = 0+$, X jumps to 1/2 or 2 with probabilities 2/3 and 1/3, respectively;
- if ever $X_{t_0} \geq 2$ then thereafter $X_t = X_{t_0} e^{r(t-t_0)}$;
- zero is an absorbing state for X;
- if ever X reaches 1/2 then it stays there for an exponential length of time, rate $3r$, and jumps to 2 with probability 1/3 and zero with probability 2/3.

Note that for the continuous time Markov process X_t, then conditional on $X_t = 1/2$,

$$\lim_{\Delta\downarrow 0} \frac{1}{\Delta}\mathbb{E}[X_{t+\Delta} - X_t] \sim 3r\left[\frac{1}{3}(2 - X_t) + \frac{2}{3}(-X_t)\right] = \frac{r}{2} = rX_t$$

Also, for this process

$$\begin{aligned} P(K) &= \max_{\tau=0,H_{1/2},H_0} \mathbb{E}[e^{-r\tau}(K - X_\tau)^+] \\ &= \max\{K-1, (2K-1)/3, K/3\} \end{aligned}$$

so we recover the put price function given at the start of the example.

8.4. **Positive gradient of P at zero, i.e. $P'(0) > 0$.** In this case $\lim_{x\downarrow 0} \varphi(x) < \infty$. It follows that $s(0) > -\infty$ and the resulting diffusion X_t can hit zero in finite time. In this case we insist that 0 is an absorbing endpoint for the diffusion.

Example 8.5. Suppose that

$$P(K) = \begin{cases} K/2 & K < 1 \\ (K+1)^2/8 & 1 \le K \le 3 \\ K - 1 & K \ge 3. \end{cases}$$

Then $\varphi(x) = 2(x+1)^{-1}$ and $x^2\sigma(x)^2 = r(x+1)(2x+1)$, so that $dX_t = rX_t + \sqrt{r(X_t+1)(2X_t+1)}\, dB_t$.

The following example covers the case of mixed linear and smooth parts to $P(K)$, and shows an example where reflection, local times and jumps all form part of the construction.

Example 8.6. Suppose $P(K)$ satisfies

$$P = \begin{cases} K/4 & K \le 1 \\ K^2/4 & 1 \le K \le 2 \\ (K-1) & K \ge 2. \end{cases}$$

Note that P' has a jump at $x = 1/2$. We have $\varphi(x) = 4 - 4x$ for $x < 1/2$ and $\varphi(x) = 1/x$ for $1/2 \le x \le 1$. We assume this formula applies on $[1, \infty)$ also.

It follows that for $x \ge 1/2$, $\eta(x)^2 \equiv (x\sigma(x))^2 = 2rx^2$. Note that since $\varphi(1/2) < \infty$ we have $H_{1/2}$ is finite (almost surely). Hence

$$dX_t = rX_t dt + \sqrt{2r} X_t dB_t \qquad t \le H_{1/2} \tag{32}$$

is consistent with the observed put prices, but we need to describe what happens when X hits $1/2$. Note that (32) has solution $X_t = e^{\sqrt{2r}B_t}$ for $t \le H_{1/2}$.

The process is not absorbed at $1/2$ as this would conflict with the idea that the discounted price process is a local martingale, or equivalently $dX_t = rX_t dt + dM_t$, for a local martingale M_t. However, adding a reflection alone would again destroy the martingale property of $e^{-rt}X_t$. What the general construction above does is that a local time reflection *and* a compensating downward jump are added. This jump takes the process to zero where it is absorbed.

Alternatively, the process can be formalised as follows. Let $I_t = -\inf_{u \leq t}\{(B_u + \ln 2/\sqrt{2r}) \wedge 0\}$. Then by Skorokhod's Lemma, $B_t + I_t$ is a reflected Brownian motion (reflected at the level $-(\ln 2/\sqrt{2r})$) and $e^{\sqrt{2r}(B_t+I_t)} \geq 1/2$.

Let N^λ be a Poisson Process rate λ, independent of B, and let T^λ be the first event time. Then the compensated Poisson Process $(N^\lambda_t - \lambda t)_{t\geq 0}$ and the compensated Poisson Process stopped at the first jump $(N^\lambda_{t\wedge T^\lambda} - \lambda(t \wedge T^\lambda))_{t\geq 0}$ are martingales. The time change $(N^\lambda_{I_t \wedge T^\lambda} - \lambda(I_t \wedge T^\lambda))_{t\geq 0}$, is also a martingale.

Take $\lambda = \sqrt{2r}$ and define X via $X_0 = 1$ and

$$dX_t = rX_t dt + \sqrt{2r}X_t \left(dB_t + dI_t - \frac{dN^{\sqrt{2r}}_{I_t}}{\sqrt{2r}} \right) \qquad t : I_t \leq T^{\sqrt{2r}}$$

Note that at the first jump time of the time-changed Poisson process, X jumps from 1/2 to zero.

By construction $(e^{-rt}X_t)_{t\geq 0}$ is a martingale.

8.5. P **is zero on an interval:** $P(K) = 0$ **for** $K \leq \underline{K}$**.** Now we find that $\varphi(x) = \infty$ for $x \leq \underline{x}$ where $\underline{x} = \underline{K}$. Depending on whether the right derivative $P'(\underline{K}+)$ is zero or positive, we may have $\varphi(\underline{x}+)$ is infinite or finite. In the former case we have that X_t does not reach $\underline{x}$ in finite time. In the latter case X_t does hit $\underline{x}$ in finite time.

The first example is typical of the case where $\varphi(\underline{x}+) = \infty$, or equivalently where there is smooth fit of P at $\underline{K}$.

Example 8.7. Suppose $X_0 = 1$ and that $P(K)$ solves

$$P(K) = \begin{cases} 0 & K \leq 1/2 \\ (2K-1)^2/8 & 1/2 \leq K \leq 3/2 \\ K-1 & K \geq 3/2. \end{cases}$$

Then P' is continuous and for $1/2 < x < 1$ we have $\varphi(x) = (2x-1)^{-1}$. We also have $\varphi(x) = \infty$ for $x \leq 1/2$. It does not matter how φ is continued on $(1, \infty)$ (provided it is decreasing and convex), but for definiteness we assume the formula $\varphi(x) = (2x-1)^{-1}$ applies there also.

It follows that $\eta(x)^2 \equiv (x\sigma(x))^2 = r(2x-1)(4x-1)/4$. Note that since $\varphi(1/2) = \infty$ we have $H_{1/2}$ (the first hitting time of 1/2) is infinite. Hence

$$dX_t = rX_t dt + \sqrt{\frac{r(2x-1)(4x-1)}{4}} dB_t \qquad t \leq H_{1/2}$$

is consistent with the observed put prices, and since the process never hits 1/2, it is not necessary to describe the process beyond $H_{1/2}$.

Now consider the other case where $P'(\underline{K}) > 0$.

Example 8.8. Suppose $X_0 = 1$ and that $P(K)$ solves

$$P(K) = \begin{cases} 0 & K \leq 1/2 \\ (2K-1)/4 & 1/2 \leq K \leq 1 \\ K^2/4 & 1 \leq K \leq 2 \\ K-1 & K \geq 2. \end{cases}$$

Then P' has a jump at $K = 1/2$.

We have $\varphi(x) = \infty$ for $x < 1/2$ and $\varphi(x) = 1/x$ for $1/2 \leq x \leq 1$. We assume this formula applies on $[1, \infty)$ also.

It follows that for $x \geq 1/2$, $\eta(x)^2 \equiv (x\sigma(x))^2 = 2rx^2$. As before we have

$$dX_t = rX_t dt + \sqrt{2r} X_t dB_t \qquad t \leq H_{1/2} \tag{33}$$

is consistent with the observed put prices, but we need to describe what happens when X hits $1/2$.

The probability that the process ever goes below $1/2$ is zero, else the put with strike $K = 1/2$ would have positive value. The process cannot be absorbed at $1/2$ as this would conflict with the idea that the discounted price process is a martingale, or equivalently $dX_t = rX_t dt + dM_t$, for a local martingale M_t. Instead, the time-change Γ includes a multiple of the local time at $s(1/2) = -2\ln 2$.

In fact g''/g is constant for $y > s(1/2)$ and equal to zero for $y < s(1/2)$. The process Γ_u is a linear combination of O^+_u and $L^{-2\ln 2}_u$, where O^+_u is the amount of time spent by the Brownian motion above $s(1/2)$ before time u. It is easy to check using Itô's formula that $e^{-r\Gamma_u} g(B_u)$ is a martingale in this case. The process $Z_t = B_{A_t}$ is 'sticky' at $s(1/2)$ (this time in the sense of a one-sided sticky Brownian motion, see Warren [14]) and this property is inherited by $X = s(Z)$.

There is a third case, where $\varphi(\underline{x}+) < \infty$, but $\varphi'(x+) = \infty$.

Example 8.9. Suppose $\varphi(x) = 2 - \sqrt{2x-1}$ for $1/2 \leq x \leq 5/2$. Equivalently,

$$P(K) = \begin{cases} 0 & K \leq 1/2 \\ 2 - \sqrt{5-2K} & 1/2 \leq K \leq 2 \\ K - 1 & K \geq 2. \end{cases}$$

Then for $1/2 < x < 5/2$ we have

$$\eta(x)^2 = (x\sigma(x))^2 = 2r(2x-1)\left(2\sqrt{2x-1} + 1 - x\right)$$

It follows that although X_t can hit $1/2$, the volatility at this level is zero, and the drift alone is sufficient to keep $X_t \geq 1/2$.

8.6. **Kink in P at K^*: $K^* < \infty$ and $P'(K^*-) < 1$.** In this case $\varphi'(x)$ is constant on an interval $(\hat{x}, x_0)$. This case is just like Section 8.2.

9. Extensions

9.1. **No options exercised immediately.** In Hypothesis 4.1, in addition to (i) and (ii) which are enforcible by no-arbitrage considerations, we also assumed (iii) that there exists a finite strike K^* such that for all strikes $K \geq K^*$ the put option is exercised immediately. Since $K^* < \infty$ is equivalent to $\varphi'(x_0) < 0$, it is apparent from the expression in (6), that provided σ is finite for some $x > x_0$, or equivalently ν gives mass to some $z > 0$, then this property will hold. However, it is interesting to consider what happens when this fails.

Suppose $P(K) > K - x_0$ for all K and $\lim_K P(K) - (K - x_0) = 0$. Then $\varphi'(x_0) = 0$, but φ is strictly decreasing on $(\underline{x}, x_0)$. The measure ν places no mass on $(0, \infty)$, the process Z_t spends no time on $(0, \infty)$ and X_t never takes values above x_0. In particular X_t is reflected (downwards) at x_0. The resulting model is consistent with observed option prices, but not with the assumption that the discounted price process is a martingale. However, by allowing non-zero dividend rates we can find a model for which the ex-dividend price process is a martingale, and for which the model prices are given by $P(K)$. See Section 9.2 below.

Now suppose $\lim_K P(K) - (K - x_0) = \delta > 0$. If $P(x_0) = x_0$ then $P(K) \equiv K$, and we have an extreme example which falls into this setting. For P as specified above we have that $\varphi(x) = 1$ on $(x_0 - \delta, \infty)$. The measure ν places no mass on $(-\delta, \infty)$ and $s(x) = x - x_0$ on this region. Except for time 0, the process Z_t spends no time in $(-\delta, \infty)$ and X_t jumps instantly to $x_0 - \delta$, and thereafter spends no time above this point.

Note that if x_0 is not specified, then this case can be reduced to the previous case by assuming $x_0 = K - \lim_K P(K)$. This obviates the need for a jump at $t = 0$.

9.2. **Time homogeneous processes with non-constant interest rates and non-zero dividend processes.** In the main body of the paper we have assumed that the interest rate r is a given positive constant, that dividend rates are zero, and that σ is a function to be determined. However, the same ideas can be used to find other time-homogeneous models consistent with observed perpetual put prices, whereby the volatility function is given, and either a state-dependent dividend rate, or a state-dependent interest rate is inferred.

Suppose X has dynamics $dX_t = (r(X_t) - q(X_t))X_t\, dt + X_t \sigma(X_t)\, dB_t$. Given put prices $P(K)$ as before, define φ via $\varphi(z) = \inf_{K: K \geq z}(K - z)/P(K)$. Then the relationship between φ and the characteristics of the price process X are such that φ solves $L\varphi = 0$, where L is given by

$$Lu = \frac{1}{2}\sigma(x)^2 x^2 u_{xx} + x(r(x) - q(x))u_x - r(x)u.$$

Note that we now allow for any of σ, q or r to depend on x. To date we have assumed that r is constant and q is zero and solved for σ, but alternatively we can assume $\sigma(x)$ is a given function, and r is a positive constant, and solve for q, or assume that q and σ are given, and solve for r.

For example, if r and σ are given constants then the (proportional) dividend rate process is given by

$$q(x) = r + \frac{x^2\sigma^2\varphi_{xx} - 2r\varphi}{2x\varphi_x}.$$

If q is negative this should be thought of as a convenience yield.

By allowing for dividend processes which are singular with respect to calender time, and which are instead related to the local time of X at level x_0, it is possible to construct candidate price processes which spend no time above x_0. For example, if $\tilde{L}$ is the local time at 1 of X, and if

$$\frac{dX_t}{X_t} = dB_t + rdt - \frac{d\tilde{L}_t}{2}$$

then X_t reflects at 1, and if $\hat{\varphi}(x) = (\mathbb{E}^1[e^{-rH_x}])^{-1}$ for $x < 1$, then $\hat{\varphi}'(1-) = 0$. This gives an example of a model consistent with the class of option prices described in Section 9.1.

9.3. **Recovering the model from perpetual calls.** The perpetual American call price function $C : [0, \infty) \to [0, x_0]$ must be non-increasing and convex as a function of the strike K, and must satisfy the no-arbitrage bounds $(x_0 - K)^+ \le C(K) \le x_0$.

If there are no dividends (and if $e^{-rt}X_t$ is a martingale) then the perpetual call prices are given by the trivial function $C(K) = x_0$.

So suppose instead that the (proportional) dividend rate q is positive. Let $\hat{\psi}$ be the increasing positive solution to

$$\frac{1}{2}x^2\sigma(x)^2\psi'' + (r-q)x\psi' - r\psi = 0$$

normalised so that $\hat{\psi}(x_0) = 1$. Then, for $z > x$, $\mathbb{E}^x[e^{-rH_z}] = \hat{\psi}(x)/\hat{\psi}(z)$, and call prices in a model where $dX_t = (r-q)X_t\,dt + X_t\sigma(X_t)\,dB_t$ are given by

$$\hat{C}(K) = \sup_\tau \mathbb{E}^{x_0}[e^{-r\tau}(X_\tau - K)^+] = \sup_{x:x\ge x_0} \frac{(x-K)}{\hat{\psi}(x)}.$$

Example. Suppose X solves $(dX_t/X_t) = (r-q)\,dt + \sigma\,dB_t$, with $X_0 = x_0$. Then $\hat{\psi}(x) = (x/x_0)^\gamma$ where $\gamma = \beta_+$ and

$$\beta_\pm = -\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\frac{2r}{\sigma^2} + \left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2}.$$

Note that since $q > 0$ we have $\gamma > 1$. Note also that $\hat{\varphi}(x) = x^{\beta_-}$.

The corresponding call prices are given by

$$\hat{C}(K) = x_0^\gamma \sup_{x:x\ge x_0}\{(x-K)x^{-\gamma}\}$$

which for $K \le (\gamma-1)x_0/\gamma$ gives $\hat{C}(K) = (x_0 - K)$, and for $K > (\gamma-1)x_0/\gamma$

$$\hat{C}(K) = x_0^\gamma \gamma^{-\gamma}(\gamma-1)^{\gamma-1}K^{1-\gamma}.$$

The example discusses the forward problem, but the discussion of the inverse problem is similar to that in the put case. Given perpetual call prices $C(K)$, for $x > x_0$ we can define ψ via $\psi(x) = \inf_{K:K\le x}(x-K)/C(K)$, and then construct a triple $\sigma(x), q(x), r(x)$ so that

$$\frac{1}{2}x^2\sigma(x)^2\psi'' + (xr(x) - q(x))\psi' - r(x)\psi = 0.$$

By combining information from put and call prices it is possible to determine a model which simultaneously matches both puts and calls.

Appendix A. Proofs

A.1. **Duality.**

Proof of Lemma 2.5. It is clear that g is non-negative and non-decreasing since f is positive and non-increasing. The lower bound on g follows from choosing $z = z_0 \wedge k$ in (13), and the upper bound follows since f is non-increasing. To show that g is convex, first note that $g(k)$ is minus the reciprocal of the slope of the tangent of the function f which passes through the point $(k, 0)$. For two given points k_1 and k_2 with $k_1 < k_2$, let $l_1(z)$ and $l_2(z)$ be the corresponding tangent lines. Let $k = \lambda k_1 + (1-\lambda)k_2$ for some $\lambda \in (0, 1)$, and let $l(z)$ be the line through the point $(0, k)$ and the intersection point of l_1 and l_2. If the intersection point is denoted $(z, l(z))$, then the convexity of f guarantees that

$$g(k) \leq \frac{k-z}{l(z)} = \frac{(1-\lambda)k_1 - (1-\lambda)z}{l_1(z)} + \frac{\lambda k_2 - \lambda z}{l_2(z)} = (1-\lambda)g(k_1) + \lambda g(k_2),$$

which proves that g is convex.

To prove the self-duality, let $z \leq z_0$. By the definition of g we have that $g(k) \geq (k-z)/f(z)$ for all $k \geq z$. Consequently,

$$F(z) = \sup_{k \geq z} \frac{k-z}{g(k)} \leq f(z).$$

For the reverse inequality, let $z \leq z_0$ and let l be a tangent line to f through the point $(z, f(z))$ (such a tangent is not necessarily unique if f has a kink at z). Assume that the point where l intersects the z-axis is given by $(k', 0)$. Then $g(k') = (k'-z)/f(z)$, so

$$F(z) = \sup_{k \geq z} \frac{k-z}{g(k)} \geq \frac{k'-z}{g(k')} = f(z),$$

which finishes the proof of (ii). The proof of (iii) can be performed along the same lines. □

A.2. **Time changes of local martingales.**

Proposition A.1. *Suppose $(\gamma_u)_{u \geq 0}$ is a martingale with respect to the filtration $\mathbb{G} = (\mathcal{G}_u)_{u \geq 0}$, and A_t is an increasing process such that A_t is a stopping time with respect to $\mathbb{G}$ for each t. Define $\tilde{\gamma}_t = \gamma_{A_t}$, and $\tilde{\mathcal{G}}_t = \mathcal{G}_{A_t}$. In general $(\tilde{\gamma}_t)_{t \geq 0}$ is not a martingale. However, if γ is a bounded martingale then $\tilde{\gamma}$ is a bounded martingale.*

Proof. Given a Brownian motion B, for $b > 0$, let H^B_b be the first hitting time of level b. Then $(\tilde{B}_b)_{b \geq 0}$ defined via $\tilde{B}_b \equiv B_{H^B_b}$ is not a martingale.

However, if γ is bounded then $\mathbb{E}[\tilde{\gamma}_t | \tilde{\mathcal{G}}_s] = \mathbb{E}[\gamma_{A_t} | \mathcal{G}_{A_s}] = \gamma_{A_s} = \tilde{\gamma}_s$ by optional sampling. □

Suppose now that we are in the setting of Section 5 where Z_t is constructed from the Brownian motion B. In particular, Γ_u is an increasing additive functional of B, and A is the right-continuous inverse to Γ.

Proof of Lemma 5.1. Intuitively a time change of Brownian motion is a local martingale, but if the additive functional Γ is constant when B is in $[a, \infty)$ then the resulting process spends no time above a and reflects there. To maintain the local martingale property we need either that the time-changed process never gets to a, or that there are arbitrarily large values at which Γ is strictly increasing.

If $[\underline{z}_\nu, \overline{z}_\nu]$ is a bounded interval, then $A_\infty \leq H^B_{\underline{z}_\nu} \wedge H^B_{\overline{z}_\nu}$ and $(Z_t)_{0 \leq t < \infty} = (B_{A_t})_{0 \leq A_t < A_\infty}$ is a bounded martingale by Proposition A.1.

Now suppose $(\underline{z}_\nu, \overline{z}_\nu) = \mathbb{R}$ and suppose that for each a, ν assigns mass to every set (a, ∞) and $(-\infty, a)$.

We have $A_{\Gamma_t} \geq t$ with equality provided Γ is strictly increasing at t. Let $\{a^+_n\}$ and $\{a^-_n\}$ be two sequences converging to $+\infty$ and $-\infty$, respectively, so that ν assigns mass to any neighborhood of a^+_n and a^-_n, and set $H_n =$

$\inf\{u : B_u \notin (a_n^-, a_n^+)\}$. Then Γ is strictly increasing at H_n. Set $T_n = \Gamma_{H_n}$. Then $A_{T_n} = H_n$. Note that Γ_u increases to infinity almost surely, and hence $\Gamma_{H_n} \uparrow \infty$. Under our hypothesis, $(M_t^{T_n})_{t\geq 0}$ given by

$$M_t^{T_n} := M_{t\wedge T_n} = B_{A_{t\wedge T_n}} = B_{A_t \wedge H_n}$$

is a bounded martingale. Hence T_n is a localisation sequence for M.

The mixed case can be treated similarly. □

Proof of Lemma 6.1. For $y \in (s(0), s(\infty))$, set $H(y) = \ln(g(y)/x_0)$ and write $h(y) = H'(y) = g'(y)/g(y)$. Then

$$\nu(dy) = \frac{1}{2r}\frac{g''(y)}{g(y)}dy = \frac{1}{2r}\left(h'(dy) + h(y)^2 dy\right)$$

and, as usual, $\nu(dy) = \infty$ for $y \notin [s(0), s(\infty)]$.

We have $H(y) = \int_0^y h(v)dv = \ln(g(y)/x_0)$ and

$$\Gamma_u = \frac{1}{2r}\int_{\mathbb{R}} L_u^y (H''(dy) + H'(y)^2 dy).$$

Let ξ be the first explosion time of Γ. Then, by the Itô-Tanaka formula (eg, Revuz and Yor [11, Theorem VI.1.5]), for $u < \xi$,

$$\begin{aligned} H(B_u) &= \int_0^u H'(B_s)dB_s + \frac{1}{2}\int_{\mathbb{R}} H''(y)L_u^y\, dy \\ &= \int_0^u h(B_s)dB_s - \frac{1}{2}\int_{\mathbb{R}} L_u^y h(y)^2\, dy + r\Gamma_u. \end{aligned}$$

Thus $g(B_u) = x_0 e^{H(B_u)} = \mathcal{E}(h(B)\cdot B)_u e^{r\Gamma_u}$, where $\mathcal{E}$ denotes the Doléans exponential, and $e^{-r\Gamma_u} g(B_u)$ is a local martingale. It follows that M is a local martingale.

Now define $J(y) = \int_0^y j(v)dv = \ln f(y)$ and

$$\tilde\Gamma_u = \frac{1}{2r}\int_{\mathbb{R}} L_u^y \left(J''(dy) + J'(y)^2 dy\right)$$

By exactly the same argument as above we find that $f(B_u) = e^{J(B_u)} = \mathcal{E}(j(B)\cdot B)_u e^{r\tilde\Gamma_u}$ and $e^{-r\tilde\Gamma_u} f(B_u)$ is a local martingale.

It remains to show that in fact $\Gamma_u = \tilde\Gamma_u$. Define $L(y) = (f(y)g(y))^{-1}$. Then $L'(y)/L(y) = -g'(y)/g(y) - f'(y)/f(y) = -(H'(y) + J'(y))$ and

$$\begin{aligned} J'(y) - H'(y) &= \frac{\varphi'(g(y))g'(y)}{\varphi(g(y))} - \frac{g'(y)}{g(y)} = \frac{g'(y)[g(y)\varphi'(g(y)) - \varphi(g(y))]}{g(y)\varphi(g(y))} \\ &= -\frac{g'(y)s'(g(y))}{g(y)f(y)} = -L(y) \end{aligned}$$

and then

$$(J'(y) - H'(y))' = (H'(y) + J'(y))(L(y)) = H'(y)^2 - J'(y)^2.$$

Finally, since L_u^y is a bounded continuous function with compact support for each fixed u, we conclude that $\Gamma_u = \tilde\Gamma_u$. □

Proof of Corollary 6.2. Recall that in our setting Γ defined via (27) grows without bound and is continuous, at least until B hits $s(0)$ or $s(\infty)$. Thus, if ξ denotes the first explosion time of Γ, then the inverse function A

is defined for every t and $A_t = \xi$ for $t \geq \Gamma_\xi$. Then, using the extension of the definition of $\tilde{M}$ beyond Γ_ξ as necessary, we have

$$\tilde{M}_t = e^{-rt} X_t = \begin{cases} M_{A_t} & t \leq \Gamma_\xi \\ M_\xi & t > \Gamma_\xi. \end{cases}$$

Recall that φ is extended to (x_0, ∞) in such a way that $\lim_{x \uparrow \infty} \varphi(x) = 0$. Therefore either $s(\infty) < \infty$ and ν assigns infinite mass to all points $z > s(\infty) = \overline{z}_\nu$, or $s(\infty) = \infty$ and there exists a sequence $a_n \uparrow \infty$ such that ν assigns mass to any neighbourhood of a_n.

Suppose, we are in the second case. If $s(0) > -\infty$ then $H^B_{s(0)} = \xi < \infty$, else $\xi = \infty$. On $H^B_{a_n} < H^B_{s(0)} = \xi$, Γ_u is strictly increasing at $u = H^B_{a_n}$ and $A_{\Gamma_{H^B_{a_n}}} = H^B_{a_n}$. Set

$$T_n = \begin{cases} \Gamma_{H^B_{a_n}}, & H^B_{a_n} < H^B_{s(0)}; \\ \infty, & H^B_{a_n} > H^B_{s(0)}, \end{cases} \tag{34}$$

where the second line is redundant if $s(0) = -\infty$. Then $A_{T_n} = H^B_{a_n} \wedge \xi$ is such that $\tilde{M}^{T_n}_t := \tilde{M}_{t \wedge T_n} = M_{A_t \wedge \xi \wedge H^B_{a_n}} \leq g(a_n)$ and T_n is a reducing sequence for $\tilde{M}$.

Now suppose $s(\infty) < \infty$ and $\bar{g} = \infty$. Choose $a_n \uparrow s(\infty)$ such that ν assigns mass to any neighbourhood of a_n. Then, on $H^B_{s(\infty)} < H^B_{s(0)}$ we have by the argument after Lemma 6.1 that $\Gamma_{H^B_{a_n}} \uparrow \infty$ almost surely, and the argument proceeds as before with T_n given by (34) a reducing sequence.

Finally, suppose $s(\infty) < \infty$ and $\bar{g} < \infty$. Then M is bounded by $\bar{g}$ and $\tilde{M}$ is a martingale. □

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