Marked Excursions and Random Trees

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1 Introduction

This article is devoted to the study of the properties of a marked Brownian excursion, and an embedding of a branching tree in an excursion with marks. The embedding depends upon the heights of the excursion at the various mark times, and the heights of minima between consecutive marks. The resulting tree is a critical binary tree with an independent branching structure. The construction also gives rise to a natural connection between a Brownian excursion with at least one mark, and a family of Brownian excursions each with exactly one mark.

The relationship between Brownian excursions and trees has been much explored in the literature, a selection of which we note here. Neveu and Pitman [9, 10] (see also Le Gall [5]) considered excursions of height at least h, and embedded a tree in the excursion based on the locations of h-*maxima* and h-*minima*, which are excursion dependent local-extremes (see [9, 10] for definitions.) LeGall [6] has described one fruitful application of the relationship between excursions and trees to the construction of super-processes.

Aldous [1, 2, 3] based a tree on the excursion value at arbitrary times chosen independently of the excursion structure. His tree, the *Brownian continuum random tree* contains an infinite number of 'leaves'. Le Gall [7] considers a tree with a finite number of leaves which correspond to times chosen uniformly over the lifetime of a Brownian excursion. This article has much in common with this last paper.

Le Gall chooses an excursion according to Itô measure, re-weighted by the length σ of the excursion, and imagines putting p points uniformly, and independently, in the interval $[0, \sigma]$. He then defines a construction of a tree based upon the excursion, and in particular the excursion values at these time-points. Here we imagine a Brownian motion and an independent Poisson process marking the time axis. We consider the first Brownian excursion to contain a mark, or equivalently we choose an excursion according to the Itô measure of marked excursions. This first marked excursion contains a random number of marks, but by the properties of the Poisson process, each mark is uniformly distributed over the lifetime of the excursion. Now we have an excursion with a (random) number of identified points and we can define an embedded tree in the manner of Le Gall [7]. We investigate the properties of the resultant tree: since the excursion and the marks have been chosen in a canonically simple fashion, the resulting tree is also particularly simple.

We extend this idea to construct a second tree in which each node of the tree corresponds to a mark in the original excursion, and in the final section use this construction to explain an observation on the Arcsin law.

In this article we consider marked excursions. A related problem involving normalised excursions (ie excursions of unit length) has been considered by Pitman [12]. Pitman derives the joint distribution of the values of a normalised Brownian excursion at times $0 < U_{(1)} < \cdots < U_{(n)} < 1$ and the minima of the process over subintervals $([U_{(i)}, U_{(i+1)}])_{1 \leq i \leq n-1}$. Here $U_{(1)}, \ldots, U_{(n)}$ are the order statistics of n independent standard uniform random variables. The choice in this paper of excursions which contain a mark introduces a size bias which is exactly the right factor to guarantee the independent branching structure of the embedded tree.

It is a pleasure to thank Jean-Francois Le Gall for discussions and correspondence on this subject, and an anonymous referee whose detailed reading of an earlier version of this manuscript ensured that this version is much clearer.

2 Preliminaries on Trees

2.1 Trees

A tree consists of a finite family of elements ordered into generations. The zeroth generation contains a single parent individual who has a random number of offspring. These offspring form the members of the first generation. Subsequently each member of the k^{th} generation has a random number of offspring who form part of the $(k+1)^{th}$ generation.

Formally, following Neveu [8], define a tree as follows; τ is a tree if τ is a finite subset of $U := \cup_{n=0}^{\infty} \mathbb{N}^n$, where $\mathbb{N}^0 := \{\emptyset\}$, such that

- $\emptyset \in \tau$;
- for $k \geq 1$, if $u = u_1 \ldots u_k \in \tau$ then $u_1 \ldots u_{k-1} \in \tau$;
- for $k \geq 1$, if $u = u_1 \ldots u_k \in \tau$ with $u_k = n$ then $u = u_1 \ldots u_{k-1} m \in \tau$ for $1 \leq m < n$.

Let $\mathbb{T}$ be the set of trees, and if $u = u_1 \ldots u_k$ let $|u| = k$.

2.2 Marked trees

A marked tree is of the form

$$\tilde{\tau} = (\tau, (L_u, u \in \tau))$$

where $\tau \in \mathbb{T}$ and $L_u \in \mathbb{R}_+, \forall u \in \tau$. The mark L_u should be interpreted as the branch length or lifetime of the individual associated with element u. Let $\tilde{\mathbb{T}}$ be the space of marked trees.

Note the unfortunate dual use of the word marked. In an excursion a mark refers simply to an identified time. In a tree a mark is a piece of additional information associated with an element. Hopefully no confusion will arise.

2.3 Binary trees

In a binary tree the number of offspring is always either zero or two. Let $\mathbb{T}_b$ be the set of binary trees. Introduce a measure μ on $\mathbb{T}_b$ by setting

$$\mu(\tau) = 2^{-\|\tau\|}$$

where $\|\tau\|$ is the cardinality of τ. The measure ν models a branching process with a binary branching distribution where each individual has zero or two offspring with probability $\frac{1}{2}$. If $p := \|\{u \in \tau : u1 \notin \tau\}\|$ then p is the number of elements of the tree with no descendents and $\|\tau\| = 2p - 1$. Such elements are termed 'leaves' of the tree.

Let $\tilde{\mathbb{T}}_b$ be the space of marked binary trees. For α positive define a measure μ_α on $\tilde{\mathbb{T}}_b$ by setting

$$\mu_\alpha(d\tilde{\tau}) \equiv \mu_\alpha(\tau, dL_u) := \mu(\tau) \prod_{u \in \tau} (\alpha e^{-\alpha L_u}) dL_u.$$

This measure corresponds to a continuous time branching process with offspring distribution the uniform measure on $\{0, 2\}$, and exponential, rate α, lifetimes. In particular, for $u \in \tau$, the law of the subtree rooted at u is the same as the law of the whole tree.

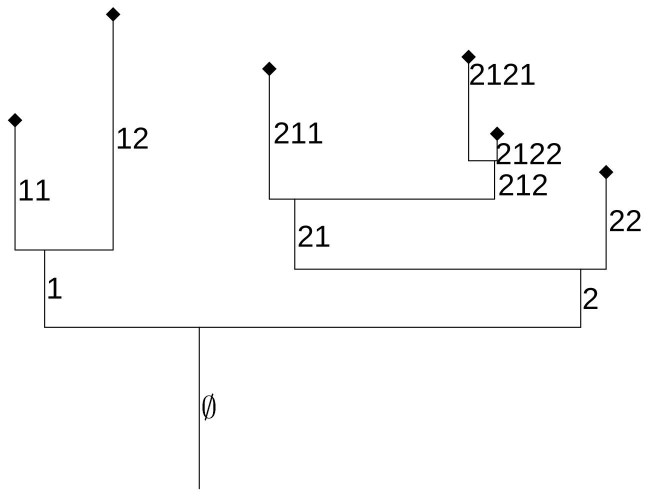

Figure 1: A representation of a marked binary tree. Vertical distances correspond to the branch lengths. Note that the horizontal scale and spacings are not part of the definition of the tree.

2.4 Binary trees and excursions

Define the set E of excursions to be the family of functions $e : \mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \Delta$ with an associated lifetime $\sigma \equiv \sigma(e) \in \mathbb{R}_+$ such that $e(0) = 0$, $\lim_{s \uparrow \sigma} e(s) = 0$, e is continuous on $[0,\sigma)$ and positive on $(0,\sigma)$, and $e(s)=\Delta, \forall s\geq\sigma$. The element Δ is a graveyard state. We will loosely define an excursion by describing it's lifetime σ and a continuous function $e:[0,\sigma)\mapsto\mathbb{R}_+$ with the appropriate properties. In the sequel we will also use the concept of an excursion started from $h>0$, which is as above except that $e(0)=h$. We will also specify time-points $0<s_1<\dots s_p<\sigma$ during the lifetime of an excursion. These points will be identified with leaves on an associated tree. Let E_p be the space of excursions, with p specified time-points, and let $\tilde{E}\equiv E_1$.

Given an excursion e, for a finite integer p, fix $0<s_1<\dots s_p<\sigma$. Then in each interval $[s_r,s_{r+1}]$ $(r=1,\dots p-1)$, there exists a time t_r such that the infimum of e over the interval is attained at t_r. Assume that t_r is unique and in the open interval (s_r,s_{r+1}), and further that the values $e(t_j)$ are distinct. When we consider Brownian excursions these assumptions will be satisfied, almost surely.

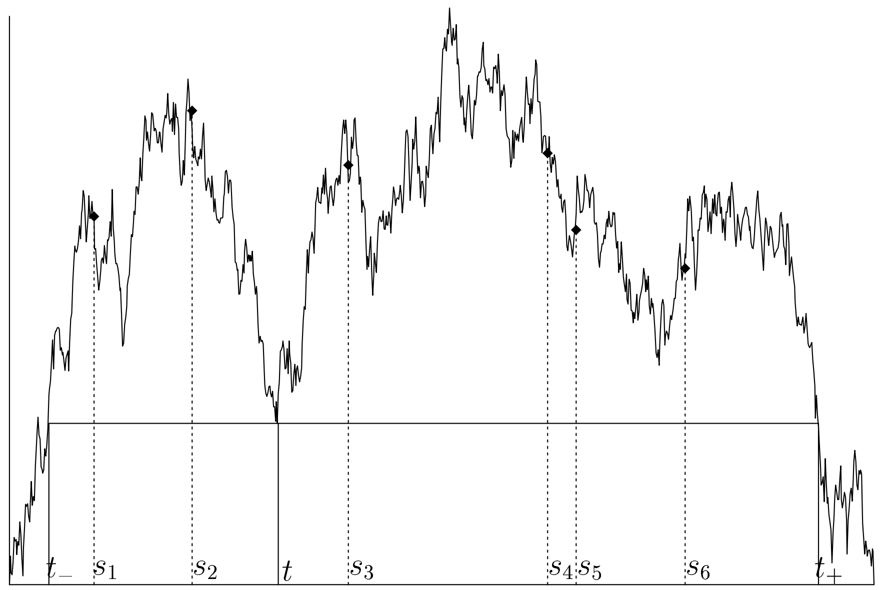

Figure 2: A plot of a typical excursion e up to its lifetime σ, with identified points $s_1<\dots s_6<\sigma$. The time t at which the infimum of e over $[s_1,s_6]$ is attained is also shown, as are t_- and t_+.

Suppose, inductively, that we are given a label $u\in\tau$, an associated excursion e_u, and an ordered set of times $0<s_1^u<\dots<s_{p(u)}^u<\sigma(e_u)$. Initially we take $u\equiv\emptyset$, $e_\emptyset\equiv e$, and $(s_1^\emptyset,\dots,s_{p(u)}^\emptyset)\equiv(s_1,\dots,s_p)$. Then

- if $p(u)=1$, set u to be a *leaf*, so that $u1\notin\tau$ and $u2\notin\tau$. Let $L_u=e_u(s_1^u)$ and $\tilde{e}_u\equiv e_u$. Note that $\tilde{e}_u\in\tilde{E}$.

- if $p(u)>1$, let t^u be the time-point between the first mark (at s_1^u) and the last mark (at $s_{p(u)}^u$) at which the excursion e_u attains its minimum value, so that $e_u(t^u)=\min_{s_1^u\leq t\leq s_{p(u)}^u}e_u(t)$. Associate with the label u the length $L_u\equiv e_u(t^u)$. Moreover, set $u1,u2\in\tau$. We wish to decompose e_u into three excursions; an excursion $\tilde{e}_u$ with exactly one labelled point, and two other excursions e_{u1}, e_{u2}, each with at least one labelled point. Define the times $t_-^u \equiv \sup_{s<s_1^u}\{e_u(s) = e_u(t^u)\}$ and $t_+^u \equiv \inf_{s>s_{p(u)}^u}\{e_u(s) = e_u(t^u)\}$. The three components of the decomposition are then as follows: $e_{u1}(s) := e_u(t_-^u + s) - e_u(t^u)$ is an excursion with lifetime $t^u - t_-^u$; $e_{u2}(s) := e_u(t^u + s) - e_u(t^u)$ an excursion with lifetime $(t_+^u - t^u)$; and $\tilde{e}_u(s)$ which has lifetime $\sigma(e_u) - (t_+^u - t_-^u)$ and is given by $\tilde{e}_u(s) := e_u(s)$ for $s \le t_-^u$ and $\tilde{e}_u(s) = e(t_+^u - t_-^u + s)$, for $s > t_-^u$.

For the label $u \in \tau$ this construction defines a mark L_u and associated excursion $\tilde{e}_u \in \tilde{E}$. Moreover, when $p(u) > 1$ the construction produces two sub-excursions labelled $u1$ and $u2$, with marks $(s_1^{u1}, \dots, s_{p(u1)}^{u1}) \equiv (s_1^u - t_-^u, \dots s_j^u - t_-^u)$ and $(s_1^{u2}, \dots, s_{p(u2)}^{u2}) \equiv (s_{j+1}^u - t^u, \dots, s_{p(u)}^u - t^u)$ respectively, where $j = \sup\{k : s_k^u < t^u\}$.

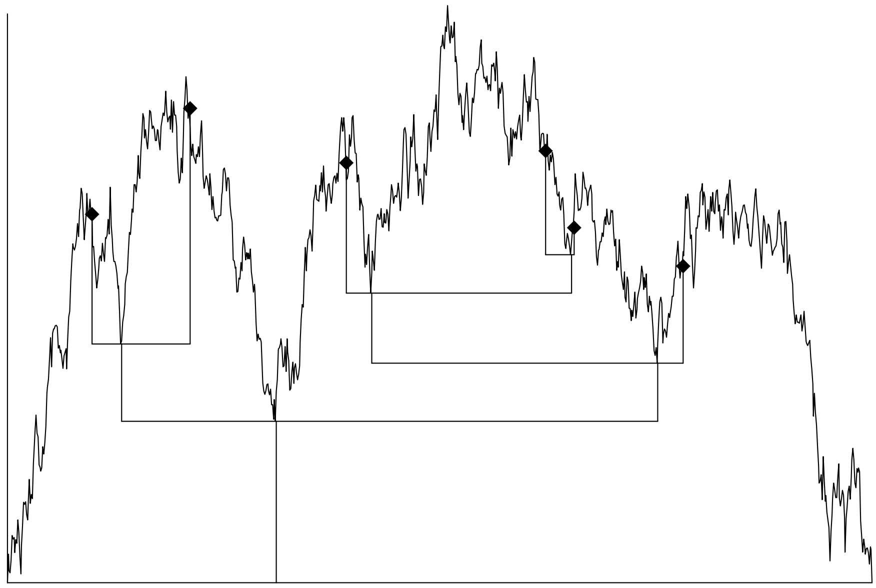

Figure 3: A λ-marked excursion, and superimposed the associated tree with six leaves, corresponding to the six marks.

Since p is finite this construction must terminate, to produce a marked tree $(\tau, (L_u, u \in \tau))$, and a family of excursions $(\tilde{e}_u, u \in \tau)$. Note that each member of this family of excursions has exactly one identified point which corresponds either to one of the original marks, or to one of the split times t_r.

The first aim of this article is to describe the law of the marked tree, and the associated excursions, when the original excursion is chosen to be a marked Brownian excursion.

3 Itô excursions and marked trees

Let B be a Brownian motion with local time process l at zero. Then, as Itô observed, B can be decomposed into excursions away from zero, indexed by l. These excursions form a Poisson process, on the space of excursions (as defined above, but also with the choice of sign), with intensity $n(de)$. By saying that 'e is a Brownian excursion' we mean that e has been chosen according to n.

Suppose further that time is marked by an independent Poisson process with constant intensity λ. For an excursion e, with duration $\sigma = \sigma(e)$, the probability that there are k marks is $(\lambda\sigma)^k e^{-\lambda\sigma}/k!$. By properties of the Poisson process the intensity of Brownian excursions e with k marks at $0 < s_1 < \cdots < s_k < \sigma$ is

$$n_{\lambda,k}(de, ds_1, \ldots, ds_k) = n(de)(\lambda\sigma)^k e^{-\lambda\sigma} \frac{ds_1}{\sigma} \ldots \frac{ds_k}{\sigma}$$

We will say that 'e is a λ-marked excursion' or just 'e is a marked excursion' if e is chosen according to n and the time domain of e contains at least one mark of the Poisson process.

For Brownian motion B, with local time process Λ and first hitting times H, and for T_λ the time of the first mark of the independent Poisson process, we have

$$\begin{aligned} \mathbb{P}(\Lambda_{T_\lambda} > z) &= \mathbb{P}(H_z < T_\lambda) \\ &= \mathbb{E}(\int_{H_z}^{\infty} \lambda e^{-\lambda t} dt) \\ &= \mathbb{E}(e^{-\lambda H_z}) = e^{-z\sqrt{2\lambda}}. \end{aligned} \tag{1}$$

Here the first equality is based on Lévy's Theorem identifying the law of the local time of Brownian motion with the law of the maximum. It follows that the rate of marked excursions is $\sqrt{2\lambda}$, and that the probability density of an excursion e with k marks at $0 < s_1 < \cdots < \sigma_k < \sigma$, conditional on the excursion having a mark, is

$$\tilde{n}_{\lambda,k}(de, ds_1, \ldots, ds_k) = n(de) \frac{\lambda^k e^{-\lambda\sigma}}{\sqrt{2\lambda}} ds_1 \ldots ds_k$$

Let $\tilde{n}_\lambda = \sum_{k\geq 1} \tilde{n}_{\lambda,k}$ be the probability density of an excursion with associated Poisson marks, conditioned to have at least one mark.

Theorem 1 *Let $\alpha \equiv 2\sqrt{2\lambda}$. Under $\tilde{n}_\lambda$ the distribution of $\tilde{\tau}(e)$ is $\mu_\alpha(d\tilde{\tau})$. Moreover each of the associated excursions is a λ-marked excursion conditioned to have exactly one mark, and the excursions associated with different individuals in the tree are independent.*

In the proof of this theorem we need a key lemma on excursions with a single mark.

Lemma 2 *A λ-marked excursion conditioned to have exactly one mark has the following structure; the height Z of the mark has an exponential distribution, rate α, the post-mark process is an independent Brownian motion, started at Z and conditioned to hit zero before being marked, and the pre-mark process is a time reversal of a further independent Brownian motion, again started at Z and again conditioned to hit zero before being marked.*

Proof of Lemma 2
We have that

$$\mathbb{P}(|B_{T_\lambda}| > z) = 2\mathbb{P}(B_{T_\lambda} > z) = \mathbb{P}(H_z < T_\lambda) = e^{-z\sqrt{2\lambda}}$$

(recall (1)) so that the height of the first mark in a Brownian motion has an exponential distribution, rate $\sqrt{2\lambda}$. Further the probability started from z of hitting zero unmarked is $e^{-z\sqrt{2\lambda}}$. Hence, the height of the mark in an excursion conditioned to have exactly one mark has density proportional to

$$(\sqrt{2\lambda}e^{-z\sqrt{2\lambda}})(e^{-z\sqrt{2\lambda}}) \equiv \sqrt{2\lambda}e^{-2\sqrt{2\lambda}z}$$

so that it has an exponential distribution, rate $\alpha = 2\sqrt{2\lambda}$. It also follows that the probability that a λ-marked excursion has exactly one mark is 1/2.

The other statements follow from the strong Markov property and the invariance of the excursion measure under time reversal (see Rogers and Williams [13, VI.49]). $\square$

Given an excursion e, started from $h > 0$, define $\grave{e}(s) \equiv \inf_{[0,s]} e(u)$. Then by Lévy's Theorem, if e is a Brownian excursion run until it first hits 0, then $((e-\grave{e})(s), s \geq 0)$ is a reflected Brownian motion started at 0 and run until the local time at 0 first reaches h. Further, if e is a Brownian motion conditioned to have no marks, then the excursions of $(e - \grave{e})$ from 0 are Brownian excursions conditioned to have no marks.

Proof of Theorem 1
Let e be a λ-marked Brownian excursion with lifetime σ. Let T_1 be the time of the first mark, and let $\xi \equiv e(T_1)$ be the height of the first mark. Then ξ has an exponential distribution with rate $\sqrt{2\lambda}$ and if $e_T(s) \equiv e(T_1 + s)$ then the process $(e_T - \grave{e}_T)$ is a reflecting Brownian motion, independent of ξ, run until the local time at 0 reaches ξ. For $s \leq \sigma - T_1$ define $Y(s) := (e_T - \grave{e}_T)(\sigma - s)$. The process Y is a reflecting Brownian motion run until it's local time at zero first reaches ξ, but we can think of it as the first part of a reflecting Brownian motion run for all time, and also labelled Y. We are interested in the last marked excursion from 0 (if any) of the process $e - \grave{e}$, this corresponds to the first marked excursion of the time reversed process Y (if that excursion occurs before the local time at zero reaches ξ.)

The first λ-marked excursion of Y occurs when the local time at 0 reaches η where η has an exponential distribution with rate $\sqrt{2\lambda}$ (again recall (1)). The probability that the excursion e has exactly one mark is precisely the probability that $\xi < \eta$, namely one-half, and moreover the law of $\eta \wedge \xi$ is exponential rate $\alpha = 2\sqrt{2\lambda}$.

Consider the case where $\xi < \eta$. Conditional on $\xi < \eta$ the excursion e contains exactly one mark at a height ξ which has an exponential rate α distribution, and we can apply the description in Lemma 2. Consequently, with probability 1/2, the marked tree consists of just the parent individual, who has lifetime ξ. Note that, conditional on $\xi < \eta$, ξ has an exponential, rate α distribution.

Now consider the case where $\eta < \xi$. The process Y contains excursions which are unmarked, followed by a first marked excursion at local time η, which begins at time t_Y^- and ends at t_Y^+. Now consider these times in terms of the original excursion e. Let $t := \sigma - t_Y^+$ and $t_+ = \sigma - t_Y^-$, and let $t_- := \sup_{u<t}\{e(u) = e(t)\}$. Then t, t_-, t_+

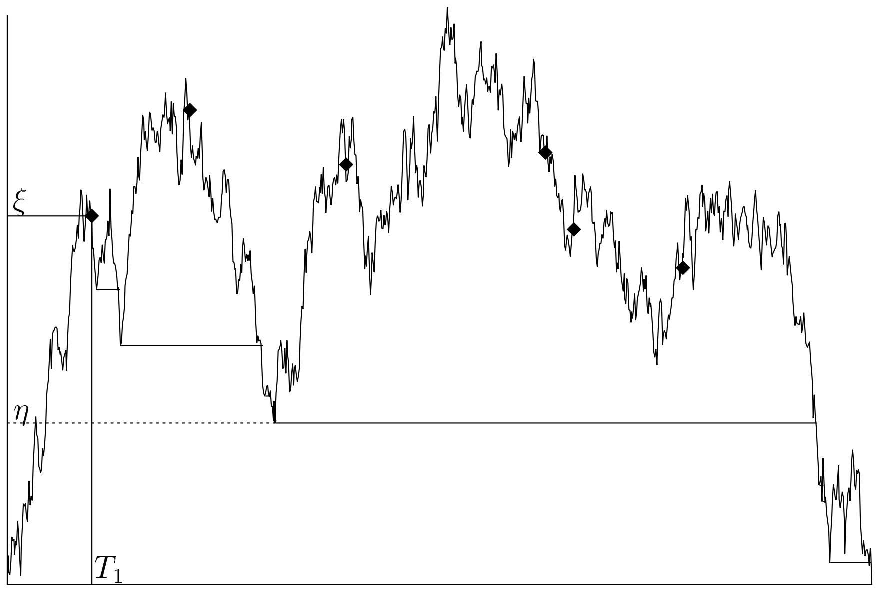

Figure 4: A Brownian excursion e, with first mark at T_1 and the process $\grave{e}$ also plotted.

have the meanings they were given in Section 2 and the excursion e is divided into three excursions, $e_1, e_2, \tilde{e}_\emptyset$ given by $e_1(s) := e(t_- + s) - e(t_-)$ for $0 \le s \le t - t_-$, $e_2(s) := e(t+s) - e(t)$ for $0 \le s \le t_+ - t$ and $\tilde{e}_\emptyset(s) = e(s)$ for $s < t_-$, and $\tilde{e}_\emptyset(s) = e(t_+ - t_- + s)$ for $s \ge t_-$. By construction the time reversal of e_2 is a marked excursion of Y, and, by invariance under time-reversal of n_λ, so is e_2. Moreover, again by the invariance under time-reversal of n_λ, and by the strong Markov property, so is e_1. Finally, by Lemma 2, $\tilde{e}_\emptyset$ is precisely a Brownian excursion with exactly one mark.

Recall that $\mathbb{P}(\eta < \xi) = 1/2$ so that the probability that the excursion contains more than one individual is one half. Then the marked tree has a parent individual (with lifetime η; conditional on $\eta < \xi$, η has an exponential distribution, rate α) who has exactly two offspring. These offspring correspond to the marked excursions e_1 and e_2, which are independent of each other, and of the excursion $\tilde{e}_\emptyset$ associated with the parent.

The theorem follows by induction. $\square$

Remark 3 It follows from the theorem that if we extend the notion of a mark or label on an element of a tree from a lifetime to an excursion with exactly one mark or identified time then we have a correspondence $e \leftrightarrow (\tau, (\tilde{e}_u, u \in \tau))$ where e is a λ-marked Brownian excursion, τ a critical binary tree, and $\tilde{e}_u$ a family of Brownian excursions conditioned to have exactly one mark.

4 Equating marks with leaves

Our goal in this section is to embed a different tree, ρ, in a marked Brownian excursion. This new tree has one node (rather than one leaf) for each mark in the excursion. The motivation for consideration of this new tree is a Ray-Knight interpretation of excursions and local times at different levels. An application is given in the next section.

Suppose, inductively, that we are given an individual labelled $u \in \rho$, an associated excursion e_u, and an ordered set of times $0 < s_1^u < \cdots < s_{p(u)}^u < \sigma(e_u)$. Initially we take $u \equiv \emptyset$, $e_\emptyset \equiv e$, and $(s_1^\emptyset, \ldots, s_{p(u)}^\emptyset) \equiv (s_1, \ldots, s_p)$. Let $\{e_u(s_1^u), \ldots, e_u(s_{p(u)}^u)\}$ be minimised at $h = e_u(s_i^u)$ for some i. Associate with the label u the lifetime $L_u = h$. If $p(u) = 1$ then this individual has no offspring. Otherwise there is at least one excursion above the height h which is marked; label the $d = d(u)$ such marked excursions sequentially $u1, u2, \ldots, ud$. These excursions correspond to direct descendents of u in the tree ρ. For $j = 1, \ldots, d$ let t^{uj} be the start of the j^{th} marked excursion e_{uj} above h, and define new mark times $s_\cdot^{uj} = s_\cdot^u - t^{uj}$ as appropriate. We are now in a position to repeat the construction for these sub-excursions. Since the number of marks p is finite this construction must terminate, to produce a marked tree $(\rho, (L_u, u \in \rho))$

Theorem 4 *Under $\tilde{n}_\lambda$ the distribution of $\rho(e)$ is that of a critical Galton-Watson process with geometric offspring distribution. In particular the measure of a tree ρ is $\nu(\rho) \equiv 2^{-(2\|\rho\|-1)}$. The marked tree has law*

$$\begin{aligned}\nu_\alpha(d\tilde{\rho}) = \nu_\alpha(\rho, dL_u) &= \prod_{u\in\rho} 2^{-(1+d(u))}(1+d(u))\alpha e^{-\alpha L_u}(1-e^{-\alpha L_u})^{d(u)} dL_u \\ &= \nu(\rho)\prod_{u\in\rho}(1+d(u))\alpha e^{-\alpha L_u}(1-e^{-\alpha L_u})^{d(u)} dL_u\end{aligned}$$

where $d(u)$ is the number of offspring of the individual labelled u.

We prove Theorem 4 by defining a surjection from $\tilde{\mathbb{T}}_b$ to $\tilde{\mathbb{T}}$. Given a marked binary tree (with marks corresponding to individual lifetimes) associate with each leaf $u = u_1 \ldots u_k \in \tau$ a total lapsed time $l_u = \sum_{j=0}^k L_{u_1 \ldots u_j}$ (the sum of its own lifetime plus the lifetimes of its ancestors).

Let u be the label of the leaf with lowest lapsed time. Let this lapsed time l_u be the lifetime of the parent $\emptyset$ of the marked tree $\tilde{\rho}$. Now consider the number of descendents of the parent individual in the binary tree τ (excluding the individual corresponding to the leaf u), who are alive at the time l_u, see Figure 5. Each of these individuals is the root of a marked binary tree (we consider only that part of the sub-tree above l_u), and to each of these individuals we can associate an offspring of the parent in ρ. In this way we can build inductively a marked tree $\tilde{\rho}$ in which each node, including the leaves, corresponds to a leaf in τ.

Proof of Theorem 4

Most of Theorem 4, and in particular the independent branching structure, follows from Theorem 1, and in particular the fact that the marks have an exponential

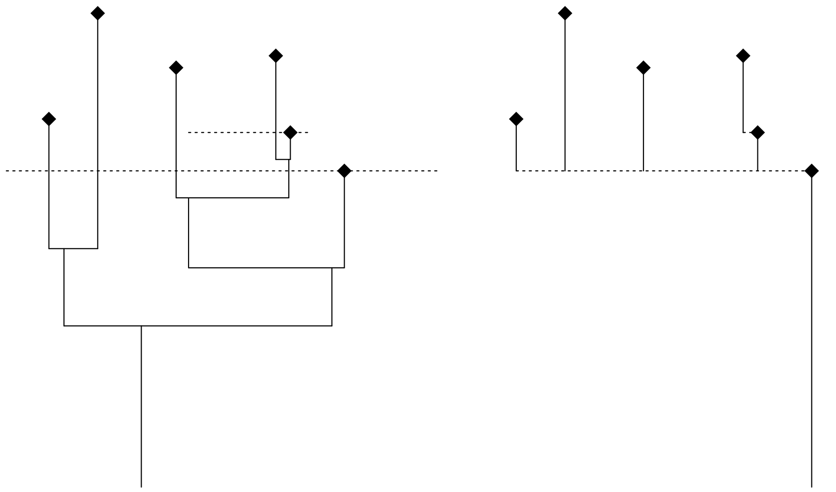

Figure 5: Constructing the general tree from a binary tree. In the new tree the parent has four offspring, the last of whom has one offspring. Again the horizontal scale is unimportant, although it has been inherited from a Brownian excursion. Note that each node in the new tree corresponds to a leaf in the old tree.

distribution, and from the mapping $\tilde{\tau} \mapsto \tilde{\rho}$. The remaining part is to identify the joint law of the lifetime and offspring distribution.

Number of offspring. Each event in the binary tree is with equal probability either a split or a death. The number of offspring of the parent in ρ is exactly the number of splits before the first death in τ (working upwards through the tree) and has a Geometric, parameter $1/2$, distribution.

Each individual alive at this first death is the root of a future tree, by the exponential lifetime property in $\tilde{\tau}$, each of these trees is independent and has the same probabilistic structure as the whole tree.

The law of the lifetime, conditioned on the number of offspring. Conditional on the parent $\emptyset$ having k offspring, the lifetime of $\emptyset$ is the sum of independent exponentials $T^{\emptyset}_{\alpha} + T^{\emptyset}_{2\alpha} + \cdots + T^{\emptyset}_{(k+1)\alpha}$. Here $T^{\emptyset}_{\alpha}$, rate α, is the time until the first event in the binary tree, $T^{\emptyset}_{2\alpha}$, rate 2α, is the subsequent time until the second event, (there now being two individuals in the binary tree), and so on.

By well known facts on the Yule process, or inductively on the number of offspring, the lifetime of the parent, conditioned on k offspring, has density

$$f_k(t) = (k+1)\alpha e^{-\alpha t}(1 - e^{-\alpha t})^k.$$

$\square$

5 Trees and the Arcsin Law

Let B_t be a Brownian motion and let $V_t = (1/t)\int_0^t I_{\{B_s>0\}}ds$ be the proportion of time that Brownian motion has spent positive by time t. Then by Brownian scaling $V_t \stackrel{\mathcal{D}}{=} V$, a random variable independent of t which has the Arcsin distribution:

$$\mathbb{P}(V \leq v) = \int_0^v \frac{du}{\pi\sqrt{u(1-u)}}; \qquad 0 \leq v \leq 1.$$

The original motivation of this study was to explain and understand the observation that, for all integers $k \geq 0$,

$$\mathbb{E}(V^k) = 2^{-2k}\begin{pmatrix} 2k \\ k \end{pmatrix} = \mathbb{P}(\text{random walk of length } 2k \text{ is always non-negative}).$$

Since V is a distribution on $[0,1]$ and is completely determined by its moments, an explanation of this observation can be viewed as an alternative proof (though hardly the most direct) of the Arcsin Law.

The final equivalence that we need relates a tree ρ with $m \geq 1$ individuals to an excursion of a simple random walk S with $2m$ steps. Given a tree ρ, define a map $f : \{1,2,\ldots,2m-1\} \mapsto \{u : u \in \rho\}$ as follows. Let $f(1) = \emptyset$. Given $f(i) = u = u_1\ldots u_k$ choose, if possible, the first child v of u which has not already been visited, and let $f(i+1) = v$, otherwise let $f(i+1) = u_1\ldots u_{k-1}$. Finally let $S_0 = 0$, $S_i = |f(i)|+1$ and $S_{2m} = \Delta$.

Standard combinatorial properties give that if ρ is a critical branching process with geometric parameter $\frac{1}{2}$ offspring distribution, then the resulting random walk is an excursion of a simple symmetric random walk. This construction, called the contour process or exploration process, is due to Harris [4], see also Le Gall [5].

Proposition 5

$$\mathbb{E}(V^k) = \mathbb{P}(\text{random walk of length } 2k \text{ is always non-negative})$$

Proof.
The proof is based upon the observation that the k^{th} moment of V is the probability that for a Brownian path with k marks, the value of the Brownian motion is positive at each of the mark-times. We combine this remark with Theorem 4 and the bijection between critical Galton-Watson branching processes and excursions of random walks to deduce our result.

Let T_{k+1} be the time of the $(k+1)^{th}$ mark. Consider Brownian motion on $[0, T_{k+1}]$ (with exactly k marks) and $V_{T_{k+1}}$. Conditional on T_{k+1}, the times $T_1 < \cdots < T_k$ are (the order statistics of) k uniform random variables on $[0, T_{k+1}]$. Thus

$$\begin{aligned}\mathbb{E}(V^k) \equiv \mathbb{E}(V^k_{T_{k+1}}) &= \int_0^{T_{k+1}} \cdots \int_0^{T_{k+1}} I_{\{B_{s_1}>0\}} \cdots I_{\{B_{s_1}>0\}} \frac{ds_1}{T_{k+1}} \cdots \frac{ds_k}{T_{k+1}} \\ &= \mathbb{P}(B_{T_i} > 0, 1 \leq i \leq k)\end{aligned}$$

Now consider the whole path of B. Use Theorem 4 to identify the j^{th} marked excursion of B, (with m_j marks), with a j^{th} realisation of a branching tree (with m_j

individuals) and a positive or negative label according as the Brownian excursion is positive or negative. The j^{th} realisation of the branching tree is in one-to-one correspondence with an excursion of a simple symmetric random walk (with $2m_j$ steps). We use the label of the tree to decide if this is a positive or negative excursion of the random walk. Now glueing these excursions together gives us a simple random walk $(S_n, n \geq 0)$. The positivity of $B_{T_1}, \dots B_{T_k}$ is equivalent to the fact that $(S_n \geq 0)$ for every $n \leq 2k$. $\square$

For other consequences of the relationship between Brownian motion sampled uniformly at random times, and simple random walks, see Pitman [11, Corollary 3].

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