Pathwise inequalities for local time: applications to Skorokhod embeddings and optimal stopping

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Abstract

We develop a class of pathwise inequalities of the form $H(B_t) \geq M_t + F(L_t)$, where B_t is Brownian motion, L_t its local time at zero and M_t a local martingale. The concrete nature of the representation makes the inequality useful for a variety of applications. In this work, we use the inequalities to derive constructions and optimality results of Vallois' Skorokhod embeddings. We discuss their financial interpretation in the context of robust pricing and hedging of options written on the local time. In the final part of the paper we use the inequalities to solve a class of optimal stopping problems of the form $\sup_\tau \mathbb{E}[F(L_\tau) - \int_0^\tau \beta(B_s)ds]$. The solution is given via a minimal solution to a system of differential equations and thus resembles the maximality principle described by Peskir. Throughout, the emphasis is placed on the novelty and simplicity of the techniques.

1 Introduction

The aim of this paper is to develop and explore a new approach to solving Skorokhod embeddings and related problems in stochastic control based on

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pathwise inequalities of the form

$$H(B_t) \geq M_t + F(L_t) \qquad \forall t \geq 0, \tag{1}$$

where B_t is Brownian motion, L_t its local time in zero, and M_t is a local martingale. Then, provided the stopping time τ is finite almost surely, and provided the stopped martingale $M_{t\wedge\tau}$ is uniformly integrable, we have

$$\mathbb{E}[H(B_\tau)] \geq \mathbb{E}[F(L_\tau)]. \tag{2}$$

There is equality in (2) if there exists equality at τ in (1).

Our aim is to find pairs (H, F) such that (1) holds and to use this pathwise inequality to deduce inequalities of the form (2). We can then investigate the optimality properties of (2). For the examples we have in mind F and H are typically convex. Further we often consider stopping rules of the form

$$\tau_\phi = \inf\{u : B_u \notin (\phi_-(L_u), \phi_+(L_u))\}$$

and then there is a 1-1 correspondence between $\phi_{+/-}$ and the law of the stopped process.

We shall consider three different approaches to the inequalities in (1) and (2).

Firstly, given H and $\phi_{+/-}$ we find F (and M) such that (1) and then (2) holds. This will build our intuition for constructing inequalities of this type.

Secondly, and more importantly, given F we find H such that (1) holds, and then for all τ satisfying suitable integrability conditions we also have $\mathbb{E}[F(L_\tau)] \leq \mathbb{E}[H(B_\tau)]$. If we let $\mathcal{T}(\mu)$ denote the set of stopping times such that $(B_{t\wedge\tau})$ is a uniformly integrable martingale, and such that $B_\tau \sim \mu$, then for all $\tau \in \mathcal{T}(\mu)$

$$\mathbb{E}[F(L_\tau)] \leq \int_{\mathbb{R}} H(x)\mu(dx).$$

In particular, for all minimal solutions of the Skorokhod embedding problem for μ in B we have a bound on $\mathbb{E}[F(L_\tau)]$. We carry out this program in Section 2. We recover results of Vallois [17, 18] for Skorokhod embeddings based on local times. See Cox and Hobson [6] for a recent study concerned with similar embeddings and Obłój [11] for an extensive survey and history of the Skorokhod embedding problem.

Thirdly, given F and H satisfying (1) then for suitable τ we have $\mathbb{E}[F(L_\tau)-H(B_\tau)] \leq 0$. This means we can consider problems of the form

$$\sup_\tau \mathbb{E}\left[F(L_\tau) - H(B_\tau)\right]$$

both for general τ and for $\tau \in \mathcal{T}(\mu)$ for given μ; further, under suitable integrability conditions, the problem can be recast (via Itô's lemma) as the more natural stopping problem

$$\sup_\tau \mathbb{E}\left[F(L_\tau) - \int_0^\tau \frac{1}{2} H''(B_s)\, ds\right].$$

This is the subject of Section 3. Similar problems, but with the local time replaced by the maximum process, have been studied by Jacka [10], Dubins, Shepp and Shiryaev [7], Peskir [15], Obłój [13] and Hobson [9]. The formulation of our solution will be similar to the *maximality principle* of Peskir [15].

One of our motivations for studying inequalities of the form (2) and the relationship to pathwise inequalities such as (1), is the interpretation of such inequalities in mathematical finance as superreplication strategies for exotic derivatives, with associated price bounds. The idea is that if we can identify a martingale stock price process S_t with a time-changed Brownian motion such that $S_T \sim B_\tau$, and if we know the prices of vanilla call options on S_T, then this is equivalent to knowing the law of B_τ. If we can also identify the martingale M in (1) with the gains from trade from a simple strategy in S, then we have a superreplicating strategy for an exotic option with payoff which is a function of the local time of S. Furthermore this strategy and associated price does not depend on any model assumptions.

For the case where the exotic option has a payoff which depends on the maximum (for example lookback and barrier options) this idea was exploited by Hobson [8] and Brown et al [3]. Financial options with payoff contingent on the local time are rare, but they can appear naturally when considering the 'naive' hedging of plain vanilla options and have recently been the subject of a study by Carr [4]. A further discussion of the application of our ideas to mathematical finance is given in Section 2.3.

Notation. We work on a filtered probability space satisfying the usual hypotheses. (B_t) denotes a real-valued Brownian motion. We stress however that throughout one can equally assume that (B_t) is a diffusion on natural scale (thus a Markov local martingale) with $B_0 = 0$ and $\langle B\rangle_\infty = \infty$ a.s.. No changes in the paper are needed.

F, H will typically denote convex functions and μ a probability measure with $\overline{\mu}(x) = \mu([x, \infty))$ denoting the right-tail. We write $X \sim \mu$ or $\mathcal{L}(X) = \mu$ to say that the law of X is μ.

2 Convex functions of the terminal local time

We begin by studying the first and second problems suggested in the Introduction. First, given H and μ we will find F and M such that (1) holds. Then we will reverse the process, so that for a given F and μ we will find H. For an appropriate H it will follow that (2) holds for all minimal τ with $B_\tau \sim \mu$. Furthermore, the function H will be optimal in the sense that there exists a stopping time τ_ϕ (which we give explicitly) for which there is equality in (2). We will first consider the well-behaved case to build up intuition and then, in Section 2.2, develop the general approach.

2.1 Symmetric terminal laws with positive densities

Let H be a symmetric, strictly convex function which is differentiable on $\mathbb{R} \setminus \{0\}$. Then, for any $a, b > 0$ we have $H(-b) = H(b) \geq H'(a)(b-a) + H(a)$, with equality if and only if $b = a$, see Figure 1.

Let ϕ be any continuous, strictly increasing function with $\phi(0) = 0$, and let ψ denote its inverse. Define $\gamma(l) = H'(\phi(l))$, $\Gamma(l) = \int_0^l \gamma(m)dm$ and $\theta(l) = H(\phi(l)) - \phi(l)H'(\phi(l))$. Then with $b = |B_t|$ and $a = \phi(L_t)$ we have for $t \geq 0$,

$$\begin{aligned} H(B_t) &\geq H'(\phi(L_t))|B_t| - H'(\phi(L_t))\phi(L_t) + H(\phi(L_t)), \\ &= M_t + F(L_t) \end{aligned} \tag{3}$$

where $M_t = M_t^{H,\phi} = |B_t|\gamma(L_t) - \Gamma(L_t)$ and $F(l) = F_{H,\phi}(l) = \Gamma(l) + \theta(l)$.

By construction $M_t^{H,\phi}$ is a local martingale (cf. Obłój [12]), so if τ is a stopping time such that $\mathbb{E}[M_\tau^{H,\phi}] = 0$ then

$$\mathbb{E}[F_{H,\phi}(L_\tau)] \leq \mathbb{E}[H(B_\tau)]. \tag{4}$$

Define $\tau_\phi = \inf\{u > 0 : |B_u| = \phi(L_u)\}$ and suppose ϕ is such that τ_ϕ is finite almost surely. Let $\mu = \mathcal{L}(B_{\tau_\phi})$. Then for any solution of the Skorokhod embedding problem for μ in B with the property that $\mathbb{E}[M_\tau^{H,\phi}] = 0$ we have

$$\mathbb{E}[F_{H,\phi}(L_\tau)] \leq \int_{\mathbb{R}} H(x)\mu(dx). \tag{5}$$

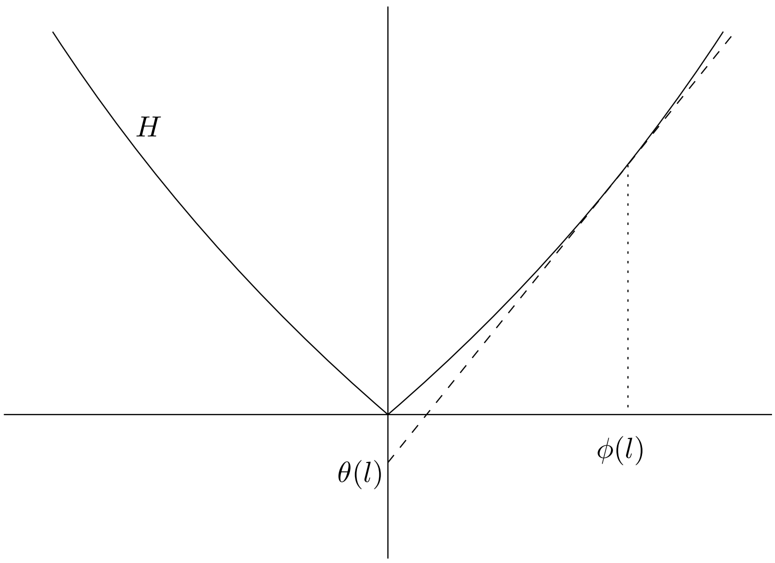

Figure 1: Since H is convex we have $H(b) \geq H(\phi(l)) + (b - \phi(l))H'(\phi(l))$. Further the intercept of this tangent with the y-axis is $\theta(l) = H(\phi(l)) - \phi(l)H'(\phi(l))$.

Thus we obtain an upper bound for the value $\mathbb{E}[F_{H,\phi}(L_\tau)]$. There is equality in (5) if $\tau = \tau_\phi$ and $\mathbb{E}[M^{H,\phi}_{\tau_\phi}] = 0$, and amongst $\tau \in \mathcal{T}(\mu)$ this is the only stopping time with this property.

Stopping times of the form τ_ϕ were used by Vallois [17] to solve the Skorokhod embedding problem. Vallois [18] proved that they maximized the expectation of convex functions of L_T and using our methodology we recover his results. We will also prove the embedding property.

Our aim now is to reverse the procedure described above. For a given convex function F and measure μ we aim to find H such that $F_{H,\phi} = F$ where ϕ is related to μ via Vallois' solution to the Skorokhod embedding problem.

To illustrate our method we begin with the simplest case and for the remainder of this section we adopt the following simplifying assumptions:

(A1.1) Suppose μ is symmetric, with finite first moment and with a positive density on $\mathbb{R}$.

(A1.2) Suppose $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is convex and increasing and has continuous first derivative F' which is bounded by K.

(A1.3) Suppose ϕ *is any continuous, strictly increasing function such that* $\phi(0) = 0$ *and such that* $\int_{0+} dl/\phi(l) < \infty$ *and* $\int^{\infty} dl/\phi(l) = \infty$.

Denote the inverse to ϕ by ψ. Define the measure $\nu \equiv \nu_\phi$ on $\mathbb{R}^+$ via

$$\overline{\nu}(l) = \nu([l,\infty)) = \exp\left(-\int_0^l \frac{dm}{\phi(m)}\right). \tag{6}$$

By the assumptions on ϕ, ν is a probability measure with density $\overline{\nu}(l)/\phi(l)$.

Define $H \equiv H_{F,\phi}$ via

$$H'(b) = \frac{1}{\overline{\nu}(\psi(b))}\int_{\psi(b)}^{\infty} F'(m)\nu(dm); \qquad H(0) = F(0). \tag{7}$$

By the assumptions on F we have that H' is well defined and bounded by K. It is easy to show that H' is increasing so that H is convex. Indeed,

$$H''(b) = \frac{\psi'(b)}{b\overline{\nu}(\psi(b))}\int_{\psi(b)}^{\infty}[F'(m) - F'(\psi(b))]\nu(dm) \tag{8}$$

$$= \frac{\psi'(b)}{b}(H'(b) - F'(\psi(b))) \tag{9}$$

which is non-negative since ψ is increasing and F is convex.

Lemma 2.1. *Suppose* $H'(x) \le K$ *and* τ *is such that* $(B_{t\wedge\tau})$ *is a uniformly integrable martingale. Then* $\mathbb{E}[M_\tau^{H,\phi}] = 0$.

Proof. Let $\sigma_n = \inf\{t : |B_t| \ge n\}$, $\rho_m = \inf\{t : L_t \ge m\}$ and $\tau_{m,n} = \tau \wedge \sigma_n \wedge \rho_m$. As the local martingale $(M^{H,\phi}_{t\wedge\tau_{m,n}} : t \ge 0)$ is bounded it is UI and $\mathbb{E}\, M^{H,\phi}_{\tau_{m,n}} = 0$ so that

$$\mathbb{E}\,\Gamma(L_{\tau_{m,n}}) = \mathbb{E}\,\gamma(L_{\tau_{m,n}})|B_{\tau_{m,n}}| = \mathbb{E}\,\gamma(L_{\tau_{\infty,n}})|B_{\tau_{\infty,n}}|\mathbf{1}_{\tau\wedge\sigma_n\le\rho_m}\ .$$

By the monotone convergence theorem, as $m \to \infty$ both sides converge, and in the limit we obtain $\mathbb{E}\,\Gamma(L_{\tau_{\infty,n}}) = \mathbb{E}\,\gamma(L_{\tau_{\infty,n}})|B_{\tau_{\infty,n}}|$. Now, as $n \to \infty$, the left hand side converges, again by the monotone convergence theorem, to $\mathbb{E}[\Gamma(L_\tau)]$, since $\tau = \tau_{\infty,\infty}$. The right hand side converges to $\mathbb{E}\,\gamma(L_\tau)|B_\tau|$ since γ is bounded and $|B_{t\wedge\tau}|$ is UI, so that finally $\mathbb{E}[M_\tau^{H,\phi}] = 0$.

□

Proposition 2.2. *(i) Define* $H \equiv H_{F,\phi}$ *via* (7). *Then,* $\forall\tau$ *such that* $(B_{t\wedge\tau})$ *is a uniformly integrable martingale,*

$$\mathbb{E}[F(L_\tau)] \le \mathbb{E}[H_{F,\phi}(B_\tau)]. \tag{10}$$

(ii) Let ϕ_μ be the inverse to ψ_μ where ψ_μ solves

$$\psi_\mu(x) = \int_0^x \frac{s}{\overline{\mu}(s)}\mu(ds). \tag{11}$$

Let $\tau_\mu \equiv \tau_{\phi_\mu} = \inf\{u > 0 : |B_u| = \phi_\mu(L_u)\}$. Then $B_{\tau_\mu} \sim \mu$, and $\mathbb{E}[F(L_{\tau_\mu})] = \int H_{F,\phi_\mu}(x)\mu(dx)$.
(iii) $\forall \tau \in \mathcal{T}(\mu)$, $\mathbb{E}[F(L_\tau)] \leq \mathbb{E}[F(L_{\tau_\mu})]$.

Proof. (i) The first part follows from (4) provided we can show that $F_{H,\phi} \equiv F$ and $\mathbb{E}[M_\tau^{H,\phi}] = 0$. This latter statement is guaranteed by Lemma 2.1. For the former, recall that $F_{H,\phi}(l) = \int_0^l H'(\phi(m))dm - \phi(l)H'(\phi(l)) + H(\phi(l))$. Setting $l = \psi(b)$, and differentiating we obtain from (9)

$$\psi'(b)F'_{H,\phi}(\psi(b)) = \psi'(b)H'(b) - bH''(b) = \psi'(b)F'(\psi(b))$$

Since $F_{H,\phi}(0) = H(0) = F(0)$ and the image of ψ is the whole of $\mathbb{R}$ we conclude that $F_{H,\phi} \equiv F$.

(ii) Note first that

$$\int_0^u \frac{dl}{\phi_\mu(l)} = \int_0^{\phi_\mu(u)} \frac{\mu(ds)}{\overline{\mu}(s)} = -\log(\overline{\mu}(\phi_\mu(u))),$$

which is finite for $u \in (0, \infty)$ and infinite for $u = \infty$. Hence ϕ_μ satisfies Assumption *(A1.3)*.

Now let ϕ be any function which satisfies Assumption *(A1.3)*, and let $\tau_\phi = \inf\{t > 0 : |B_t| \geq \phi(L_t)\}$. By an excursion theory argument (cf. Obłój and Yor [14])

$$\mathbb{P}(L_{\tau_\phi} > l) = \exp\Big(- \int_0^l \frac{ds}{\phi(s)}\Big) \tag{12}$$

and therefore $\tau_\phi < \infty$ a.s. We have $|B_{\tau_\phi}| = \phi(L_{\tau_\phi})$ and as remarked earlier equality is achieved in (3), so that $\mathbb{E}\, F(L_{\tau_\phi}) = \mathbb{E}\, H(B_{\tau_\phi})$.

It remains to show that for the choice $\phi = \phi_\mu$, the law of B_{τ_μ} is μ. Write ρ for the law of $|B_{\tau_\mu}|$. To see directly that $\rho = 2\mu_{|\mathbb{R}_+}$ write $\overline{\rho}(x) = \mathbb{P}(L_{\tau_\mu} \geq \psi_\mu(x))$ which can be computed via (12) and (11). We want to give, however, a natural approach to recover (11) where we only suppose that $\mathbb{E}\,|B_{\tau_\mu}| < \infty$. We know from the comments about equality in (5) that for a wide class of functions H,

$$\int_0^\infty F_{H,\phi}(\psi(x))\rho(dx) = \mathbb{E}\, F_{H,\phi}(L_{\tau_\mu}) = \mathbb{E}\, H(|B_{\tau_\mu}|) = \int_0^\infty H(x)\rho(dx). \tag{13}$$

This holds in particular for $H(x) = (|x| - k)^+$ and then the right-hand side is finite. We have

$$\begin{aligned} F_{H,\phi}(\psi(x)) &= \int_0^{\psi(x)} H'(\phi(u))du + H(x) - xH'(x) \\ &= \int_0^x H'(u)d\psi(x) + H(x) - xH'(x), \end{aligned}$$

which substituted into (13) yields

$$\int_0^\infty \rho(dx) \int_0^x H'(u)d\psi(u) = \int_0^\infty xH'(x)\rho(dx).$$

Changing the order of integration we conclude

$$\int_0^\infty \overline{\rho}(x)H'(x)d\psi(x) = \int_0^\infty xH'(x)\rho(dx).$$

Given that the family of functions $H'(x)x$ contains the functions $f_k(x) = x\mathbf{1}_{x\geq k}$, for all $k \geq 0$, and that this family is rich enough to determine probability measures on $\mathbb{R}_+$, it follows that

$$\frac{d\psi(x)}{x} = \frac{\rho(dx)}{\overline{\rho}(x)}.$$

In particular, if $\psi \equiv \psi_\mu$ so that ψ solves (11) then $d(\log(\overline{\rho}(x))) = d(\log(\overline{\mu}(x)))$ and thus $\overline{\rho}(x) = 2\overline{\mu}(x)$ where the constant 2 arises form the fact that $\overline{\rho}(0) = 1 = 2\overline{\mu}(0)$. Since $\mathcal{L}(B_{\tau_\phi})$ is symmetric we conclude $B_{\tau_\mu} \sim \mu$.

(iii) This follows immediately from (i) and (ii). □

Remark. From the definition of ψ_μ we have for its inverse ϕ_μ

$$\phi'_\mu(l)d[\ln \overline{\mu}(\phi_\mu(l))] = \frac{1}{\phi_\mu(l)} = d[\ln \overline{\nu}_\mu(l)]$$

where ν_μ is defined from (6) using ϕ_μ. It follows that $\overline{\nu}_\mu(l) = 2\overline{\mu}(\phi_\mu(l))$. Hence (7) can be rewritten as

$$H'(b) = \frac{1}{\overline{\mu}(b)} \int_b^\infty F'(\psi(x))\mu(dx); \qquad H(0) = F(0).$$

Corollary 2.3. *Suppose μ satisfies Assumption* (A1.1) *and F is convex. Then, for all $\tau \in \mathcal{T}(\mu)$, $\mathbb{E}[F(L_\tau)] \leq \mathbb{E}[F(L_{\tau_\phi})]$. In particular, the assumptions that F' is continuous and $F' \leq K$ can be removed.*

Proof. It is clear from the definition of H via (7), and the proof of convexity in (8) that we do not need the derivative F' continuous, but just that the integrals in (7) and (8) are well defined. Further, for any increasing convex function F we define F_K via $F_K(0) = F(0)$ and $F_K' = F' \wedge K$. We have shown so far that for any solution to the Skorokhod embedding problem $\mathbb{E}\, F_K(L_{\tau_\phi}) \geq \mathbb{E}\, F_K(L_\tau)$. Taking limits as $K \to \infty$, via monotone convergence theorem, we obtain the general optimal property of Vallois' stopping time: $\mathbb{E}\, F(L_{\tau_\phi}) \geq \mathbb{E}\, F(L_\tau)$ for smooth symmetric terminal distributions $B_\tau \sim B_{\tau_\phi} \sim \mu$. □

Example 2.4. Suppose $\overline{\mu}(l) = e^{-2\alpha^2 l}/2$. Then $\psi_\mu(b) = \alpha^2 b^2$, $\phi_\mu(l) = \sqrt{l}/\alpha$ and $\overline{\nu}(l) = 2\overline{\mu}(\phi(l)) = e^{-2\alpha\sqrt{l}}$.

Suppose now that $H(x) = Ax^2 + B|x|$, with $A, B \geq 0$. Then

$$F_{H,\phi}(l) = \frac{4}{3\alpha} A l^{3/2} + \left(B - \frac{1}{\alpha^2} A\right) l.$$

Note that F is convex and increasing if and only if $B \geq A/\alpha^2$. Conversely, if $F(l) = Cl^{3/2}$, then $H_{F,\phi}(x) = (3\alpha/4)Cx^2 + (3/4\alpha)C|x|$.

2.2 Arbitrary centered terminal laws

We want to extend the results of the previous section to arbitrary terminal laws. We need to be able to deal with two issues: atoms in μ and asymmetry of μ. We deal with the former by parameterising the fundamental quantities in terms of the quantiles of μ and we deal with the latter by introducing separate functions ϕ_+ and ϕ_- on the positive and negative half-spaces respectively.

In this section when we take inverse functions we always mean the right-continuous versions. We also use the notation $\phi_\pm$ to indicate the pair (ϕ_+, ϕ_-), this should cause no confusion.

Let $\phi_+ : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and $\phi_- : \mathbb{R}_+ \to \mathbb{R}_-$ a decreasing function. To develop an analogue to (3) we will parameterize the negative half-line with ϕ_- and the positive half-line with ϕ_+ so that

$$\begin{cases} H(x) & \geq & \gamma_+(l)x + \theta_+(l), \quad x > 0, \\ H(z) & \geq & \gamma_-(l)z + \theta_-(l), \quad z < 0, \end{cases}$$

where (see Figure 2)

$$\begin{cases} H'(\phi_+(l-)) \leq \gamma_+(l) \leq H'(\phi_+(l+)) & \theta_+(l) = H(\phi_+(l)) - \phi_+(l)\gamma_+(l), \\ H'(\phi_-(l+)) \leq \gamma_-(l) \leq H'(\phi_-(l-)) & \theta_-(l) = H(\phi_-(l)) - \phi_-(l)\gamma_-(l). \end{cases}$$

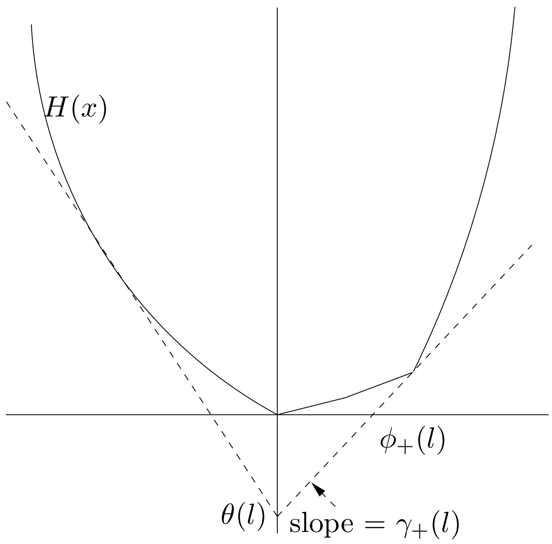

Figure 2: Specification of the various functions, H, ϕ_+, γ and θ

Substituting B_t and L_t we obtain

$$\begin{aligned} H(B_t) &\geq \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0} \\ &= M_t^{H,\phi} + \Gamma(L_t) + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0}, \end{aligned} \tag{14}$$

where $\Gamma(l) = \int_0^l (\gamma_+(m) - \gamma_-(m))/2 \, dm$ and $M_t^{H,\phi} = \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- - \Gamma(L_t)$ is a local martingale. If we choose the various quantities such that $\theta_+(l) = \theta_-(l) = \theta(l)$, then we have

$$H(B_t) \geq M_t^{H,\phi} + F_{H,\phi,\gamma}(L_t) \tag{15}$$

where $F_{H,\phi,\gamma}(l) = \Gamma(l) + \theta(l)$.

Note that when H' is not continuous different choices of functions $\gamma_\pm(l)$ (or equivalently different choice of tangents to H) will lead to different functions $F_{H,\phi,\gamma}(l)$ and different inequalities.

As before, our goal is to reverse the procedure: given F and a centered probability measure μ on $\mathbb{R}$, we aim to choose H, ϕ and γ such that $F_{H,\phi,\gamma} \equiv F$ and $B_{\tau_\phi} \sim \mu$ where

$$\tau_\phi = \inf\{u > 0 : B_u^+ = \phi_+(L_u) \ \text{ or } \ B_u^- = -\phi_-(L_u)\} \tag{16}$$

Define

$$\Delta(l) = \int_0^l \left(\frac{1}{2\phi_+(m)} + \frac{1}{2|\phi_-(m)|} \right) dm.$$

Assumption (A2)**.** *Suppose F is convex and increasing and suppose ϕ_+ and ϕ_- are increasing positive and decreasing negative functions respectively such that $\Delta(l)$ is finite for each $l > 0$, but increases to infinity with l.*

Fix ϕ_+ and ϕ_- satisfying Assumption (A.2), and let ψ_+, ψ_- denote their respective inverses. Define $\nu = \nu_\phi$ via $\overline{\nu}(l) = \exp(-\Delta(l))$. Given F, define the increasing function Σ via

$$\Sigma(l) = \frac{1}{\overline{\nu}(l)} \int_l^\infty F'(m)\nu(dm).$$

Where $F'(l)$ exists, define $\delta(l) = \Sigma(l) - F'(l)$. Then δ is defined almost everywhere in l and is positive.

Define

$$A_+(l) = \Sigma(0) + \int_0^l \frac{\delta(m)}{\phi_+(m)}dm, \qquad C(l) = -F(0) + \int_0^l \delta(m)dm,$$
$$A_-(l) = -\Sigma(0) - \int_0^l \frac{\delta(m)}{|\phi_-(m)|}dm,$$

and the function H via,

$$H(x) = \begin{cases} \sup_{l>0}\{xA_+(l) - C(l)\} & x > 0 \\ F(0) & x = 0 \\ \sup_{l>0}\{xA_-(l) - C(l)\} & x < 0 \end{cases} \tag{17}$$

Remark. In fact, the only condition we need on $A_+(0)$ and $A_-(0)$ is that $A_+(0) - A_-(0) = 2\Sigma(0)$ and $A_+(0)$ and $A_-(0)$ are undetermined except through this difference. However, we fix both of them using an antisymmetry condition. A different convention for the choice of $A_+(0)$ would lead to a modification $H(x) \mapsto H(x) + xk$ for some constant k. Since for $\tau \in \mathcal{T}(\mu)$ we have $k\,\mathbb{E}[B_\tau] = 0$ and this has no effect on inequalities such as (2).

Lemma 2.5. *H is convex. The suprema for $x > 0$ and $x < 0$ in* (17) *are attained at $l = \psi_+(x)$ and $l = \psi_-(x)$ respectively. Further $H'(\phi_+(l-)) \leq A_+(l) \leq H'(\phi_+(l+))$ and $H'(\phi_-(l+)) \leq A_-(l) \leq H'(\phi_-(l-))$.*
Finally, $F_{H,\phi,A} = F$.

Proof. We have

$$xA_+(l) - C(l) = x\Sigma(0) + F(0) + \int_0^l \delta(m)\left(\frac{x}{\phi_+(m)} - 1\right)dm$$

which is maximised by $l = \psi_+(x)$ since thereafter the integrand is negative. Note that H is continuous at 0.
For the final statement observe that by definition $F_{H,\phi,A}(l) \equiv \Gamma_A(l) + \theta(l)$. We have $\theta_+(l) = \theta_-(l) = -C(l)$ and

$$\begin{aligned} A_+(l) - A_-(l) &= 2\Sigma(0) + 2\int_0^l \delta(m)\frac{\nu(dm)}{\overline{\nu}(m)} \\ &= 2\Sigma(0) + 2\int_0^l \frac{\nu(dm)}{\overline{\nu}(m)^2}\int_m^\infty [F'(m) - F'(l)]\nu(dm) \\ &= 2\Sigma(0) + 2\int_0^l \Sigma'(m)dm = 2\Sigma(l). \end{aligned}$$

In consequence, $F_{H,\phi,A}(l) = \int_0^l \Sigma(m)dm - C(l) = F(0) + \int_0^l F'(m)dm = F(l)$. □

We can now deduce our theorem which makes precise the ideas outlined in the Introduction.

Theorem 2.6. *Suppose F and ϕ satisfy Assumption (A.2). Define $H \equiv H_{F,\phi,A}$ via* (17)*. Then, $\forall\tau$ such that $(B_{t\wedge\tau})$ is a uniformly integrable martingale*

$$\mathbb{E}[F(L_\tau)] \leq \mathbb{E}[H(B_\tau)]. \tag{18}$$

Proof. Suppose $F' \leq K$ (the result for the general case can be deduced as in Corollary 2.3). Then, by a slight generalisation of Lemma 2.1, $\mathbb{E}[M_\tau^{H,\phi}] = 0$. The result now follows from (15). □

Our goal is to prove that there can be equality in (18). Moreover, if given a centred distribution μ we can find a stopping rule such that $B_\tau \sim \mu$ and there is equality in (18) then, as in Proposition 2.2, we have a tight bound on $\mathbb{E}[F(L_\tau)]$ over solutions of the Skorokhod embedding problem for μ. The existence and form of an embedding of μ based on the local time, and its optimality in the sense of maximising convex functions, is due to Vallois [17, Théorème 3.1], and [18, Théorème 1].

Let μ be a centered probability distribution with no atom in zero and let $\mu(\mathbb{R}^-) = a_* > 0$. Let G denote the cumulative distribution function of μ so that $G(x) = \mu((-\infty, x])$. For $a_* \leq a \leq 1$ define $\alpha(a)$ via

$$\int_{a_*}^a G^{-1}(c)dc + \int_{\alpha(a)}^{a_*} G^{-1}(c)dc = 0.$$

Then $0 < \alpha(a) < a_*$, $\alpha(a_*) = a_*$, $\alpha(1) = 0$ and α is a strictly decreasing absolutely continuous function with $\alpha'(c) = G^{-1}(c)/G^{-1}(\alpha(c))$.

Define $\xi = \xi_\mu$ via

$$\xi(a) = 2\int_{a_*}^{a} \frac{G^{-1}(c)}{\alpha(c) + (1-c)} dc, \qquad a_* \le a \le 1;$$

and

$$\xi(a) = 2\int_{a}^{a_*} \frac{G^{-1}(c)}{c + 1 - \alpha^{-1}(c)} dc, \qquad 0 \le a \le a_*.$$

Then, ξ is an absolutely continuous function which is convex on $a \ge a_*$ and concave on $a \le a_*$. Note also that $\xi(1) = \infty = -\xi(0)$.

Define $\psi_\mu(x)$ via $\psi_\mu(x) = \xi(G(x))$ so that ψ_μ is an increasing function on $\mathbb{R}$. If μ is symmetric then $\alpha(c) = 1 - c$ and for $x > 0$ we obtain the following generalisation of the formula (11):

$$\psi_\mu(x) = -\int_{[0,x]} s\, d(\ln \overline{\mu}(s)).$$

Theorem 2.7. *Let μ be a centered probability distribution on $\mathbb{R}$ with $\mu(\{0\}) = 0$. Let $\phi_\mu : \mathbb{R} \mapsto \mathbb{R}$ be the inverse to ψ_μ defined above, and define $\phi_+ : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\phi_- : \mathbb{R}^+ \mapsto \mathbb{R}^-$ via $\phi_\pm(l) = \phi_\mu(\pm l)$. Define τ_ϕ as in* (16). *Then $B_{\tau_\phi} \sim \mu$ and there is equality in* (18).

Proof. First we show that $\overline{\nu}(\xi(a)) = \alpha(a) + 1 - a$ for $a > a_*$. We have

$$\begin{aligned} \ln \overline{\nu}(\xi(a)) = -\Delta(\xi(a)) &= -\int_{a_*}^{a} \xi'(c)\Delta'(\xi(c))dc \\ &= \int_{a_*}^{a} \frac{-1}{(\alpha(c) + 1 - c)} \left(1 - \frac{G^{-1}(c)}{G^{-1}(\alpha(c))}\right) dc \\ &= \ln(\alpha(a) + 1 - a) \end{aligned}$$

where we use the fact that identities $\phi_+(\xi(c)) = G^{-1}(c)$ and $\phi_-(\xi(c)) = G^{-1}(\alpha(c))$ hold dc almost everywhere on $(a_*, 1)$. This implies

$$\lim_{m\uparrow\infty} \Delta(m) = \infty$$

as required by Assumption (A.2).

For $x > 0$ we have

$$\begin{aligned}\mathbb{P}(B_{\tau_\phi} > x) &= \int_0^\infty \overline{\nu}(l)\frac{dl}{2\phi_+(l)}\mathbf{1}_{\{\phi_+(l)>x\}} \\ &= \int_{(x,\infty)} \frac{\overline{\nu}(\xi(G(y)))}{2y} d\xi(G(y)) \\ &= \int_{G(x)}^1 \frac{\overline{\nu}(\xi(c))}{\alpha(c)+(1-c)} dc = 1 - G(x).\end{aligned}$$

Calculations for $a \in (0, a_*)$ and for $x < 0$ are similar.
Equality in (18) follows from the definition of τ_ϕ and resulting equality in (14). □

Corollary 2.8. *Suppose μ is centered with $\mu(\{0\}) = 0$ and F is convex. Then, for all $\tau \in \mathcal{T}(\mu)$, $\mathbb{E}[F(L_\tau)] \leq \mathbb{E}[F(L_{\tau_\phi})]$.*

Finally, we relax the condition that $\mu(\{0\}) = 0$. If μ places mass at zero then we can construct an embedding of μ as follows. Let Z be a Bernoulli random variable with $\mathbb{P}(Z = 0) = \mu(\{0\})$ which is independent of B — if necessary we expand the probability space so that it is sufficiently rich as to support Z — and, given $X \sim \mu$, let $\tilde{\mu}$ be the law of X conditioned to be non-zero.

On $Z = 0$ set $\tau = 0$. Otherwise, on $Z = 1$, let τ be the stopping rule defined via $\tilde{\phi}$ and (16), where $\tilde{\phi}$ is defined from $\tilde{\mu}$ using the algorithm described following Theorem 2.6.

It is clear $B_\tau \sim \mu$. Also it is clear that with H defined relative to F and $\tilde{\phi}$, (18) still holds. Further, by considering the cases $Z = 0, Z = 1$ separately we see that there can be equality in (18).

Independent randomisation is necessary for a stopping rule to attain equality in (18). Otherwise, if we insist that the stopping times are adapted to the minimal filtration generated by B then the best that is possible is to find a sequence of times τ_n such that $B_{\tau_n} \sim \mu$ and $\lim_{n\uparrow\infty} \mathbb{E}[F(L_{\tau_n})] = \int H(x)\mu(dx)$.

2.3 Financial Applications

Let S_t be the time-T forward-price process of a financial asset. (To keep notation simple we express all prices in terms of monetary units at time T.) Consider the following 'naive' hedging strategy for a European call option with maturity T and strike $K \geq S_0$: borrow K and trade such that the portfolio holdings are $\max\{S_t, K\}$. In particular, the first time, if ever, that the forward exceeds K buy the forward; if subsequently the forward price falls below K then sell; whereupon the process is repeated.

Such a strategy was called the *stop-loss start-gain strategy* by Seidenverg [16]. At maturity this strategy yields $K + (S_T - K)^+$ and paying back K we have replicated the call payoff at no cost. Therefore, for no arbitrage to hold the price of an out-of-the-money call would have to be zero, while in practice such calls have positive value. The answer to the apparent paradox is that when S_t has unbounded variation trading continuously at level K accumulates local time at that level, and the strategy is not self-financing.

This resolution of the paradox, identified by Carr and Jarrow [5], shows that local time related quantities can arise naturally in financial markets. Other products closely linked with local time include corridor variance swaps, or more generally products dependent on number of downcrossings of an interval (see also Carr [4]). When exposed to the risk related to the local time, in addition to model-based prices one would want to have *model-free* bounds on the risk-quantifying products, and our study can be interpreted in this way. Analogous studies based on the supremum process (and yielding the Azéma-Yor solution to the Skorokhod embedding problem) led to *model-free* bounds on prices of look-back and barrier options (cf. Hobson [8] and Brown et al [3]).

In today's markets plain vanilla options are traded liquidly and it is an established practice to use them to calibrate models. From Breeden and Litzenberger [2] and subsequent works, we know that differentiating the maturity-T call prices twice with respect to the strike we recover the probability distribution of S_T under the risk-neutral measure.

To apply our results directly we define a shifted process $P_t := S_t - S_0$ with initial value $P_0 = 0$. Under the risk-neutral measure, S_t is a martingale and the distribution of P_T is a centered distribution on $[-S_0, \infty)$ which we denote by μ. As we now show our results give bounds on the value of a contingent claim paying $F(L_T(P))$ at time T, where F is some convex function and $(L_t(P))$ is the local time in zero of (P_t).

The process $(P_t : t \leq T)$ can be written as a time changed Brownian motion $(B_{\tau_t} : t \leq T)$ where $\tau = \tau_T$ is a stopping time such that $B_\tau \sim \mu$ and $(B_{\tau \wedge s} : s \geq 0)$ is a UI martingale. Furthermore, $L_T(P)$ is equal to the stopped Brownian local time L_τ (cf. Obłój [12]). Theorems 2.6 and 2.7 imply that $\Theta = \int H(x)\mu(dx)$, where H is given explicitly via (17) for $\phi_\pm$ as in Theorem 2.7, is the upper *model-free* bound on the expected value of $F(L_T(P))$.

Associated with the price bound is a superreplicating portfolio, consisting of a static portfolio paying $H(P_T)$ and a dynamic hedge. The European payoff $H(P_T)$ can be written as a static portfolio of puts and calls. The dynamic component is given by a self-financing portfolio G_t whose increase is given by $dG_t = -\Delta_t dS_t$ where

$$\Delta_t = H'(\phi_+(L_t))\mathbf{1}_{P_t>0} + H'(\phi_-(L_t))\mathbf{1}_{P_t>0}. \tag{19}$$

Note that G_t is simply the time-change of the martingale $-M_t^{H,\phi}$. Then (15) implies that $F(L_T(P)) \leq H(P_T) + G_T$ a.s. and we have exhibited a superreplicating portfolio.

This approach gives an upper bound on the potential model-based prices of options contingent upon local time. The pricing mechanism in which the price of the security paying $F(L_T(P))$ is set to be Θ may be too conservative, but it does have the benefit of being associated with a super-hedging strategy which is guaranteed to be successful, pathwise. A selling price $\tilde{\Theta} < \Theta$ can only be justified if the forward price process is known to belong to some subclass of models. Even in this case the seller can still use the hedging mechanism described above and be certain that his potential loss is bounded below by $\Theta - \tilde{\Theta}$ regardless of all other factors.

3 Optimal Stopping Problems

In this section we consider related optimal stopping problems. In particular, we consider solutions to problems of the form:

$$\sup_{\tau} \mathbb{E}\left[F(L_\tau) - \int_0^\tau \beta(B_s)\, ds\right], \tag{20}$$

subject to the expectation of the integral term being finite. In many ways, this problem can be considered as a relative of the problem considered in Peskir [15] and the form of our solution resembles the *maximality principle* introduced by Peskir [15]. We assume (initially) only that F and β are both non-negative; we will make stronger assumptions later as required.

The approach we use will be based on the representations used in previous sections, where we have made extensive use of the fact that we can construct a local martingale M_t such that $F(L_t) \leq H(B_t) + M_t$. In this section, we can interpret a related martingale as the Snell envelope for the optimal stopping problem; specifically, we are typically able to provide both a meaningful description of the optimal strategy for our problem, and also to write down explicitly the Snell envelope. We believe that being able to get such explicit descriptions of these objects is a strong advantage of this approach.

In this section, we will outline the principle behind this approach, and provide two results, the first of which allows us to provide an upper bound on the problem under very mild conditions on F and β. The second result gives the value of the problem and an optimal solution under some regularity conditions on F and β. We then demonstrate through examples that in fact we can find the optimal solution in more general cases. A final example shows how this technique might be used to derive inequalities concerning the local time.

Writing $H''(x) = 2\beta(x)$, we get:

$$H(B_t) = H(0) + \int_0^t H'(B_s)\, dB_s + \int_0^t \beta(B_s)\, ds. \tag{21}$$

In this section, we will further assume that

$$H(x) = \int_0^x \int_0^y 2\beta(z)\, dz\, dy,$$

and therefore H' is continuous and $H(0) = H'(0) = 0$.

Using the results from previous sections, (14) says

$$\begin{aligned} H(B_t) &\geq \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- + \theta_+(L_t)\mathbf{1}_{B_t \geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0} \qquad (22)\\ &= M_t^{H,\phi} + \Gamma(L_t) + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0}, \end{aligned}$$

where

$$\begin{cases} \gamma_+(l) = H'(\phi_+(l)) & \theta_+(l) = H(\phi_+(l)) - \phi_+(l)H'(\phi_+(l)), \\ \gamma_-(l) = H'(\phi_-(l)) & \theta_-(l) = H(\phi_-(l)) - \phi_-(l)H'(\phi_-(l)); \end{cases}$$

and $\Gamma(l) = \int_0^l (\gamma_+(m) - \gamma_-(m))/2\ dm$. In particular

$$M_t^{H,\phi} = \gamma_+(L_t)B_t^+ - \gamma_-(L_t)B_t^- - \Gamma(L_t)$$

is a local martingale. Combining (21) and (22) we deduce

$$\int_0^t \beta(B_s)\, ds \geq M_t^{H,\phi} - \int_0^t H'(B_s)\, dB_s + \Gamma(L_t) + \theta_+(L_t)\mathbf{1}_{B_t\geq 0} + \theta_-(L_t)\mathbf{1}_{B_t<0}.$$

Moreover, suppose we can find a solution $(\nu, \phi_+(\cdot), \phi_-(\cdot))$ to both

$$F(l) \leq \nu + \int_0^l \frac{H'(\phi_+(u)) - H'(\phi_-(u))}{2}\, du + H(\phi_+(l)) - \phi_+(l)H'(\phi_+(l)), \tag{23}$$

and

$$F(l) \leq \nu + \int_0^l \frac{H'(\phi_+(u)) - H'(\phi_-(u))}{2} \, du + H(\phi_-(l)) - \phi_-(l) H'(\phi_-(l)). \tag{24}$$

(Note that, unlike in previous sections, we make no assumption that the functions ϕ_+, ϕ_- are monotonic.) We can now write:

$$F(L_t) - \int_0^t \beta(B_s) \, ds \leq \nu - M_t^{H,\phi} + \int_0^t H'(B_s) \, dB_s. \tag{25}$$

We note that $N_t^{H,\phi} = \int_0^t H'(B_s) \, dB_s - M_t^{H,\phi}$ is a local martingale with $N_0^{H,\phi} = 0$. In addition, under the assumption that a stopping time τ satisfies

$$\mathbb{E}\left[\int_0^\tau \beta(B_s) \, ds\right] < \infty \tag{26}$$

we deduce from (25) that $N_{t\wedge\tau}^{H,\phi}$ is bounded below by an integrable random variable, so that it is a supermartingale. Taking expectations, we conclude:

$$\mathbb{E}[F(L_\tau) - \int_0^\tau \beta(B_s) \, ds] \leq \nu$$

for all stopping times τ satisfying the integrability constraint (26). In particular we have proved the following result.

Proposition 3.1. *Suppose $F(\cdot)$ and $\beta(\cdot)$ are non-negative functions, then for any solution $(\nu, \phi_+(\cdot), \phi_-(\cdot))$ to*

$$\begin{aligned} F(l) &\leq \nu + \int_0^l \frac{H'(\phi_+(u)) - H'(\phi_-(u))}{2} \, du \\ &\quad + H(\phi_+(l)) - \phi_+(l) H'(\phi_+(l)) \qquad (27) \\ &\leq \nu + \int_0^l \frac{H'(\phi_+(u)) - H'(\phi_-(u))}{2} \, du \\ &\quad + H(\phi_-(l)) - \phi_-(l) H'(\phi_-(l)), \qquad (28) \end{aligned}$$

we have

$$\sup_{\tau \in \mathcal{T}_\beta} \mathbb{E}\left[F(L_\tau) - \int_0^\tau \beta(B_s) \, ds\right] \leq \nu.$$

The arguments which formed the proof of Proposition 3.1 will be important in the sequel. One of our aims will be to obtain an expression for the value of (20) rather than merely a bound. To do this we will need to have equality in (22), (23) and (24), as well as a suitable integrability constraint on the local martingale $N^{H,\phi}$.

If we have equality in (23) and (24) and if we can differentiate suitably then we must have

$$\phi'_+(l) = \frac{\frac{1}{2}(H'(\phi_+(l)) - H'(\phi_-(l))) - F'(l)}{\phi_+(l)H''(\phi_+(l))}, \tag{29}$$

$$\phi'_-(l) = \frac{\frac{1}{2}(H'(\phi_+(l)) - H'(\phi_-(l))) - F'(l)}{\phi_-(l)H''(\phi_-(l))}, \tag{30}$$

together with a constraint on the initial values

$$H(\phi_+(0)) - \phi_+(0)H'(\phi_+(0)) = H(\phi_-(0)) - \phi_-(0)H'(\phi_-(0)). \tag{31}$$

Further, equality is attained in (22) exactly on the set where $B_t^+ = \phi_+(L_t)$ or $B_t^- = -\phi_-(L_t)$. Also, since $H(\cdot)$ is convex, the function $H(x) - xH'(x)$ is decreasing in x for $x > 0$, and increasing for $x < 0$. Consequently, we can choose

$$\nu = F(0) - H(\phi_+(0)) + \phi_+(0)H'(\phi_+(0)) \tag{32}$$

(and then also $\nu = F(0) - H(\phi_-(0)) + \phi_-(0)H'(\phi_-(0))$ where we note that ν is increasing if considered as a function of $\phi_+(0)$, and decreasing as a function of $\phi_-(0)$. In particular, we should attempt to minimise ν to get a bound which may be attained by the optimal stopping time.

Remark. Since ν is a function of (or determines) our choice of $\phi_-(0), \phi_+(0)$, it seems reasonable to ask how to interpret the solutions of (29)–(31) (that is, ones with different choices of initial value). In this context there are two possibilities, assuming the relationship in (32) holds.

If we choose $\phi_+(0)$ (and $|\phi_-(0)|$) too small, then we find that the candidate solutions to (29)–(31) hit zero at a finite value m, and thereafter the equations no longer make sense. (However the stopping time τ_ϕ would be optimal for the problem (20) with objective function $F(l \wedge m)$. We will make extensive use of this fact in the sequel.)

Conversely we can ask what happens if we choose initial values for $\phi_+(0)$ and $|\phi_-(0)|$ which are too large. In that case, we can still define a stopping time τ_ϕ but it will not satisfy the integrability constraint (26).

For the main result of this section it will be convenient to introduce the class of finite, positive solutions of (29)–(31). We therefore define the set

$$\Phi = \left\{ (\phi_+(\cdot), \phi_-(\cdot)) : \begin{array}{c} \phi_+, \phi_- \text{ are solutions of (29)–(31)}, \\ |\phi_+(l)|, |\phi_-(l)| < \infty \qquad \forall l \geq 0, \\ |\phi_+(l)|, |\phi_-(l)| > 0 \qquad \forall l > 0 \end{array} \right\}.$$

Theorem 3.2. *Suppose $F(\cdot)$ has a continuous derivative which is strictly positive on $(0,\infty)$, and that $\beta(\cdot) > 0$ is continuous. We write*

$$\mathcal{T}_\beta = \left\{ \tau : \tau \text{ is a stopping time, } \mathbb{E}\left[\int_0^\tau \beta(B_s)\,ds\right] < \infty \right\},$$

and the value of the optimal stopping problem

$$V = \sup_{\tau\in\mathcal{T}_\beta} \mathbb{E}\left[F(L_\tau) - \int_0^\tau \beta(B_s)\,ds\right]. \tag{33}$$

Suppose the set Φ is non-empty. Then Φ contains a minimal (in $\phi_+(0)$) element $(\phi_+(\cdot), \phi_-(\cdot))$, and one of the following is true:

(i) The supremum in (33) is attained by the stopping time:

$$\tau = \inf\{t \geq 0 : B_t \notin (\phi_-(L_t), \phi_+(L_t))\} \tag{34}$$

with corresponding value

$$V = F(0) + \phi_+(0)H'(\phi_+(0)) - H(\phi_+(0)) = F(0) + \int_0^{\phi_+(0)} z\beta(z)dz. \tag{35}$$

(ii) The stopping time defined in (34) is not in $\mathcal{T}_\beta$, but there exists a sequence of stopping times $\tau_N \uparrow \tau$ such that $\tau_N \in \mathcal{T}_\beta$ whose corresponding values

$$V_N = \mathbb{E}\left[F(L_{\tau_N}) - \int_0^{\tau_N} \beta(B_s)\,ds\right]$$

increase to V, which again is given by (35).

Suppose $\Phi = \emptyset$ and

$$\int_{\mathbb{R}_+} |z|\beta(z)\,dz = \infty = \int_{\mathbb{R}_-} |z|\beta(z)\,dz. \tag{36}$$

Then the value of the problem is infinite.

Remark. Observe that $(\phi_+(\cdot), \phi_-(\cdot))$, a solution to (29)–(31), is minimal in ϕ_+ among solutions which do not hit the origin and which remain finite, if and only if it is maximal in ϕ_-.

Under (36), we can state the conclusions of the theorem in the following way:
The value V in (33) is finite if and only if there exists a minimal solution

(ϕ_+, ϕ_-) *to* (29)–(31) *which does not hit the origin, in which case* V *is given by* (35).
This formulation is parallel to the *maximality principle* described by Peskir [15]. Note also that the solution in *(ii)* is what Peskir [15] calls an *approximately optimal* solution.

As we do not rely on Markovian techniques to solve (33) we only consider it for (B_t, L_t) starting at $(0, 0)$, but the generalisation to an arbitrary starting point is straightforward.

Our aim with this result is not to prove the strongest possible version of Theorem 3.2 since this seems to come at the price of having to account for a variety of idiosyncrasies that these solutions might display. Instead we believe that most, if not all of the restrictions on F and β can be weakened in different ways. Examples 3.3 and 3.4 explore this further.

Proof of Theorem 3.2. First note that if $\phi^1_\pm$ and $\phi^2_\pm$ are two solutions of (29)–(31) defined on $[0, m)$ with $|\phi^1_\pm(0)| \geq |\phi^2_\pm(0)|$ then $|\phi^1_\pm(l)| \geq |\phi^2_\pm(l)|$ for all $l < m$.

Suppose the set Φ is non-empty. By taking the (pointwise) infimum of ϕ_+ in Φ and the (pointwise) supremum of ϕ_- we obtain a solution to (29)–(31) which is finite for all $l \geq 0$. To deduce that this is an element of Φ, we need to show that this solution remains strictly positive. However, since this is the infimum of a sequence of strictly positive solutions, the minimal solution may only be equal to zero at a point for which $\phi'(l) = 0$, and we note that $F'(\cdot) > 0$ rules out this possibility in (29).

Moreover, under the regularity assumptions on $\beta(\cdot)$ and $F(\cdot)$, solutions of (29)–(31) can be constructed for any initial choice of $\phi_+(0) \geq 0$ below the starting point of our minimal solution. Note that equality in (23) and (24) implies

$$H(\phi_+(l)) - \phi_+(l)H'(\phi_+(l)) = H(\phi_-(l)) - \phi_-(l)H'(\phi_-(l))$$

and therefore ϕ_+ hits zero if and only if ϕ_- does. It is clear that the resulting functions (defined to be zero beyond the singularity m) are in fact solutions of (29)–(30) for the function $F(l \wedge m)$. Our proof will rely on the fact that (when $\phi_+(0) > 0$) the minimal solution in Φ is the supremum (for ϕ_+, infimum for ϕ_-) of these solutions. This is a consequence of the fact that $F'(l)$ is bounded, since by the smoothness of F and H, we know that the minimal ϕ_+ is bounded away from 0 and ∞ on any compact set, and therefore may be uniformly approximated from below by choosing a suitably close starting point. Since ϕ_+ is minimal, these approximating solutions must eventually hit 0.

We now consider these approximating solutions, writing $(\phi_+^m(l), \phi_-^m(l))$ for the solution which hits zero at $l = m$. We also write the associated stopping rules τ_m defined by (34) and ϕ^m. We argue that τ_m are optimal for the problems (33) posed for $F(l \wedge m)$. In view of the arguments which led to Proposition 3.1, the only property we need to demonstrate is that the supermartingale $N_{t\wedge\tau_m}^{H,\phi} = \int_0^{t\wedge\tau_m} H'(B_s)\,dB_s - M_{t\wedge\tau_m}^{H,\phi}$ is in fact a UI martingale. The important point to note here is that, by construction, the stopping times τ_m are smaller than $\inf\{t \geq 0 : L_t \vee |B_t| \geq J\}$ for some J. It follows immediately that $M_{t\wedge\tau_m}^{H,\phi}$ is a UI martingale. On the other hand the local martingale $\int_0^{t\wedge\tau_m} H'(B_s)\,dB_s$ has quadratic variation which is bounded by

$$\int_0^{\tau_m} H'(B_s)^2\,ds \leq K\tau_m,$$

for some $K > 0$. We know that $B_{t\wedge\tau_m}$ satisfies the conditions of Azema-Gundy-Yor theorem [1, Theorem 1b] since τ_m is bounded by the hitting time of $\{-J, J\}$. As a consequence, $\int_0^{t\wedge\tau_m} H'(B_s)\,dB_s$ also satisfies the conditions of the Azema-Gundy-Yor theorem and thus is a uniformly integrable martingale.

This procedure results in a sequence of stopping times, optimal for the problems posed with gain function $F(l \wedge m)$, and with values increasing to

$$V = F(0) - H(\phi_+(0)) + \phi_+(0)H'(\phi_+(0))$$

from which we deduce that we are in either case *(i)* or *(ii)*.

We must also consider the case where the minimal solution is a solution with $\phi_+(0) = \phi_-(0) = 0$, and corresponding stopping time $\tau \equiv 0$. It is then trivial to apply Proposition 3.1 to this solution to deduce that this is the optimal solution, with $V = F(0)$.

It remains to prove the final statement of the theorem. Let $(\phi_-(\cdot), \phi_+(\cdot))$ be a solution of (29)–(31) with some non-zero starting point. Since $\Phi = \emptyset$ this solution has to either hit zero or explode in finite time. We show that the latter is impossible. Define

$$G(z) = 2\int_{\phi_-(z)}^{\phi_+(z)} |u|\beta(u)\,du \ .$$

We have

$$\begin{aligned}
G'(z) &= 2\phi_+'(z)\phi_+(z)\beta(\phi_+(z)) + 2\phi_-'(z)\phi_-(z)\beta(\phi_-(z)) \\
&= H'(\phi_+(z)) - H'(\phi_-(z)) - 2F'(z) \\
&\le 2\int_{\phi_-(z)}^{\phi_+(z)} \beta(u)\,du \\
&\le 2\int_{\phi_-(0)}^{\phi_+(0)} \beta(u)\,du + \frac{2}{\phi_+(0)}\int_0^{\phi_+(z)} u\beta(u)\,du \\
&\quad + \frac{2}{|\phi_-(0)|}\int_{\phi_-(z)}^0 |u|\beta(u)\,du \\
&\le C_1 + C_2 G(z)
\end{aligned}$$

where $C_1 = 2\int_{\phi_-(0)}^{\phi_+(0)} \beta(u)du$ and $C_2^{-1} = \min\{\phi_+(0), |\phi_-(0)|\}$. It follows from Gronwall's Lemma that $G(u) \le C_1 e^{C_2 u}$; combining this with (36) we see that neither ϕ_+ nor ϕ_- can explode. We conclude that $\phi_\pm$ are bounded and hit zero in finite time. In consequence, the stopping time associated via (34) is in $\mathcal{T}_\beta$. We can apply Proposition 3.1 to see that

$$V \ge F(0) + \lim_{\phi_+(0)\uparrow\infty} \left[\phi_+(0)H'(\phi_+(0)) - H(\phi_+(0))\right]$$

and the value of the problem is infinite since (36) ensures $\int_0^x z\beta(z)dz = xH'(x) - H(x)$ increases to infinity. □

We finish this section with three examples. The first and second examples aim to demonstrate that, although we require relatively strong conditions in order to apply Theorem 3.2, the techniques and principles of the result are much more generally applicable. The first example also shows that the condition (36) cannot be weakened in general. A final example connects the results with established inequalities concerning the local time.

Example 3.3. The initial example considers the case where $F(l) = l$ and $\beta(x) > 0$. In this setting, there are three possible types of behaviour, depending on the value of

$$c = \int_{\mathbb{R}} \beta(x)\,dx.$$

For $c < 1$, every solution of (29) (resp. (30)) will have a strictly negative (resp. positive) gradient, and will therefore hit the origin in finite time. Consider $\tau = \inf\{t \ge 0 : L_t = 1\}$ and $\tau_N = \inf\{t \ge 0 : |B_t| \ge N \text{ or } L_t = 1\}$.

Then $\mathbb{P}(B_{\tau_N} = 0) = \exp(-1/N)$, and by Itô and monotone convergence, and using the fact that H' is increasing and bounded

$$\begin{aligned} \mathbb{E}\left[\int_0^\tau \beta(B_s)\,ds\right] &= \lim_{N\to\infty} \mathbb{E}\,H(B_{\tau_N}) \\ &= \lim_{N\to\infty} \left[\frac{H(N)+H(-N)}{2}\right]\left(1-\mathrm{e}^{-\frac{1}{N}}\right) \\ &= \lim_{N\to\infty} \left(\frac{H'(N)-H'(-N)}{2}\right) \\ &= c. \end{aligned}$$

As a consequence, there is a positive, finite gain of $1-c$ from simply running until the local time reaches 1. This process can then be continued, waiting until the local time reaches an arbitrary level, giving an infinite value to the problem.

The interesting case occurs when $c = 1$. In this setting, we see that the above argument fails — the strategy of waiting until the local time reaches 1 has no average gain. However there is still value to the problem; we can apply Proposition 3.1 (when we interpret $\phi_+ = \phi_- = \infty$ suitably) to deduce that the value of the problem is at most

$$\max\left(\lim_{x\to\infty}\left[xH'(x)-H(x)\right], \lim_{x\to-\infty}\left[xH'(x)-H(x)\right]\right), \tag{37}$$

and we note that this expression can be infinite. Now consider the payoff from running to exit of the interval $(-\alpha N, N)$ for some $\alpha \in (0,\infty)$. From the martingale property of $L_t - |B_t|$ this is easily seen to be

$$\frac{2\alpha N}{1+\alpha} - H(N)\frac{\alpha}{1+\alpha} - H(-\alpha N)\frac{1}{1+\alpha}.$$

Using the fact that $1 = \frac{1}{2}(H'(\infty) - H'(-\infty)) \geq \frac{1}{2}(H'(N) - H'(-\alpha N))$, we can bound the last expression from below by

$$\frac{\alpha}{1+\alpha}\left[NH'(N)-H(N)\right] + \frac{1}{1+\alpha}\left[(-\alpha N)H'(-\alpha N)-H(-\alpha N)\right].$$

Since α was arbitrary, we conclude that the solutions for sufficiently large N and sufficiently large/small α are approximately optimal and the value in (37) is obtained in the limit.

Finally we consider the case $c > 1$. We want to find a pair ϕ_-, ϕ_+ such that $\frac{1}{2}(H'(\phi_+) - H'(\phi_-)) = 1$ and such that (31) holds. We can rewrite this in terms of β:

$$\int_{\phi_-}^{\phi_+} s\beta(s)ds = 0 \quad \text{subject to} \quad \int_{\phi_-}^{\phi_+} \beta(s)ds = 1. \tag{38}$$

If this has a finite solution then we can apply Theorem 3.2 to conclude that the value function is given by $\phi_+ H'(\phi_+) - H(\phi_+) = \phi_- H'(\phi_-) - H(\phi_-)$ and the optimal strategy is to stop on exit from a finite interval.

Otherwise (38) has no solution with both ϕ_+ and ϕ_- finite and either

$$\int_{-\infty}^{\phi_+} s\beta(s)ds \geq 0 \quad \text{for } \phi_+ \text{ the solution of} \quad \int_{-\infty}^{\phi_+} \beta(s)ds = 1, \tag{39}$$

or

$$\int_{\phi_-}^{\infty} s\beta(s)ds \leq 0 \quad \text{for } \phi_- \text{ the solution of} \quad \int_{\phi_-}^{\infty} \beta(s)ds = 1, \tag{40}$$

For the former case we must have

$$\lim_{x \downarrow -\infty} xH'(x) - H(x) < \lim_{x \uparrow \infty} xH'(x) - H(x) \tag{41}$$

whereas for (40) we must have the reverse. (Note that if both sides of (41) are infinite then we must have a finite solution to (38).) Assuming (39) holds, it follows from Proposition 3.1 that $\phi_+ H'(\phi_+) - H(\phi_+)$ is an upper bound on the value function: that this bound can be attained in the limit follows from consideration of stopping rules which are the first exit times from intervals of the form $(-N, \phi_+)$. If the inequality in (41) is reversed then the value function is $\phi_- H'(\phi_-) - H(\phi_-)$ where ϕ_- solves the integral equality in (40).

Using this setup one can easily construct an example when the value is finite even though $\int_{\mathbb{R}} |z|\beta(z)dz = \infty$ which shows that both $\int_{\mathbb{R}_-} |z|\beta(z)dz = \infty$ and $\int_{\mathbb{R}_+} z\beta(z)dz = \infty$ are needed in general to ensure the last statement of Theorem 3.2.

Example 3.4. The main aim of this example is to demonstrate that the above ideas can lead to meaningful solutions to optimal stopping problems even if the functions F and H and the resulting minimal solution to (29)–(31) are not 'nice.' In particular, we shall give an example where the minimal solution is finite only on a bounded interval which excludes the origin, and where F is not non-decreasing, but where the value of the problem is finite.

Specifically, we consider the optimal stopping problem (33) where the (slightly contrived) functions F and β are defined by:

$$F'(l) = \begin{cases} (3-2l)\,\mathrm{e}^{-\frac{1}{2}l^2(l-2)^2} & : l < 2 \\ 1 & : l \geq 2 \end{cases},$$

with $F(0) = 0$ and

$$2\beta(x) = H''(x) = |x|^{-3}\mathrm{e}^{-\frac{1}{2x^2}}.$$

We also obtain:

$$H'(x) = \mathrm{e}^{-\frac{1}{2x^2}}, \qquad H(x) = x\mathrm{e}^{-\frac{1}{2x^2}} - \int_{\frac{1}{x}}^{\infty} \mathrm{e}^{-\frac{1}{2}z^2}\,dz.$$

Noting that the problem is symmetric (and therefore dropping the subscripts to denote positive and negative solutions), (29) becomes:

$$\phi'(l) = \begin{cases} \phi(l)^2\left(1-(3-2l)\exp\left\{\frac{1}{2\phi(l)^2} - \frac{1}{2}l^2(2-l)^2\right\}\right) & : l < 2 \\ \phi(l)^2\left(1-\exp\left\{\frac{1}{2\phi(l)^2}\right\}\right) & : l \geq 2 \end{cases} \tag{42}$$

Define the function $\phi_0(l)$ via

$$\phi_0(l) = \begin{cases} \frac{1}{l(2-l)} & : l \in (0,2) \\ \infty & : \text{ otherwise} \end{cases},$$

then ϕ_0 is a solution of (42) for $l < 2$. Moreover if for $l > 2$ we use the natural definitions $H'(\phi) = 1$ and $H(\phi) - \phi H'(\phi) = -\sqrt{\pi/2}$ when $\phi = \infty$ then

$$F(l) = \int_0^l H'(\phi_0(u))\,du + H(\phi_0(l)) - \phi_0(l)H'(\phi_0(l)) + \sqrt{\frac{\pi}{2}}$$

for all $l \geq 0$.

We will show that the stopping time $\tau = \inf\{t \geq 0 : |B_t| \geq \phi_0(L_t)\}$ is *approximately optimal* in the sense described above. Specifically, we consider a set of solutions ϕ_m increasing to ϕ_0 which have expected values increasing to $\sqrt{\frac{\pi}{2}}$, and show further that no solution can improve on this bound. The solutions $\phi_m(l)$ and $\phi_0(l)$ are shown in Figure 3.

It is straightforward to check that the solutions ϕ_m do indeed increase to ϕ_0, and that each ϕ_m is a well defined solution to (33) where the function

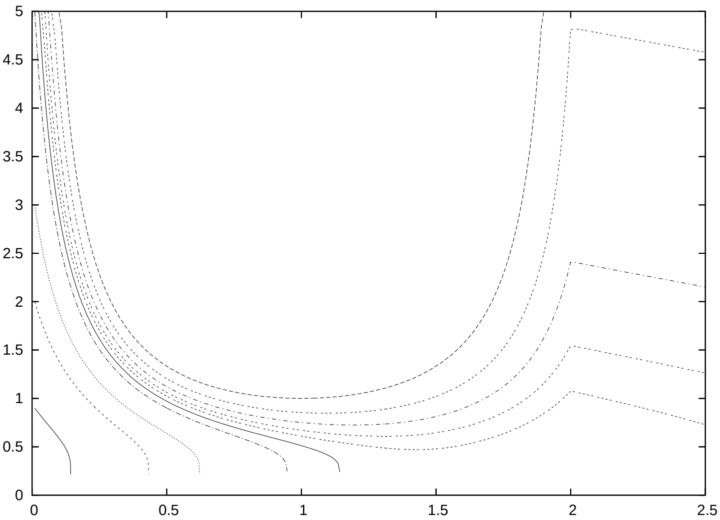

Figure 3: Solutions to (42) with different initial values. The top curve shows the function $\phi_0(l)$.

F is replaced by $F(l) \wedge m'$ for some m'. The resulting sequence provides an increasing set of stopping times in $\mathcal{T}_\beta$, with values increasing to $\sqrt{\frac{\pi}{2}}$.

To conclude that this does give the supremum, we suppose for a contradiction that there exists $\tau \in \mathcal{T}_\beta$ with larger expected value. Then for sufficiently large m' we have

$$\mathbb{E}\left[F(L_\tau) \wedge m' - \int_0^\tau \beta(B_s)\, ds\right] > \sqrt{\frac{\pi}{2}}$$

contradicting the optimality of the τ_m.

Finally, we note that the value of the problem for $F(l) = (l - l_0)_+$ is $(\sqrt{\frac{\pi}{2}} - l_0)_+$, so that $F(l) = l$ has the same value as our original problem, while $F(l) = (l - \sqrt{\frac{\pi}{2}})_+$ has zero value. The value for $F(l)$ actually follows from the Example 3.3 since $H'(\infty) = -H'(-\infty) = 1$ and this also provides additional intuition behind the above results.

Example 3.5. We end by demonstrating how our techniques may be used to recover the well known inequality

$$\mathbb{E}\, L_\tau^p \leq p^p\, \mathbb{E}\, |B_\tau|^p, \qquad p > 1, \tag{43}$$

valid for all stopping times τ such that the $\mathbb{E}\,|B_\tau|^p = \frac{p(p-1)}{2}\,\mathbb{E}\int_0^\tau |B_s|^{p-2} ds$. Fix $p > 1$ and consider $F(l) = \frac{l^p}{p}$ and $\beta_c(x) = c|x|^{p-2}/2$ for some $c > 0$. The function $H_c(x) = \frac{c}{p(p-1)}|x|^p$ is symmetric so that $\phi_- = \phi_+$. One can easily verify that $\phi(x) = ax$ satisfies (29) if and only if $ca^p - \frac{c}{p-1}a^{p-1} + 1 = 0$ and that this equation has two solutions only for $c \geq c_{min} = p^p(p-1)$. As ϕ is linear $\int_{0+} ds/\phi(s) = \infty$ and the resulting stopping time $\tau_V = 0$ a.s. Thus the value V of the optimal stopping problem (33) associated with F and β_c is zero. Consequently

$$\mathbb{E}\,\frac{L_\tau^p}{p} \leq \frac{c}{p(p-1)}\,\mathbb{E}\,|B_\tau|^p, \qquad \tau \in \mathcal{T}_\beta, c \geq c_{min} \tag{44}$$

and we recover (43) on taking $c = c_{min}$.

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