Stochastic Volatility Models

The Black-Scholes model has various drawbacks. SV models aim to address these.

Overview

Revision ① Stochastics: Itô's formula, diffusion, SDEs, Martingale. Local martingale. Brownian martingale representation theorem, ~~CMG~~ Cameron-Martin Girsanov change of measure, ~~Black Scholes formula~~.

Revision ② The Black-Scholes model
Assumptions, theory, criticisms

Level-dependent volatility models
- CEV model
- Displaced diffusion model
- Rady's quadratic diffusion co-efficient model
- Compound options model.

Dupire's local volatility model
Stochastic Volatility Models
- General specification
- Common models
  - Hull & White / Wiggins
  - Scott
  - Scott / Stein & Stein
  - Hull-White / Heston
  - 3/2 model
  - Maghsoodi (?)
  - SABR
  - Models where $\rho = 1$ (?) or $\rho = -1$. Hobson-Rogers (Heston-Nandi)
- Pricing

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$

$\mathcal{F}_t$ is the information at time t

$X = (X_t)_{t \geq 0}$ is a stochastic process if X is $\mathbb{F}$-adapted ~~i.e.~~

X_t is $\mathcal{F}_t$-measurable $\forall t$.

T is a stopping time if $\{\omega : T(\omega) \leq t\} \in \mathcal{F}_t \quad \forall t$.

$M = (M_t)$ is a $(\mathbb{F}, \mathbb{P})$ martingale if

M is $\mathbb{F}$-adapted

$\mathbb{E}|M_t| < \infty \qquad \forall t$

$\mathbb{E}[M_t | \mathcal{F}_s] = M_s \qquad \forall s < t.$

M is a local martingale if there exists $T_n \uparrow \infty$ a.s. such that M^{T_n} is a martingale for each n, where M^{T_n} is the stopped process

$$M_t^{T_n} = M_{t \wedge T_n}.$$

Lévy's Theorem

If W is a continuous local martingale and $[W]_t = t$ then W is Brownian motion.

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$

Diffusions (informally)

A diffusion is a continuous Markov process.
The movement over the next small time interval h has mean μ and variance σ^2 where μ and σ depend on the history only through the current level of the process.

$$dX = \sigma(X,t)\,dW + \mu(X,t)\,dt$$

If $\sigma(x,t) = \sigma(x)$ and $\mu(x,t) = \mu(x)$ then X is time-homogeneous.

$$[X]_t = \lim_{n\uparrow\infty} \sum_{k=0}^{n-1} \left(X_{kt/n} - X_{k(t-1)/n}\right)^2$$ is the quadratic variation

$$[X]_t = \int_0^t \sigma(X_s,s)^2\,ds \qquad d[X]_t = dt\,\sigma(X_t,t)^2$$

Itô's formula.

Let $f: \mathbb{R} \to \mathbb{R}$ be C^2. Let X be a semi-martingale (eg a diffusion)

$X_s = M_s + A_s$ $\quad M$ = martingale $\quad A$ = Finite Variation

Then

$$f(X_t) = f(X_0) + \underbrace{\int_0^t f'(X_s)\,dX_s}_{\underbrace{\int_0^t f'(X_s)\,dM_s}_{\text{(local) martingale}} + \underbrace{\int_0^t f'(X_s)\,dA_s}_{\text{Classical Integral (Lebesgue)}}} + \int_0^t \frac{1}{2} f''(X_s)\,d[X]$$

~~If X is a diffusion then~~

$f: \mathbb{R}^n \to \mathbb{R}$

$$f(X_t^{(1)}, X_t^{(2)}, \ldots X_t^{(n)}) = f(X_0^{(1)}, \ldots X_0^{(n)}) + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dX_t^{(i)}$$
$$+ \int_0^t \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}\,d[X^{(i)}]_t + \int_0^t \sum_{i=1}^n \sum_{j=i+1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\,d[X^{(i)}, X^{(j)}]_t$$

~~Mdr~~ Itô's formula and local martingales

If M is a continuous local martingale, then $\left(\int_0^t \theta_s\,dM_s\right)_{t\geq 0}$ is a continuous local martingale.

Suppose $\mathcal{F}_t = \sigma((B_s)_{0 \le s \le t})$

Brownian martingale representation theorem

Suppose M is a $(\mathbb{F}, \mathbb{P})$-(local) martingale. Then $\exists$ an adapted integrand Γ such that

$$M_t = M_0 + \int_0^t \Gamma_s \, dB_s .$$

Cameron-Martin Girsanov Change of measure.

~~Suppose $\Omega = C[0,T]$~~

~~$\mathcal{F}_s = \sigma(\{\omega(u) : 0 \le u \le s\})$~~

~~$\mathbb{P} = W$~~

We work on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

Suppose W is a $\mathbb{P}$-Brownian motion.

$$\text{Define } Z_t = \exp\left(+\int_0^t c_s \, dW_s - \tfrac{1}{2}\int_0^t c_s^2 \, ds\right)$$

Suppose $Z = (Z_t)_{0 \le t \le T}$ is a $\mathbb{P}$-martingale. Define $\mathbb{Q}$ via $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[1_A Z_T]$.

Then $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and under $\mathbb{Q}$, $W^{\mathbb{Q}}$ given by

$$W_t^{\mathbb{Q}} = W_t - \int_0^t c_s \, ds$$

is a Brownian motion.

Suppose $c = 1$

$$\mathbb{E}^{\mathbb{Q}}[Z_T^{-1}] = \mathbb{E}^{\mathbb{Q}}\left[e^{-cW_T + \frac{1}{2}c^2 T}\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-c(W_T - cT) - \frac{1}{2}c^2 T}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-cW_T^{\mathbb{Q}} - \frac{1}{2}c^2 T}\right] = 1.$$

The theorem has a direct analogue for $\mathbb{R}^d$-Brownian motions.

The Black-Scholes model

Asset price follows geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t} \qquad \frac{dS_t}{S_t} = \sigma dW_t + \mu dt$$

The volatility parameters σ and μ are known and constant.

The interest rate is known and constant - money can be invested in a risk-less bank account at rate r. Same rate for borrowing and lending.

Agents are price takers

Markets are perfect, continuous, and frictionless.

Remark: theory generalises easily to σ, μ, r functions of t (provided $\sigma > 0$).

Theorem

The fair price for a contingent claim with payoff $H = H(S_T)$ is $\mathbb{E}^{\mathbb{Q}}[e^{-rT}H(S_T)]$ where under $\mathbb{Q}$, $dS = S(\sigma dW^{\mathbb{Q}} + r\,dt)$

$\mathbb{Q}$ is called the risk-neutral measure

Proof

Define $Z = (Z_t)_{0 \le t \le T}$ via $Z_t = e^{-\lambda W_t - \frac{1}{2}\lambda^2 t}$ where $\lambda = \frac{\mu - r}{\sigma}$.

Then Z is a martingale.

Define $\mathbb{Q}(A)$ by $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[1_A Z_T]$. Then $W^{\mathbb{Q}}$ given by $W^{\mathbb{Q}}_t = W_t + \lambda t$ is a $\mathbb{Q}$-Brownian motion and $\frac{dS}{S} = \sigma(dW^{\mathbb{Q}} - \lambda dt) + \mu dt = \sigma dW^{\mathbb{Q}} + r\,dt$

Set $M_t = \mathbb{E}^{\mathbb{Q}}[e^{-rT}H(S_T) \mid \mathcal{F}_t]$. Then M is a $\mathbb{Q}$-martingale. Then $M_t = M_0 + \int_0^t \Gamma_s dW^{\mathbb{Q}}_s$ for some Γ. Set $C_t = e^{rt}M_t$. Then $C_T = e^{rT}M_T = H(S_T)$, $C_0 = M_0$.

$$dC_t = d(e^{rt}M_t) = re^{rt}M_t dt + e^{rt}dM_t = rC_t dt + e^{rt}\Gamma_t dW^{\mathbb{Q}}_t = rC_t dt + \frac{e^{rt}}{\sigma}\Gamma_t\left(\frac{dS}{S} - r\,dt\right)$$

Set $H = e^{rt}\Gamma_t / \sigma S_t$

$$dC_t = rC_t dt + H\,dS_t - rHS_t dt$$
$$= H\,dS + r(C - HS_t)dt$$

C is the gain from trade from holding H ... [illegible]

Black–Scholes formula and PDEs

Consider $C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} H(S_T) \mid \mathcal{F}_t\right] = e^{rt} M_t$

We must have $C_t = C(S_t, t)$. Then $e^{-rt} C(S_t,t)$ is a $\mathbb{Q}$-martingale.

Applying Itô: $d\left[e^{-rt} C(S_t, t)\right] =$

$$= e^{-rt}\left(-rC + C' dS + \frac{1}{2} C'' d[S] + \dot{C} dt\right)$$

$$= e^{-rt} C'(S_t, t)\, \sigma\, dW_t^{\mathbb{Q}} + e^{-rt}\left[-rC + C' r S + \frac{1}{2}\sigma^2 S^2 C'' + \dot{C}\right] dt$$

Since $e^{-rt} C(S_t, t)$ is a $\mathbb{Q}$-martingale we must have C solves the pde

$$\boxed{\begin{array}{c} \dot{C} - rC + rSC' + \frac{1}{2}\sigma^2 S^2 C'' = 0 \\ C(T, S) = H(S) \end{array}}$$

See Call Price Example (SV7) INSERT HERE

Some remarks

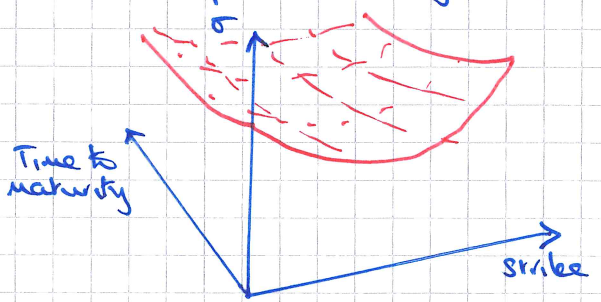
Option prices don't depend on μ. Instead they depend on σ the volatility.
σ is the focus of interest.

μ can only be reliably estimated from very long runs of data.
σ can be estimated from a very short run, assuming it is constant.

Call prices are increasing in σ. Given σ can write down price of call.
~~Implied price~~ Alternatively, given price of call can write down <u>implied volatility</u>

Option prices often quoted in terms of volatility.

Can derive the implied volatility surface



(in practice, interpolate from a finite number of strikes)

The Black-Scholes formula for a Call. (SV7)

$$C(S_0, 0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+ \right] \qquad S_T = S_0 e^{\sigma W_T + (r - \frac{1}{2}\sigma^2)T}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\left(S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - K e^{-rT}\right)^+ \right]$$

$$= S_0\, \mathbb{E}^{\mathbb{Q}}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} ; e^{\sigma W_T - \frac{1}{2}\sigma^2 T} > \frac{Ke^{-rT}}{S_0} \right] - e^{-rT} K\, \mathbb{P}^{\mathbb{Q}}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} > \frac{Ke^{-rT}}{S_0} \right]$$

$$\mathbb{P}^{\mathbb{Q}}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} > \frac{Ke^{-rT}}{S_0} \right] = \mathbb{P}\left[\frac{W_T}{\sqrt{T}} > \frac{\frac{1}{2}\sigma^2 T + \ln(Ke^{-rT}/S_0)}{\sigma\sqrt{T}} \right]$$

$$= \Phi\left(\frac{\ln S_0/K + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

Write $Z = e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$. Define $\tilde{\mathbb{Q}}$ by $\tilde{\mathbb{Q}}(A) = \mathbb{E}^{\mathbb{Q}}[Z 1_A]$

Then

$$\mathbb{E}^{\mathbb{Q}}\left[Z 1_{(Z > Ke^{-rT}/S_0)} \right] = \tilde{\mathbb{Q}}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} > \frac{Ke^{-rT}}{S_0} \right]$$

Under $\tilde{\mathbb{Q}}$, $\tilde{W}_t = W_t - \sigma t$ is a Brownian motion.

$$e^{\sigma W_T - \frac{1}{2}\sigma^2 T} = e^{\sigma \tilde{W}_T + \frac{1}{2}\sigma^2 T}$$

$$\tilde{\mathbb{Q}}\left[e^{\sigma W_T - \frac{1}{2}\sigma^2 T} > \frac{Ke^{-rT}}{S_0} \right] = \tilde{\mathbb{Q}}\left[\sigma \tilde{W}_T + \frac{1}{2}\sigma^2 T > \ln \frac{Ke^{-rT}}{S_0} \right]$$

$$= \tilde{\mathbb{Q}}\left[\frac{\tilde{W}_T}{\sqrt{T}} > \frac{\ln Ke^{-rT}/S_0 - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right]$$

$$= \Phi\left(\frac{\ln S_0/K + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$\therefore \quad C(S_0, 0) = S_0 \Phi(d_+) - e^{-rT} K \Phi(d_-)$$

$$d_\pm = \frac{1}{\sigma\sqrt{T}} \left\{ \ln S_0/K + (r \pm \tfrac{1}{2}\sigma^2)T \right\}$$

Note: call prices are increasing in volatility $\quad \frac{\partial C}{\partial \sigma} = S_0 \Phi'(d_+) \frac{\partial d_+}{\partial \sigma} - Ke^{-rT} \Phi'(d_-) \frac{\partial d_-}{\partial \sigma}$

But

$$\frac{S_0 \Phi'(d_+)}{Ke^{-rT}\Phi'(d_-)} = \frac{S_0}{Ke^{-rT}} \exp\left(-\frac{1}{2}\left[d_+^2 - d_-^2 \right] \right) = \frac{S_0}{Ke^{-rT}} \exp\left(-\frac{(d_+ + d_-)(d_+ - d_-)}{2} \right)$$

$$= \frac{S_0}{Ke^{-rT}} \exp\left(-\frac{(\ln S_0/Ke^{-rT} + rT)}{\sigma\sqrt{T}} \cdot \sigma\sqrt{T} \right) = 1$$

Thus $\frac{\partial C}{\partial \sigma} = S_0 \Phi'(d_+) \frac{\partial}{\partial \sigma}(d_+ - d_-) = S_0 \Phi'(d_+)\sqrt{T} > 0$

Issues with the Black-Scholes model

1) The model assumes σ is constant, but it is not
   - historical time series show fluctuations
   - implied volatilities not constant over maturity or strike.
     Instead see a smile or smirk, flattening over time.

2) Implied volatility is a better predictor of realised vol than historical volatility.

3) Today there are several candidate volatilities. Which one to use?

4) "Implied volatility is the wrong number to put in the wrong formula to get the right price."

Response: Drop the assumption of constant volatility.

Either: move to a richer class of models which better fit reality
- Level-dependent volatility models
- Stochastic volatility
- Lévy models

or: Construct models which exactly match options data.

Level Dependent Volatility Models

Idea: to capture the stylized fact that volatilities increase as prices go down.

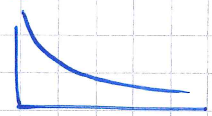
Cox (1975) Constant Elasticity of Variance Model

$$dS = \sigma S^{\beta}\,dW + rS\,dt$$

$\beta \in (0,1)$

$$\frac{dS}{S} = \sigma S^{\beta-1}\,dW + r\,dt$$

$\sigma S^{\beta-1}$

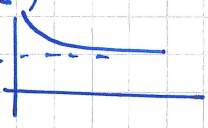


Rubinstein (1983) Displaced diffusion

$$dS = \sigma(S + J)\,dB + rS\,dt$$

$$\frac{dS}{S} = \sigma\left(1 + \frac{J}{S}\right)dB + r\,dt$$

$\sigma(1 + \frac{J}{S})$ 1



Firm has risky assets and debt.

Risky assets	V	$\cancel{dV = \sigma V\,dB + rV\,dt}$
Debt	D	$\cancel{dD = rD\,dt}$
Equity	$E = V - D$	$\cancel{dE = \sigma V\,dB + rE\,dt}$

$$\cancel{= \sigma(E + D)\,dB + rE\,dt} \quad = \cancel{\sigma(E_t + D_0 e^{rt})\,dB_t + rE_t\,dt}$$

If instead some of the risky assets are used to pay off debt.

$$dV = \sigma V\,dB + rV\,dt - rD\,dt$$

$$dD = rD\,dt - rD\,dt = 0$$

$$dE = \sigma(E + D)\,dB + rE\,dt$$

Rady (1997) Quadratic diffusion. $X_t = S_t e^{-rt}$

$$dX = \sigma(X - J)(K - X)\,dW$$

$J < K$

Then $J < X_t < K$

Geske (1979) Compound Options.

Firm has debt D to be repaid at time T_∞.

~~Asset price S is a call on~~
Firm has assets V whose value follows an GBM $\quad dV = \sigma V dW + rV dt$

Stock price is an option on the values of the firms assets with strike D, maturity T_∞

ie
$$S_t = \mathbb{E}\left[e^{-r(T_\infty - t)}(V_{T_\infty} - D)^+ \mid \mathcal{F}_t\right]$$

$$= V_t \, \Phi\left(\frac{\ln V_t/De^{-r(T_\infty-t)} + \frac{1}{2}\sigma^2(T_\infty - t)}{\sigma\sqrt{T-t}}\right) - De^{-r(T_\infty-t)}$$

$$- De^{-r(T_\infty - t)} \, \Phi\left(\frac{\ln V_t/De^{-r(T_\infty-t)} - \frac{1}{2}\sigma^2(T_\infty - t)}{\sigma\sqrt{T-t}}\right)$$

$$S_t = C_{BS}(V_t, t\,;\, D, T_\infty) = C_{BS}(V_t, t)$$

Here $e^{-rt}\, \overset{\text{discounted}}{C(V_t, t)} = \mathbb{E}\left[e^{-r(T_\infty - t)}(V_T - K)^+ \mid \mathcal{F}_t\right]$ is a martingale

$$d[e^{-rt}C(V_t, t)] = e^{-rt}\left\{-rC\,dt + C'\,dV + \tfrac{1}{2}C''(dS)^2 + \dot{C}\,dt\right\}$$

$$\therefore \; -rC + rVC' + \frac{\sigma^2}{2}V^2C'' + \dot{C} = 0$$

$$\boxed{C(V_{T_\infty}, T_\infty) = (V_{T_\infty} - D)^+}$$

Then
$$dC(V_t, t) = C'\,dV + \tfrac{1}{2}V\sigma^2\,dt + \dot{C}\,dt$$
$$= C'[dV - rV\,dt] + rC\,dt = C'\,dV + r[C'V - C]\,dt$$

Then S solves
$$dS_t = \sigma V C'(V_t, t)\,dW_t + rS\,dt$$
$$dS_t = \sigma\left\{C(\cdot, t)^{-1}(S_t)\right\}C'\left(C(\cdot, t)^{-1}(S_t), t\right)dW + rS\,dt$$

This is the displaced diffusion model.

It looks intractable, but it is OK for pricing Call Options.

Level dependent models

Model is complete, so unique option prices as discounted expectations.

Can have better fit — to volatility skews, but not to smiles.

When back at same price level, must see same ~~price~~ volatility surface.