Dupire's Level Dependent Volatility Model

Idea: Suppose we have option prices. Can we build a model which exactly fits those option prices. ~~Without these~~

$$dS = \Sigma_t dW_t + b_t dt$$

- Option prices are calculated under $\mathbb{Q}$, so we look for a model under $\mathbb{Q}$

$$dS = \Sigma_t dW_t^{\mathbb{Q}} + rS_t dt$$

- Try first for a (time and space inhomogeneous) model

$$dS_t = S_t \sigma(S_t, t)\, dW_t^{\mathbb{Q}} + rS_t\, dt.$$

Suppress the $\mathbb{Q}$ and write $dS_t = S_t \sigma(S_t, t)\, dW + rS\, dt$; $\mathbb{E}$ not $\mathbb{E}^{\mathbb{Q}}$.

We ~~assume~~ assume we have a double continuum of option prices

$$\{C(K,T)\}_{T \geq 0,\, K \geq 0}$$

- We suppose they are smooth so that we can find $\frac{\partial C}{\partial T}$ or $\frac{\partial C}{\partial K}$ etc
- Change of viewpoint: before for fixed (K,T) we found a PDE for C as function of (t, S_t). Now we fix $t=0$ and S_0 and consider C as ~~a~~ a function of option characteristics.

$$C(K,T) = \mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = e^{-rT}\mathbb{E}(S_T - K)^+$$

Differentiating wrt K $\qquad \frac{\partial}{\partial K}(S_T - K)^+ = -1_{(S_T > K)} \qquad \frac{\partial^2}{\partial K^2}(S_T - K) = -\delta_K(S_T)$

$$\left\{\begin{aligned} \frac{\partial C}{\partial K} &= -\mathbb{E}\left[e^{-rT}\, 1_{(S_T > K)}\right] = -e^{-rT}\,\mathbb{E}\left[1_{(S_T>K)}\right] \\ \frac{\partial^2 C}{\partial K^2} &= +e^{-rT}\,\mathbb{E}\left[\delta(S_T - K)\right] = e^{-rT} p(T,K) \\ &= e^{-rT} \int_0^\infty p_T(s)\,\delta_K(s)\,ds = e^{-rT} p_T(K). \end{aligned}\right\}$$

BETTER

$C(K,T) = \mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = e^{-rT}\int_0^\infty (s-K)^+ p_T(s)\,ds$ where $p_T(s)$ is the density of S_T

$$\frac{\partial C(K,T)}{\partial K} = -e^{-rT}\int_0^\infty 1_{(S_T > K)}\, p_T(s)\,ds = -e^{-rT}\int_K^\infty p_T(s)\,ds = -e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[1_{(S_T>K)}\right]$$

$$= -e^{-rt}\mathbb{Q}(S_T > K)$$

$$\frac{\partial^2}{\partial K^2} C(K,T) = +e^{-rT} p_T(K)$$

All assuming $\mathcal{L}(S_T)$ has a density

For any regular f and $\Delta > 0$

$$e^{-r(T+\Delta)} f(S_{T+\Delta}) = e^{-rT} f(S_T) + \int_T^{T+\Delta} e^{-ru} f'(S_u)\,dS_u + \int_T^{T+\Delta} \frac{1}{2} e^{-ru} f''(S_u)\,d[S]_u - r\int_T^{T+\Delta} e^{-ru} f(S_u)\,du$$

$$= e^{-rT} f(S_T) + \int_T^{T+\Delta} e^{-ru}\, r S_u f'(S_u)\,du + \int_T^{T+\Delta} e^{-ru} \sigma(S_u,u) S_u\,dW_u + \int_T^{T+\Delta} du\, \frac{1}{2} e^{-ru} f''(S_u) S_u^2 \sigma(S_u,u)^2 - r\int_T^{T+\Delta} e^{-ru} f(S_u)\,du$$

$$\frac{\mathbb{E}\left[e^{-r(T+\Delta)} f(S_{T+\Delta}) - e^{-rT} f(S_T)\right]}{\Delta}$$

$$= \frac{1}{\Delta}\mathbb{E}\left[\int_T^{T+\Delta} r e^{-ru} S_u f'(S_u)\,du\right] + \frac{1}{\Delta}\mathbb{E}\left[\int_T^{T+\Delta} e^{-ru} \frac{S_u^2}{2} \sigma(S_u,u)^2 f''(S_u)\,du\right] - \frac{r}{\Delta}\mathbb{E}\left[\int_T^{T+\Delta} e^{-ru} f(S_u)\,du\right]$$

Take $f(s) = (s-K)^+$

$$\frac{\partial C(\cancel{S},T,K)}{\partial T} = r e^{-rT}\mathbb{E}\left[S_T 1_{(S_T>K)}\right] + e^{-rT}\frac{K^2}{2}\sigma(K,T)^2 p(T,K) - r e^{-rT}\mathbb{E}\left[(S_T-K)^+\right]$$

$$= e^{-rT} r K\, \mathbb{E}\left[1_{(S_T>K)}\right] + e^{-rT}\frac{K^2}{2}\sigma(K,T)^2 p(T,K) - rC$$

$$\dot{C} = -rKC' + \frac{1}{2}K^2\sigma(K,T)^2 C'' - rC$$

$$\boxed{\sigma(K,T)^2 = \frac{\dot{C} + rKC' + rC}{\frac{1}{2}K^2 C''}} = \frac{2(\dot{C} + rK')}{K^2 C''}$$

Theorem

Suppose $C \in C^{2,1}$, $C \downarrow K$, $\dot{C} + rKC' > 0$, C convex in K. $C(0,T) = S_0$, $\lim_{K\uparrow\infty} C(K,T) = 0$

There exists a unique diffusion such that $e^{-rt}S_t$ is a martingale and such that call prices are given by C.

Advantages : Provides a perfect fit. Unique model can be used to price path-dependent derivatives eg Asians, Barriers,

: Overfitting, we don't have a double continuum of option prices, so must start by interpolating. Our answer will depend on how we do this.

Diffusion Models of Volatility

The idea is to model the volatility as a process in it's own right (typically as a diffusion,

Allows for historic variation in volatility levels, and future uncertainty over volatility

Can incorporate leverage, either explicitly, or through correlation between price level and volatility.

Model (S, Y) is a bi-variate diffusion under $\mathbb{P}$

$$(*) \begin{cases} dS = S\{\sigma(Y)\,dW + \mu(Y)\,dt\} & S_0 = s \\ dY = \alpha(Y)\,dB + \beta(Y)\,dt & Y_0 = y \end{cases}$$

$$d[B, W] = \rho(Y)\,dt$$

- More generally we could allow σ and μ to depend on S.
- We have written down the model under $\mathbb{P}$. Sometimes want to write it down under $\mathbb{Q}$.
- Making μ and σ depend on S is not necessary for leverage. Can use correlation instead.

$$\text{If } \rho < 0, \text{ then } \quad S\uparrow \overset{\sim}{\Leftrightarrow} W\uparrow \sim B\downarrow \sim Y\downarrow$$
$$(\text{and } \sigma(\cdot) > 0) \quad S\downarrow \qquad\qquad Y\uparrow$$

Different Parametrizations

The model $(*)$ can be parametrized in many different ways

Let $Z = \psi(Y)$, ie $Z_t = \psi(Y_t)$. Then $Y_t = \psi^{-1}(Z_t)$

Then
$$\begin{aligned} dZ_t &= \psi'(Y_t)\,dY_t + \tfrac{1}{2}\psi''(Y_t)\,d[Y]_t \\ &= \psi'(Y_t)\alpha(Y_t)\,dB_t + \left[\tfrac{1}{2}\psi''(Y_t)\alpha(Y_t)^2 + \psi'(Y_t)\beta(Y_t)\right]dt \\ &= \psi'(\psi^{-1}(Z_t))\,\alpha(\psi^{-1}(Z_t))\,dB_t + \left[\tfrac{1}{2}\psi''(\psi^{-1}(Z_t))\,\alpha(\psi^{-1}(Z_t))^2 + \{\psi'(\psi^{-1})\cdot\beta(\psi^{-1})\}(Z_t)\right]dt \\ &= \alpha_Z(Z_t)\,dB_t + \beta_Z(Z_t)\,dt \end{aligned}$$

There are two natural choices : $\Sigma = \sigma(Y)$

or $V = \sigma(Y)^2$

In the former case

$$dS_t = S_t\left[\Sigma_t dW_t + \mu_\Sigma(\Sigma_t)dt\right]$$
$$d\Sigma_t = a_\Sigma(\Sigma_t)dB_t + b_\Sigma(\Sigma_t)dt \qquad d[B,W] = \rho_\Sigma(\Sigma_t)dt$$

In the latter case

$$dS_t = S_t\left[\sqrt{V_t}\,dW_t + \mu_V(V_t)dt\right]$$
$$dV_t = a_V(V_t)dB + b_V(V_t)dt$$

Choices of Model

We choose models to be tractable, so we can calculate transition densities ~~option prices/expec~~
have realistic properties (eg $\rho \le 0$)
provide a good fit to data.

~~We reqt~~ If we model via V, we should require $V \geq 0$.

If we model using Σ we have the issue that we see $[S]_t$ and therefore Σ_t^2, but may not be able to observe Σ.
Question: Is $\mathcal{F}_t^S = \sigma((S_u); 0 \le u \le t)$ equal to $\mathcal{F}_t^{B,W} = \mathcal{F}_t^{S,\Sigma} = \sigma((S_u, \Sigma_u); 0 \le u \le t)$?

Some models

1) Σ is exponential brownian motion

If $\rho = 0$ Hull-White

If $\rho \neq 0$ Wiggins

2) $\Sigma = e^u$ where u is an OU process. Scott

$$du = \gamma dB + (\beta - \lambda u)dt \qquad \beta \text{ is non-centrality}$$

$$\Sigma = e^u \qquad d\Sigma = e^u du + \tfrac{1}{2}e^u d[u]$$
$$= \gamma\Sigma dB + \Sigma(\beta - \lambda\ln\Sigma)dt + \frac{\gamma^2}{2}\Sigma dt$$
$$= \gamma\Sigma dB + \Sigma\left(\beta + \frac{\gamma^2}{2} - \lambda\ln\Sigma\right)dt$$

Σ is a mean reverting process.

3) Σ is an OU process.

Note: we can instantaneously have $\Sigma = 0$, and Σ is $\mathbb{R}$-valued.

From observing S we can see $V = \Sigma^2$, but we do not know if $\Sigma = +\sqrt{V}$ or $\Sigma = -\sqrt{V}$

$$d\Sigma = \gamma \, dB + (\beta - \lambda \Sigma) \, dt$$

Exercise
$$dU_t^{(i)} = \theta \, dB_t^{(i)} - \frac{\lambda}{2} U_t^{(i)} dt$$
Find an SDE for Q where
$$Q_t = \sum_{i=1}^{n} \{U_t^{(i)}\}^2$$

4) Square-root Model (Heston)

$$dV_t = a\sqrt{V_t}\, dB_t + (b - cV_t)\, dt$$

$$\frac{dS_t}{S_t} = \sqrt{V_t}\, dW_t + \mu(V_t)\, dt$$

Then $\Sigma_t = \sqrt{V_t}$

$$d\Sigma_t = \frac{1}{2\sqrt{V_t}} dV_t - \frac{1}{8}\frac{(dV_t)^2}{V_t^{3/2}}$$
$$= \frac{1}{2\Sigma_t}\left(a\Sigma_t dB_t + (b - c\Sigma_t^2)\, dt\right) - \frac{1}{8}\frac{a^2\Sigma_t^2}{\Sigma_t^3} dt$$
$$= \frac{a}{2} dB_t + \left(\frac{b}{2} - \frac{a^2}{8}\right)\frac{1}{\Sigma_t} dt - \frac{c\Sigma_t}{2} dt$$

5) $3/2$ Model

$$dV_t = aV_t^{3/2} dB_t + (bV - cV^2)\, dt$$

$$d\Sigma_t = \frac{a}{2}\Sigma_t^2 dB_t + \left(\frac{b}{2}\Sigma - \frac{c}{2}\Sigma^3\right) dt - \frac{a^2}{8}\Sigma^3 dt$$
$$= \frac{a}{2}\Sigma_t^2 dB_t + \left(\frac{b}{2}\Sigma - \left(\frac{c}{2} + \frac{a^2}{8}\right)\Sigma^3\right) dt$$

6) Heston–Nandi Model.

Square-root model, but take $\rho = -1$.

(Hagan, Kumar, Lesniewski & Woodward) (2002)

7) SABR model

Let $F_t = S_t e^{-rt}$ (Forward price) $\qquad dF_t = \Sigma_t F_t^{\beta} dW_t$

$$d[B, W] = \rho\, dt \qquad d\Sigma_t = \alpha \Sigma_t \, dB_t$$

Stochastic – alpha–beta–rho model.
Designed for small maturities. Allegedly easy to relate parameters to the shape of the implied volatility smile.

Option pricing in Stochastic Volatility Models

~~Suppose $dS = S\sqrt{V}\,dW + \mu(S,V)\,dt$ $\quad dV = a(V)\,dB + b(V)\,dt = a(V)\{\rho(V)\,dW + \sqrt{1-\rho(V)^2}\,dW^\perp\} + b(V)\,dt$~~

Definition

$\mathbb{Q}$ is an equivalent martingale measure if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $(e^{-rt}S_t)$ is a $\mathbb{Q}$ – martingale. (For an ELMM – $(e^{-rt}S_t)$ is a $\mathbb{Q}$ – local martingale)

A natural approach is to write down price under an EMM

1) Suppose the model is written down under $\mathbb{Q}$

$$dS = S\sqrt{V}\,dW + rS\,dt \qquad dV = a(V)\,dB + b(V)\,dt$$
$$= a(V)\{\rho(V)\,dW + \sqrt{1-\rho(V)^2}\,dW^\perp\} + b(V)\,dt$$

W and B are $\mathbb{Q}$ – brownian motions

$$C_t := \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}H(S_T)\,|\,\mathcal{F}_t\right]$$
$$C_t = C(S_t, V_t, t) = e^{rt}\mathbb{E}[H(S_T)\,|\,\mathcal{F}_t]$$

$\therefore \left(e^{-rt}C(S_t, V_t, t) =: M_t\right)_{t\geq 0}$ is a martingale

eg $H(S_T) = (S_T - K)_+$ $\quad K > 0$ and $T > 0$ are fixed constants.

Since M is a martingale we can derive an ODE for C:

$$dM_t = d[e^{-rt}C_t] = -re^{-rt}C(S_t,V_t,t)\,dt + e^{-rt}\left\{C_S\,dS + C_V\,dV + \frac{C_{SS}}{2}d[S] + C_{SV}\,d[S,V] + C_{VV}\,d[V]\right\}$$

$$= e^{-rt}C_S S\sqrt{V}\,dW + e^{-rt}C_V a(V)\{\rho(V)\,dW + \sqrt{1-\rho(V)^2}\,dW^\perp\}$$
$$+ e^{-rt}dt\left[-rC + rSC_S + b(V)C_V + \frac{S^2V}{2}C_{SS} + S\sqrt{V}\rho(V)C_{SV}a(V) + \frac{1}{2}a(V)^2C_{VV}\right]$$

C solves

$$-rC + rSC_S + b(V)C_V + \frac{S^2V}{2}C_{SS} + S\sqrt{V}a(V)\rho(V)C_{SV} + \frac{1}{2}a(V)^2C_{VV} = 0$$

subject to

$$C(S,V,T) = H(S)$$
$$C(0,V,T) = 0$$
$$\lim_{V\uparrow\infty} C(S,V,T) = S \qquad (\text{if } H(S) = (S-K)^+)$$

Note

$$dC_t = d[e^{rt}M_t] = rC_t\,dt + e^{rt}dM_t$$
$$= C_S S\sqrt{V}\,dW + C_V a(V)\{\rho(V)\,dW + \sqrt{1-\rho(V)^2}\,dW^\perp\} + rC_t\,dt$$
$$= \left\{C_S + \frac{C_V a(V)}{S\sqrt{V}}\rho(V)\right\}(dS - rS\,dt) + rC\,dt + C_V a(V)\sqrt{1-\rho(V)^2}\,dW^\perp$$

Set $\textcircled{H} = C_S + \frac{C_V a(V) \rho(V)}{S\sqrt{V}}$

$$dC_t = \textcircled{H}\, dS_t + r(C - \textcircled{H} S_t)\, dt + a(V) C_V \sqrt{1-\rho(V)^2}\, dW_t^\perp$$

If we hold $\textcircled{H}$ shares

$$H(S_T) = C(S_T, V_T, T) = C_0 + \int_0^T \textcircled{H}_u\, dS_u + \int_0^T r(C - \textcircled{H}_u S_u)\, du + \int_0^T a(V_u) C_V \sqrt{1-\rho^2}\, dW^\perp$$

$$\underbrace{C_0 + \int_0^T \textcircled{H}_u\, dS_u + \int_0^T r(C_u - \textcircled{H}_u S_u)\, du}_{\text{Terminal wealth ; gains from trade}} = H(S_T) - \int_0^T a(V_u) C_V(S_u, V_u, u) \sqrt{1-\rho^2(V_u)}\, dW^\perp$$

Option payout + hedging error

A) $\int_0^T a(V_u) C_V \sqrt{1-\rho(V_u)^2}\, dW^\perp$ is unhedgeable. Ignore it. Hope that it cancels out if we have a large portfolio of options. Sometimes win, sometimes loose.

B) Suppose there was a second option which traded on the market, with price $P = P(S_t, V_t, t)$.

We could write

$$dP_t = \Phi\, dS_t + r(P - \Phi S_t)\, dt + a(V)\sqrt{1-\rho(V)^2}\, dW_t^\perp P_V$$

Consider holding a portfolio of

$\textcircled{H}_t - \frac{\Phi_t C_V}{P_V}$ shares

C_V / P_V units of the second option

$C - \left(\textcircled{H} - \frac{\Phi_t C_V}{P_V}\right) S_t - \frac{C_V}{P_V} P$ cash

Change in value of this portfolio is

$$\left(\textcircled{H}_t - \frac{\Phi_t C_V}{P_V}\right) dS + \frac{C_V}{P_V} dP + r\left(C - \left(\textcircled{H} - \frac{\Phi C_V}{P_V}\right) S - \frac{C_V}{P_V} P\right) dt$$

$$= \textcircled{H}\, dS + r(C - \textcircled{H} S)\, dt + \frac{C_V}{P_V}\left[dP - \Phi\, dS + \Phi S r\, dt - rP\, dt\right] = dC_t$$