Option Pricing in SV models

Suppose we write down a model under $\mathbb{P}$

$$\frac{dS}{S} = \sqrt{V}\, dW + \mu\, dt \qquad dV = a(V)\, dB + b(V)\, dt$$

$$\mu = \mu(S,V)$$

$$dB = \rho(V)\, dW + \rho^{\perp} dW^{\perp}$$

$$\rho^{\perp} = (1-\rho(V)^2)^{1/2}$$

Suppose $Z = (Z_t)_{0 \le t \le T}$ is given by

$$Z_t = \exp\left(\cancel{-} \int_0^t \theta_s\, dW_s \cancel{-} \int_0^t \phi_s\, dW_s^{\perp} - \frac{1}{2}\int_0^t (\theta_s^2 + \phi_s^2)\, ds \right)$$

where θ and ϕ are adapted processes.

We have $Z_t \geq 0$. Suppose $\int_0^T (\theta_s^2 + \phi_s^2)\, ds < \infty$ a.s.

Then $Z_T > 0$ a.s. [illegible]

We have $dZ_t = Z_t(\theta\, dW_t \cancel{-} \phi\, dW_t^{\perp})$

Then Z_t is a local martingale.

Hence $Z = (Z_t)_{0 \le t \le T}$ is a supermartingale.

Problem Sheet

An important question is: when is Z a true-martingale.

$\mathbb{E}[Z_T] = 1$. Then Z is a mg.

Sufficient condition 1) θ and ϕ are bounded processes on $[0,T]$

2) Novikov $\mathbb{E}\left[e^{1/2 \int_0^T (\theta_s^2 + \phi_s^2) ds} \right] < \infty$

<u>Girsanov's Theorem</u>

Suppose $Z = Z^{\theta,\phi}$ is a martingale and $\int_0^T(\theta^2 + \phi^2)\, ds < \infty$ a.s. Define $\mathbb{Q}$ on $\mathcal{F}_T$ by via $\frac{d\mathbb{Q}}{d\mathbb{P}}$ $= \mathbb{Q}(A) = \mathbb{E}[Z_T 1_A]$

Then $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and under $\mathbb{Q}$ $W^{\mathbb{Q}}$ and $W^{\mathbb{Q},\perp}$ are ~~martingales~~ brownian motion where

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s\, ds \qquad W_t^{\mathbb{Q},\perp} = W_t^{\perp} + \int_0^t \phi_s\, ds.$$

Under $\mathbb{Q}$

$$\frac{dS}{S} = \sqrt{V}(dW_t^{\mathbb{Q}} \cancel{-} \theta\, dt) + \mu\, dt$$
$$= \sqrt{V}\, dW_t^{\mathbb{Q}} + (-\theta\sqrt{V} + \mu)\, dt$$

$$dV = a(V)(\rho\, dW + \rho^{\perp} dW^{\perp}) + b(V)\, dt$$
$$= a(V)[\rho(V)\, dW^{\mathbb{Q}} + \rho^{\perp}(V)\, dW^{\mathbb{Q},\perp}]$$
$$+ [-a(\rho\theta + \rho^{\perp}\phi) + b]\, dt$$

$\mathbb{Q}$ is an equivalent local martingale measure iff

$$-\theta\sqrt{V} + \mu = r \qquad \text{ie } \theta = +\frac{(\mu - r)}{\sqrt{V}} = +\frac{(\mu - r)}{\Sigma}$$

ϕ is undetermined.

We say $\theta = \frac{\mu - r}{\Sigma}$ is the market price of price risk

Hence,

There is a family of ELMMs. For each such $\mathbb{Q}^{\theta,\phi}$ we have $\theta = \frac{\mu - r}{\Sigma}$.

ϕ is undetermined.

Under $\mathbb{Q}^{\frac{\mu-r}{\Sigma},\phi}$ $\quad dV = a(V)\,dB^{\mathbb{Q}} + \left[b - a(V)\left\{\rho(V)\frac{(\mu - r)}{\Sigma} + \rho^{\perp}(V)\,\phi(V)\right\}\right]dt$

For the <u>minimal</u> martingale measure $\phi = 0$

rationale — change those Brownian motions (risks) which we have to for risk neutrality, but do not change the drifts on any orthogonal BMs.

<u>Another approach.</u>

Let h be an increasing concave function.

The h-minimal ELMM is the choice ϕ such that

$$\mathbb{E}\left[h\left(Z_T^{\mu - r/\Sigma,\, \phi}\right)\right]$$

is maximised.

Example $h(z) = z^p$

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Option pricing if S and Σ are independent

$$\frac{dS}{S} = \Sigma_s \, dW_s + r\,ds$$

$$d\Sigma_s = a(\Sigma_s)\,dB_s + b(\Sigma_s)\,ds$$

B, W are independent
This is under the pricing measure

$$\mathbb{E}_0\left[e^{-rT}(S_T - K)^+\right]$$

$$= \mathbb{E}\left[e^{-rT}\left(S_0 e^{rT} e^{\int_0^T \Sigma_s dW_s - 1/2\int_0^T \Sigma_s^2 ds} - K\right)^+\right]$$

$$= \mathbb{E}\left[e^{-rT}\left(S_0 e^{\overline{W}_{A_T} - A_T/2} - K\right)^+\right]$$

$$A_T = \int_0^T \Sigma_s^2 ds$$

$$\overline{W}_{A_t} = \int_0^t \Sigma_s \, dW_s$$

$$[\overline{W}]_{A_t} = A_t$$

$$[\overline{W}]_t = t$$

$\therefore$ By Lévy's Theorem $\overline{W}$ is a BM

Tower property $\mathcal{G} \subseteq \mathcal{H}$

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}]$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{-rT}\left(S_0 e^{\overline{W}_{A_T} - A_T/2} - K\right)^+ \,\middle|\, \sigma((\Sigma_s)_{0\le s\le T})\right]\right]$$

$$= \mathbb{E}\left[C_{BS}\left(K, T; r, \frac{A_T}{T}\right)\right]$$

where $C_{BS}(K,T;r,\sigma^2) = \mathbb{E}\left[e^{-rT}\left(S_0 e^{rT + \sigma W_T - 1/2\sigma^2 T} - K\right)^+\right]$

This is an average value of Black-Scholes prices.

As a special case, suppose $\Sigma = \Sigma(s)$ is just a deterministic function.

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = C_{BS}\left(K, T; r, \frac{1}{T}\int_0^T \Sigma(s)^2 ds\right)$$

Options on Volatility, and VIX

The key quantity in the value of European options with maturity T is $\int_0^T \sigma_s^2 \, ds$.

Hence a desire for securities with payoff which is contingent on $\int_0^T \sigma_s^2 \, ds$ or $\int_0^T \frac{d[S]}{S^2}$.

Dupire / Neuberger formula.

Let F denote the forward price of the asset

$$F_t = e^{-r(T-t)} S_t$$

Suppose S is a continuous semi-martingale

$$dS_t = \sigma_t S_t \, dW_t + r S_t \, dt$$

Then F is a continuous local martingale

$$dF_t = \sigma_t F_t \, dW_t$$

$$\int_0^T \frac{d[F]_t}{F_t^2} = \int_0^T \sigma_t^2 \, dt$$

Now consider $Y_t := -2 \ln F_t$

$$dY_t = -\frac{2 dF_t}{F_t} + \frac{d[F]_t}{F_t^2}$$

$$Y_T = Y_0 - \int_0^T \frac{2}{F_t} dF_t + \int_0^T \frac{d[F]_t}{F_t^2}$$

$$\int_0^T \frac{1}{F_t^2} d[F]_t = \underbrace{-2 \ln F_T / F_0}_{\text{Static payoff}} + \underbrace{\int_0^T \frac{2}{F_s} dF_s}_{\text{Gains from trade}}$$

Breeden-Litzenberger formula

Consider $g : \mathbb{R}_+ \to \mathbb{R}$

$$\begin{aligned} g(y) &= g(y_0) + \int_{y_0}^y g'(z) \, dz \\ &= g(y_0) + \int_{y_0}^y g'(y_0) + \int_{y_0}^y dz \int_{y_0}^z g''(w) \, dw \\ &= g(y_0) + g'(y_0)(y - y_0) + \int_{y_0}^y dw \, g''(w) \int_w^y dz \\ &= g(y_0) + g'(y_0)(y - y_0) + \int_{y_0}^y dw \, g''(w)(y - w) \end{aligned}$$

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$$\langle g, f \rangle = \int_0^\infty g(y) f(y) \, dy = g(y_0) \int_0^\infty f(y) dy + g'(y_0) \int_0^\infty (y - y_0) f(y) \, dy$$

$$+ \int_{y_0}^\infty f(y) \, dy \int_{y_0}^y dw \, g''(w)(y - w)$$

$$+ \int^{y_0} f(y) \, dy \int^{y_0} g''(w)(w - y$$

$$= g(y_0)\int_0^\infty f(y)\,dy + g'(y)\int_0^\infty (y-y_0) f(y)\,dy$$
$$+ \int_{y_0}^\infty d\omega\, g''(\omega)\int_\omega^\infty f(y)\,dy\,(y-\omega)$$
$$+ \int_0^{y_0} d\omega\, g''(\omega)\int_0^\omega f(y)(y-\omega)$$

Now, suppose f is the density of a RV Y with mean y_0

$$\mathbb{E}[g(Y)] = g(y_0) + \int_{y_0}^\infty d\omega\, g''(\omega)\,\mathbb{E}[(Y-\omega)^+] + \int_0^{y_0} d\omega\, g''(\omega)\,\mathbb{E}[(\omega-Y)^+]$$

Define $C(\omega) = C_Y(\omega) = \mathbb{E}[(Y-\omega)^+]$

$P(\omega) = P_Y(\omega) = \mathbb{E}[(\omega-Y)^+]$

$$\mathbb{E}[g(Y)] = g(y_0) + \int_{y_0}^\infty d\omega\, g''(\omega)\, C(\omega) + \int_0^{y_0} d\omega\, P(\omega)\, g''(\omega)$$

$$\mathbb{E}\left[-2\ln F_T/F_0\right] = \int_{F_0}^\infty d\omega\, \frac{2}{\omega^2} C(\omega) + \int_0^{F_0} d\omega\, \frac{2}{\omega^2} P(\omega)$$

$g(f) = -2\ln f/F_0$

$g'(f) = -\frac{1}{f}$ $\qquad g''(f) = \frac{2}{f^2}$

$$g(y) = g(y_0) + g'(y_0)(y-y_0) + \int_{y_0}^\infty d\omega\, g''(\omega)(y-\omega)^+ + \int_0^{y_0} d\omega\, g''(\omega)(\omega-y)^+$$

Let $\mathcal{C}(\omega)$ be the payoff from a ~~spot~~ call with strike ω

$\mathcal{P}(\omega)$ put

$$g(y) = g(y_0) + g'(y_0)(y-y_0) + \int_{y_0}^\infty d\omega\, g''(\omega)\,\mathcal{C}(\omega) + \int_0^{y_0} d\omega\, g''(\omega)\,\mathcal{P}(\omega)$$

CONSTANT

PAYOFF FROM $g'(y_0)$ UNITS OF THE FORWARD

PAYOFF FROM A PORTFOLIO OF CALLS

PAYOFF FROM A PORTFOLIO OF PUTS