Branching Markov Processes and Evolution Semigroups

Introduction to the Theory

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(Markov property) Let $B(E_\dagger)$ be the space of bounded measurable functions on $E_\dagger$. For all $x \in E_\dagger$, $s, t \geq 0$ and $f \in B(E_\dagger)$, we have,

$$\mathbf{E}_x[f(\xi_{t+s})|\mathcal{F}_t] = \mathbf{E}_y[f(\xi_s)],$$

where $y = \xi_t$ $\mathbf{P}_x$-almost surely.

Expectation Semigroup

Definition

Let $B^+(E)$ be the space of bounded non-negative measurable functions on E. For $s, t \geq 0$, $g \in B^+(E)$ and $x \in E$ define,

$$\mathrm{P}_t[g](x) = \mathbf{E}_x[g(\xi_t)\mathbb{1}_{(t<\zeta)}], \text{for } t \geq 0 \text{ and } g \in B^+(E).$$

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We use the notation $\mathrm{P} = (\mathrm{P}_t, t \geq 0)$ to denote the associated semigroup to the Markov process $(\xi, \mathbf{P})$, accordingly, we later referred to it as a P-Markov process.

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Suppose that $\gamma : E \mapsto \mathbb{R}$, then we can also define

$$\mathrm{P}^\gamma[g](x) = \mathbf{E}_x\left[e^{\int_0^t \gamma(\xi_s)ds} g(\xi_t)\right] \quad x \in E, t \geq 0.$$

Dynkins Lemma

Theorem

Suppose that $|\gamma| \in B^+(E)$, $g \in B^+(E)$, *and* $\sup_{s \leq t} |h_s| \in B^+(E)$, *for all* $t \geq 0$.

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then it also solves

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The converse statement is also true if $(\chi_t, t \geq 0)$ *solves* (1) *with* $sup_{s \geq t}|\chi_s| \in B^+(E)$, *for all* $t \geq 0$.

Branching Regime

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- Given their point of creation, particles move independently according to a P-Markov proces on E.
- In an event, which we refer to as "branching", particles positioned at x die at rate $\gamma(x)$, where $\gamma \in B^{+}(E)$, and instantaneously, new particles are created in E according to a point process.
- We can think of a point process simply as a random variable N, representing the number of offspring, and $(x_1, \cdots, x_N)$ in E representing their locations.

Offspring Counting Measure

Definition

We define the offspring counting measure via random counting measures as follows,

$$\mathbf{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A) \quad A \in \mathscr{B}(E),$$

where $\mathscr{B}(E)$ is the collection of Borel sets in E.

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- We also allow the possibility of death of the parent without offspring i.e. $\mathscr{P}_x(N = 0) > 0$ for some or all $x \in E$. Note that the law of N can depend on x, the point of death of the parent.

Branching Mechanism

Definition

We define the branching mechanism to be

$$\mathrm{G}[f](x) := \gamma(x)\mathscr{E}_x\left[\prod_{i=1}^{N} f(x_i) - f(x)\right], \quad x \in E,$$

where

$$f \in B_1^+(E) := \{f \in B_1^+(E) : ||f|| \leq 1\},$$

and we recall that $\gamma \in B_1^+(E)$. Here, we use $\|\cdot\|$ to be the usual supremum norm on $B_1^+(E)$.

Local Branching

In the case of local branching, we obtain (by letting $f(x_i) = f(x) = s$) for all $i \in \{0, \cdots, N\}$

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as $\mathscr{E}_x[s^N]$ is the probability generating function of N and where, for $k \geq 1$ and $x \in E$, $p_k(x)$ denotes the probability that a particle branching at site x produces k offspring. We refer to $(p_k(x), k \geq 0)$ as the offspring distribution at site $x \in E$.

Counting Measures

Let

$$\mathscr{M}_c(E) := \{\sum_{i=1}^{n} \delta_{x_i} : n \in \mathbb{N}, x_i \in E, i = 1, \cdots n\}$$

represents the space of finite counting measures.

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Definition (Branching Markov Process)

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$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \quad t \geq 0,$$

In particular, X is Markovian in $\mathscr{M}_c(E)$ and its probabilities will be denoted by $\mathbb{P} := (\mathbb{P}_\mu, \mu \in \mathscr{M}_\mu(E))$.

Non-linear Semigroup

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is the natural analytical object that gives us a complete understanding of the law of our BMP. We also have the branching Markov property. That is, if we define

$$\mathscr{F}_t = \sigma(x_i(s), i = 1, \cdots, N_s, s \leq t), \quad t \geq 0,$$

then

$$\mathbb{E}\left[e^{-X t+s[f]}\middle|\mathscr{F}_t\right] = \prod_{i=1}^{N_t} \mathrm{v}_s[f](x_i(t)).$$

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From here the semigroup property,

$$\mathrm{v}_{t+s}[f](x) = \mathrm{v}_t[\mathrm{v}_s[f]](x), \quad s, t \geq 0, x \in E, f \in B^+(E),$$

follows.

Non-linear Semigroup Evolution

Moreover, for $f \in B^+(E)$ and $x \in E$,

$$\mathrm{v}_t[f](x) = \hat{\mathrm{P}}_t[e^{-f}](x) + \int_0^t \mathrm{P}_s[\mathrm{G}[\mathrm{v}_{t-s}[f]]](x)ds, \quad t \geq 0, \tag{2}$$

where $(\hat{\mathrm{P}}, t \geq 0)$ is the adjusted semigroup which returns a value of 1 on the event of killing, i.e., when the particle is absorbed at the boundary.

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$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{v}_t[f](x)\Big|_{t=0} &= \frac{\partial}{\partial t}\hat{\mathrm{P}}_t[e^{-f}](x)\Big|_{t=0} + \frac{\partial}{\partial t}\int_0^t \mathrm{P}_s[\mathrm{G}[\mathrm{v}_{t-s}[f]]](x)ds\Big|_{t=0} \\ &= \mathscr{L}\mathrm{v}_0[f](x) + \mathrm{G}[\mathrm{v}_0[f]](x)\end{aligned}$$

where we have used the Fundamental Theorem of Calculus and the chain rule to achieve the result.

Non-linear Semigroup Evolution

Note that by the semigroup property we can obtain the relation,

$$\frac{\partial}{\partial t}\mathrm{v}_t[f](x) = \frac{\partial}{\partial t}\mathrm{v}_{t+s}[f](x)\bigg|_{s\downarrow 0}$$

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&= \frac{\partial}{\partial s}\mathrm{v}_s[\mathrm{v}_t[f]](x)\bigg|_{s\downarrow 0}.
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It follows that we can rewrite equation (2) as,

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It follows that we can rewrite equation (2) as,

$$\frac{\partial}{\partial t}\mathrm{v}_t[f](x) = \mathscr{L}\mathrm{v}_t[f](x) + \mathrm{G}[\mathrm{v}_t[f]](x), \quad t \geq 0, x \in \mathbb{R}^d. \tag{3}$$

Branching Brownian Motion

By taking our Markov process to be Brownian motion (in any dimension) on some domain $E = \mathbb{R}^d$ and the branching mechanism is local with no spatial dependence, (3) collapses to

$$\frac{\partial}{\partial t} v_t[f](x) = \frac{1}{2}\Delta \mathrm{v}_t[f](x) + \gamma \left[\sum_{i=1}^{\infty} p_k \mathrm{v}_t[f](x)^k - \mathrm{v}_t[f](x) \right], \quad t \geq 0, x \in \mathbb{R}^d,$$

where $(p_k, k \geq 1)$ is the offspring distribution.

FKPP Equation

In particular, in one dimension with dyadic branching, we recover the Fisher–Kolmogorov–Petrsovskii–Piscunov (FKPP) equation

$$\frac{\partial}{\partial t}\mathrm{v}_t = \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathrm{v}_t + \gamma \mathrm{v}_t(\mathrm{v}_t - 1), \quad t \geq 0.$$

Dyadic Branching Brownian Motion

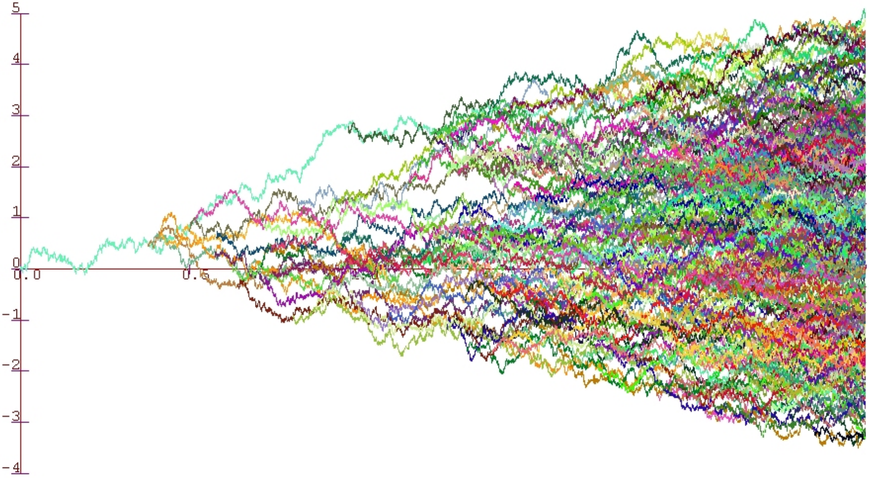

Figure: Dyadic Branching Brownian Motion [2]

Mean Semigroup

Definition (Mean Semigroup)

We define the mean semigroup of our process to be,

$$\psi[f](x) = \mathbb{E}_{\delta_x}[X_t[f]], \quad x \in E, f \in B^+(E), t \geq 0.$$

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It can be shown that the mean semigroup is a semigroup. Set

$$\mathrm{F}[f](x) = \gamma(x)\mathscr{E}_x\left[\sum_{i=1}^{N} f(x_i) - f(x)\right] := \gamma(x)(\mathrm{m}[f](x) - f(x)), \quad x \in E.$$

We assume that $\mathscr{E}_x[N] < \infty$, call this (G1).

Linear Evolution Equation

Lemma

Under (G1), the mean semigroup $(\psi_t, t \geq 0)$ *satisfies*

$$\psi_t[f](x) = \mathrm{P}_t[f](x) + \int_0^t \mathrm{P}_s[\mathrm{F}\psi_{t-s}[f]](x)ds, \quad t \geq 0, x \in E, f \in B^+(E). \tag{4}$$

As an operator from $B^+(E)$ *to itself,* $(\psi_t, t \geq 0)$ *is uniquely determined by* (4)

Regime of the Many-to-One

- Suppose that $\xi = (\xi_t, t \geq 0)$, with probabilities $\mathbf{P} = (\mathbf{P}_x, x \in E)$, is the Markov process corresponding to the semigroup P.

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- Let us introduce a new Markov process $\hat{\xi} = (\hat{\xi}_t, t \geq 0)$ which evolves as the process ξ , but at rate $\gamma(x)\mathrm{m}[1](x)$ the process is sent to a new position in E, such that for all Borel $A \subset E$, the new position is in A with probability $\mathrm{m}[1A](x)/\mathrm{m}[1](x)$.

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- We will refer to the latter as *extra jumps*. Note the law of the extra jumps is well defined thanks to the assumption (G1), which we earlier remarked ensures that $\sup_{x\in E} \mathrm{m}[1](x) < \infty$.
- We denote the probabilities of $\hat{\xi}$ by $(\hat{\mathbf{P}}_x, x \in E)$.

Many-to-One

Lemma (Many-to-One)

Write $\mathrm{B}(x) = \gamma(x)(\mathrm{m}[1](x) - 1)$, $x \in E$. *For* $f \in B^+(E)$ *and* $t \geq 0$, *under (G1), we have*

$$\psi_t[f](x) = \hat{\mathbf{E}}_x\left[\exp\left(\int_0^t \hat{\mathrm{B}}(\hat{\xi}_s)ds\right) f(\hat{\xi}_s)\right] \qquad (5)$$

Proof of Many-To-One

Proof.

First note that (4) is equivalent to

$$\begin{aligned}\psi_t[f](x) &= \mathrm{P}_t[f](x) + \int_0^t \mathrm{P}_s[\gamma(\mathrm{m}[\psi_{t-s}[f]] - \psi_{t-s}[f])](x)ds \\ &= \mathrm{P}_t[f](x) + \int_0^t \mathrm{P}_s[\gamma \mathrm{m}[1](\frac{\mathrm{m}[\psi_{t-s}[f]]}{\mathrm{m}[1]} - \psi_{t-s}[f])](x)ds \\ &\quad + \int_0^t \mathrm{P}_s[\mathrm{B}\psi_{t-s}[f]](x)ds\end{aligned}$$

At the same time, suppose we denote the right-hand side of (5) by $\hat{\psi}_t[f](x)$, $t \geq 0$.

Proof of Many-To-One

Proof (Cont.)

By conditioning this expectation on the first extra jump, we get, for $f \in B^+(E)$, $x \in E$ and $t \geq 0$,

$$\hat{\psi}_t[f](x) =$$

$$\hat{\mathbf{E}}_x\left[e^{-\int_0^t \gamma(\hat{\xi}_s)\mathrm{m}[1](\hat{\xi}_s)ds} e^{\int_0^t \mathrm{B}(\hat{\xi}_s)ds} f(\hat{\xi}_t)\right]$$

$$+ \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{m}[1](\hat{\xi}_s)\gamma(\hat{\xi}_s)e^{-\int_0^s \gamma(\hat{\xi}_u)\mathrm{m}[1](\hat{\xi}_u)du} e^{\int_0^s \mathrm{B}(\hat{\xi}_u)du} \frac{\mathrm{m}[\hat{\psi}_{t-s}[f]](\hat{\xi}_s)}{\mathrm{m}[1](\hat{\xi}_s)} ds\right]$$

Proof of Many-To-One

Proof (Cont.)

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$$+ \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{m}[1](\hat{\xi}_s)\gamma(\hat{\xi}_s)e^{-\int_0^s \gamma(\hat{\xi}_u)\mathrm{m}[1](\hat{\xi}_u)du} e^{\int_0^s \mathrm{B}(\hat{\xi}_u)du} \frac{\mathrm{m}[\hat{\psi}_{t-s}[f]](\hat{\xi}_s)}{\mathrm{m}[1](\hat{\xi}_s)} ds\right]$$

$$= \hat{\mathbf{E}}_x[f(\hat{\xi}_t)] + \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{m}[1](\hat{\xi}_s)\gamma(\hat{\xi}_s)\frac{\mathrm{m}[\hat{\psi}_{t-s}[f]](\hat{\xi}_s)}{\mathrm{m}[1](\hat{\xi}_s)} ds\right]$$

$$+ \int_0^t \hat{\mathbf{E}}_x[\mathrm{B}(\hat{\xi}_{t-s}) - \gamma(\hat{\xi}_{t-s})\mathrm{m}[1](\hat{\xi}_{t-s}))\hat{\psi}_{t-s}[f](\hat{\xi}_s)]ds$$

(by Dynkins Lemma)

Proof of Many-To-One

Proof.

$$= \hat{\mathbf{E}}_x[f(\hat{\xi}_t)] + \hat{\mathbf{E}}_x\left[\int_0^t \mathrm{m}[1](\hat{\xi}_s)\gamma(\hat{\xi}_x)\left(\frac{\mathrm{m}[\hat{\psi}_{t-s}[f]](\hat{\xi}_s)}{\mathrm{m}[1](\hat{\xi}_s)} - \hat{\psi}_{t-s}[f](\hat{\xi})\right)ds\right]$$
$$+ \int_0^t \hat{\mathbf{E}}_x[\mathrm{B}(\hat{\xi}_s)\hat{\psi}_{t-s}[f](x)ds.$$

Notice that $\hat{\psi}$ solves (4), and by the fact that solutions to (4) are unique and ψ solves (4) as well, then $\hat{\psi} = \psi$. □

References I

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