Stability and criticality

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Contents

Recap

- Particles move in E according to a Markov process $(\xi, \mathbf{P}_x)$. The associated semigroup is

$$\mathtt{P}_t[f](x) = \mathbf{E}_x[f(\xi_t)\mathbb{1}_{(t<\zeta)}]$$

- When at $x \in E$, at rate $\gamma(x)$, the particle is killed and sent to the cemetery state $\dagger \notin E$.

- At this point, new particles are created according to the point process $(\mathcal{Z}, \mathcal{P}_x)$, where

$$\mathcal{Z} = \sum_{i=1}^{N} \delta_{x_i}.$$

For convenience, we define $m[f](x) = \mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i)\right]$.

Recap

- The branching process is defined as

$$X_t := \sum_{i=1}^{N_t} \delta_{x_i(t)}.$$

- The law of $(X_t)_{t\geq 0}$ is characterised via the non-linear semigroup

$$v_t[g](x) := \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-X_t[g]}\right],$$

where

$$X_t[g] = \int_E g(y) X_t(\mathrm{d}y) = \sum_{i=1}^{N_t} g(x_i(t)).$$

Recap

We are also interested in the mean (linear) semigroup

$$\psi_t[g](x) := \mathbb{E}_{\delta_x}\left[X_t[g]\right].$$

Many-to-one lemma

There exists a Markov process $(\hat{\xi}, \hat{\mathbf{P}})$ taking values in $E \cup \{\dagger\}$ such that

$$\psi_t[g](x) = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t)\mathbf{1}_{(t<\tau)}\right],$$

where $B(x) = \gamma(x)(m[1](x) - 1)$.

Recap

- Recall that $m[f](x) = \mathcal{E}_x[\mathcal{Z}[f]]$.
- Recall also that $(\hat{\xi}_t)_{t\geq 0}$ evolves according to ξ and at rate $\gamma(x)m[1](x)$ jumps to a new location in $A \subset E$ with probability $m[\mathbf{1}_A](x)/m[1](x)$.
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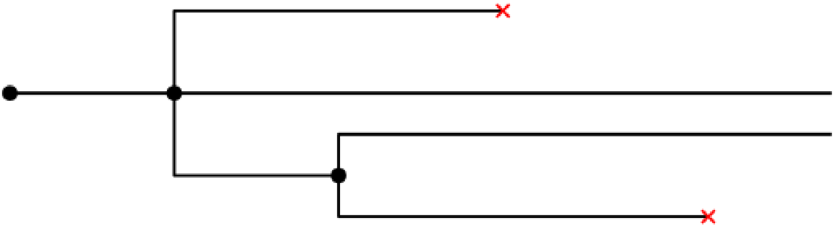
\end{aligned}$$

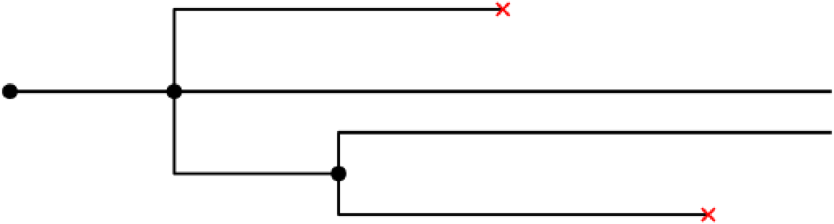
Contents

Example: continuous time Galton Watson process

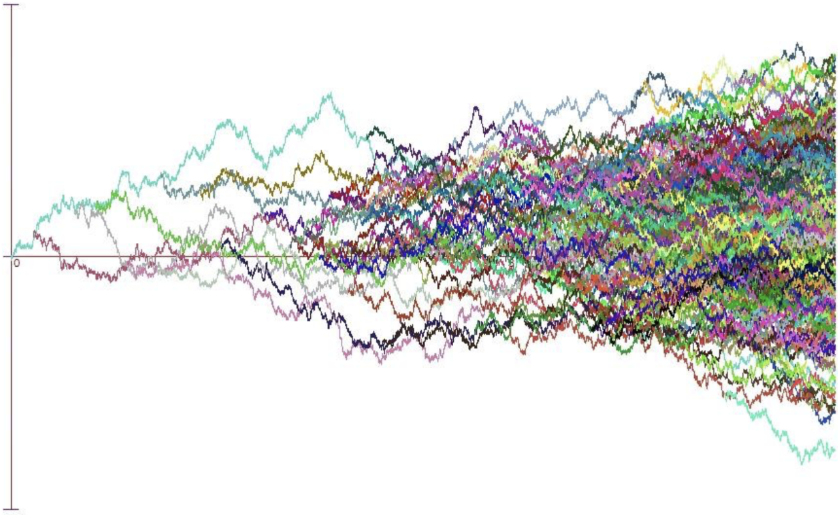


Example: continuous time Galton Watson process

Many-to-one: $\mathbb{E}[N_t] = \mathrm{e}^{(m-1)t}$.

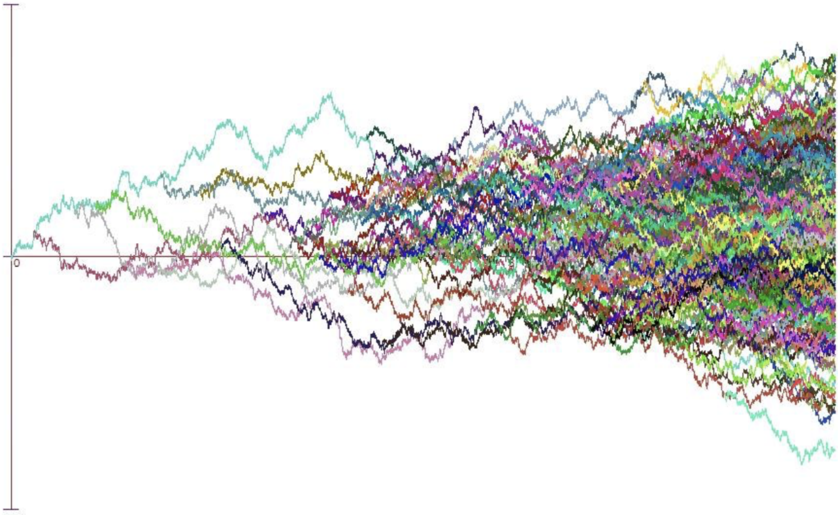


Example: BBM

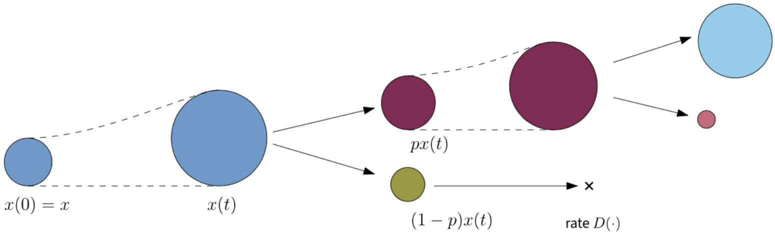


Example: BBM

Many-to-one: $\mathbb{E}_{\delta_x}[X_t[f]] = \mathrm{e}^{\beta t}\hat{\mathbf{E}}_x[f(B_t)]$



Example: growth-fragmentation



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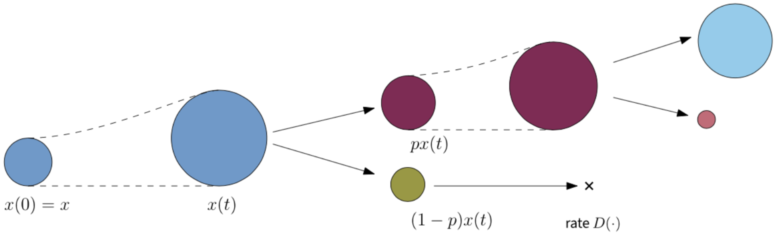
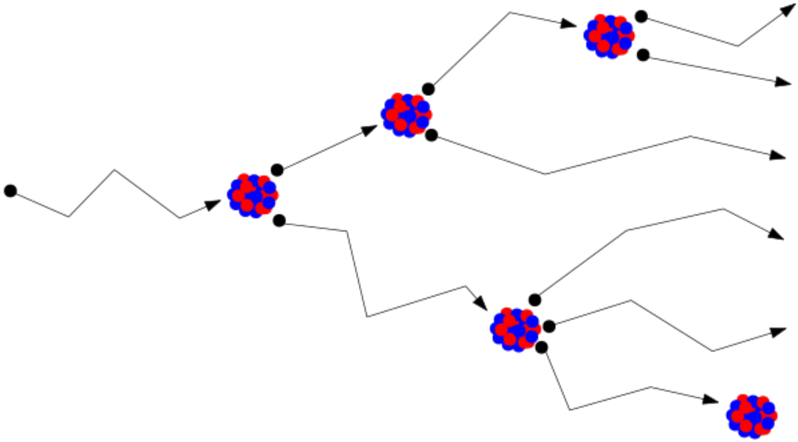
Many-to-one: $\mathbb{E}_{\delta_x}[X_t[f]] = \hat{\mathbf{E}}_x\left[\mathrm{e}^{\int_0^t \gamma(Y_s)\mathrm{d}s} f(Y_t)\mathbf{1}_{(t<\tau)}\right].$

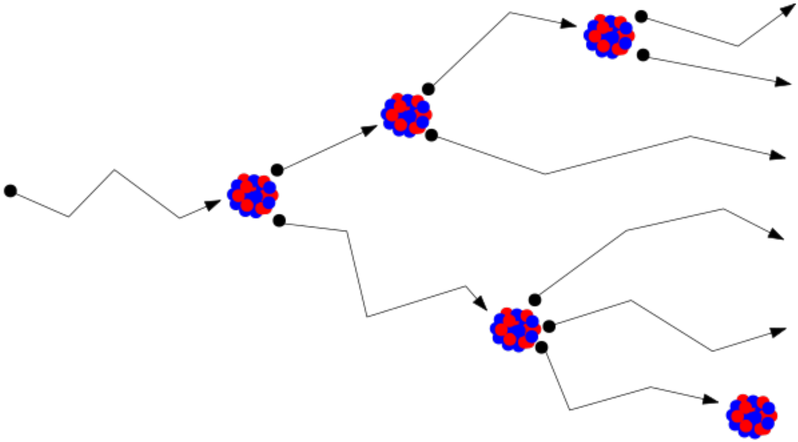


Example: neutron transport



Example: neutron transport

Many-to-one: $\mathbb{E}_{\delta_x}[X_t[f]] = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t \sigma_f(R_s,\Upsilon_s)(m[1](R_s,\Upsilon_s)-1)\mathrm{d}s} f(R_t,\Upsilon_t)\mathbf{1}_{(t<\tau)} \right].$



Observations

- The quantity $B(x) = \gamma(x)(m[1](x) - 1)$ "keeps track" of the mass in the branching process, i.e.

$$\mathbb{E}_{\delta_x}[N_t] = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} \mathbf{1}_{t<\tau} \right].$$

- If $\sup_{x \in E} B(x) < 0$, then we can interpret $|B|$ as a killing rate:

$$\hat{\mathbf{P}}_x(t < T | \sigma(\hat{\xi}_s, s \leq t)) = \mathrm{e}^{-\int_0^t |B(\xi_s)|\mathrm{d}s}.$$

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Contents

Stability

Aim: to find

- $\lambda_* \in \mathbb{R}$,
- a positive function $\varphi \in B^+(E)$,
- a probability measure η on E

such that

$$\psi_t[\varphi] = \mathrm{e}^{\lambda_* t}\varphi, \quad \eta[\psi_t[g]] = \mathrm{e}^{\lambda_* t}\eta[g],$$

and

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \quad \text{as } t \to \infty.$$

Stability

$$\psi_t[g](x) \sim \mathrm{e}^{\lambda_* t}\varphi(x)\eta[g], \quad \text{as } t \to \infty.$$

- Subcritical: if $\lambda_* < 0$, the average mass decays at rate $-\lambda_*$.
- Critical: if $\lambda_* = 0$, the average mass remains constant.
- Supercritical: if $\lambda_* > 0$, the average mass in the system grows at rate λ_*.

Example: branching Markov chains

- Consider the case where $(\xi_t, t \geq 0)$ is a continuous time Markov chain on $E = \{1, \ldots, n\}$ with transition matrix $(P_{i,j}(t))_{i,j \in E}$

- At rate γ, particles produce two offspring locally.

- What is the long-term average behaviour of the branching process?

- The key to answering this is the many-to-one:

$$\psi_t[g](i) = \mathrm{e}^{\gamma t} \mathbf{E}_i[g(\xi_t)]$$

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Example: branching Markov chains

Perron Frobenius theorem

Let A be a non-negative, irreducible square matrix. Then the following hold.

- There is a simple positive real eigenvalue λ and such that all other eigenvalues have absolute value less than or equal to λ.
- The (unique up to scaling) left- and right-eigenvectors, φ and η resp., corresponding to λ are positive.
- $\lim_{n\to\infty} A^n/\lambda^n = \varphi\eta^T$ where the left and right eigenvectors for A are normalized so that $\eta^T\varphi = 1$.

Example: branching Markov chains

Assuming the chain is irreducible, Perron Frobenius theory tells us that there exist $\lambda_c \leq 0$ and vectors φ, η such that

$$P(t)\varphi = \mathrm{e}^{\lambda_c t}\varphi, \quad \eta^T P(t) = \mathrm{e}^{\lambda_c t}\eta^T,$$

and

$$P_{i,j}(t) \sim \mathrm{e}^{\lambda_c t}\varphi(i)\eta(j) + o(\mathrm{e}^{\lambda_c t}), \quad t \to \infty.$$

Example: branching Markov chains

Using the fact that $\psi_t = \mathrm{e}^{\gamma t} P(t)$, we have

- $P(t)\varphi = \mathrm{e}^{\lambda_c t}\varphi \quad \Longrightarrow \quad \psi_t[\varphi] = \mathrm{e}^{(\gamma+\lambda_c)t}\varphi$;
- $\eta^T P(t) = \mathrm{e}^{\lambda_c t}\eta^T \quad \Longrightarrow \quad \eta^T\psi_t = \mathrm{e}^{(\gamma+\lambda_c)t}\eta^T$;
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Example: BBM

- Consider BBM on a domain $D \subset \mathbb{R}^d$ with branching rate β, mean offspring m and where particles are killed on ∂D.

- The mean semigroup $\psi_t[f](x) = \mathbb{E}_{\delta_x}[X_t[f]]$ satisfies

$$\frac{\partial \psi_t}{\partial t} = \frac{1}{2}\Delta\psi_t + \beta(m-1)\psi_t.$$

- What is the long-term average behaviour of the branching process?

- Again, the key is the many-to-one formula:

$$\psi_t[f](x) = \mathrm{e}^{\beta(m-1)t}\hat{\mathbf{E}}_x[f(B_t)\mathbf{1}_{t<\tau}].$$

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- Under certain assumptions, the eigenvalues of $-\frac{1}{2}\Delta$ are all real and can be written $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$. The associated eigenfunctions $\{\varphi_i\}_{i\geq 1}$ form an orthonormal basis of $L^2(D)$ and the first eigenfunction is strictly positive.

- Letting $p_t^D(x, y)$ denote the transition density of $(B_t)_{t<\tau_D}$, we have

$$p_t^D(x, y) = \sum_{i\geq 1} \mathrm{e}^{-\lambda_i t}\varphi_i(x)\varphi_i(y)$$

- This implies that

$$\mathbf{E}_x[f(B_t)\mathbf{1}_{t<\tau}] = \int_D f(y)p_t^D(x, y)\mathrm{d}y \sim \mathrm{e}^{-\lambda_1 t}\varphi(x)\eta[f].$$

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Example: BBM

- Since $\frac{1}{2}\Delta\varphi_i = -\lambda_i\varphi_i$, it follows that

$$(\tfrac{1}{2}\Delta + \beta(m-1))\varphi_i = (-\lambda_i + \beta(m-1))\varphi_i.$$

- Similarly,

$$\psi_t[f](x) = \mathrm{e}^{\beta(m-1)t}\mathbf{E}_x[f(B_t)\mathbf{1}_{t<\tau_D}] \sim \mathrm{e}^{(\beta(m-1)-\lambda_1)t}\varphi(x)\eta[f].$$

Stability in the general case

- Again, the key will be the many-to-one formula:

$$\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right].$$

- Since $\gamma, m[1] \in B^+(E)$, it follows that $\bar{B} := \sup_{x\in E} B(x) < \infty$.

- Hence, we may define

$$\psi_t^\dagger[f](x) := \mathrm{e}^{-\bar{B}t}\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t (B(\hat{\xi}_s)-\bar{B})\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right]$$
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Stability in the general case

- Again, the key will be the many-to-one formula:

$$\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t B(\hat{\xi}_s)\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right].$$

- Since $\gamma, m[1] \in B^+(E)$, it follows that $\bar{B} := \sup_{x\in E} B(x) < \infty$.

- Hence, we may define

$$\psi_t^\dagger[f](x) := \mathrm{e}^{-\bar{B}t}\psi_t[f](x) = \hat{\mathbf{E}}_x \left[\mathrm{e}^{\int_0^t (B(\hat{\xi}_s)-\bar{B})\mathrm{d}s} g(\hat{\xi}_t) \mathbf{1}_{t<\tau} \right]$$
$$=: \hat{\mathbf{E}}_x \left[g(\hat{\xi}_t) \mathbf{1}_{t<\kappa} \right]$$

QSDs

- Let $(Y_t)_{t\geq 0}$ be a time-homogeneous Markov process on $E \cup \{\dagger\}$ with probabilities $(\mathbf{P}^{\dagger}_x, x \in E)$ and semigroup $(\psi^{\dagger}_t)_{t\geq 0}$.
- Assume that $\kappa := \inf\{t > 0 : X_t = \dagger\} < \infty$, $\mathbf{P}^{\dagger}_x$-almost surely for all $x \in E$.
- Assume further that for all $x \in E$, $\mathbf{P}^{\dagger}_x(t < \kappa) > 0$.

QSDs

Definition

A **quasi-stationary distribution** (QSD) is a probability measure η on E such that

$$\eta = \lim_{t \to \infty} \mathbf{P}^{\dagger}_{\mu}(X_t \in \cdot | t < \kappa)$$

for some initial probability measure μ on E.

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Proposition

A probability measure η is a QSD if and only if, for any $t \geq 0$,

$$\eta = \mathbf{P}^{\dagger}_{\eta}(Y_t \in \cdot | t < \kappa).$$

QSDs

- Méléard, S., & Villemonais, D. (2012). Quasi-stationary distributions and population processes.
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QSDs

Theorem (Champagnat, Villemonais)

Under Assumption A, there exists a constant $\lambda_c < 0$, a function $\varphi \in B^+(E)$ and a probability measure η on E such that

$$\psi_t^\dagger[\varphi] = \mathrm{e}^{\lambda_c t}\varphi, \qquad \eta[\psi_t^\dagger[g]] = \mathrm{e}^{\lambda_c t}\eta[g].$$

Moreover, there exist constants $C, \varepsilon > 0$ such that

$$\sup_{x \in E, g \in B_1^+(E)} |\mathrm{e}^{-\lambda_c t}\varphi(x)^{-1}\psi_t^\dagger[g] - \eta[g]| \leq C\mathrm{e}^{-\epsilon t}.$$

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Since $\psi_t = \mathrm{e}^{\bar{B}t}\psi_t^\dagger$, the same conclusion then holds for ψ_t with λ_c replaced by $\lambda_* = \lambda_c + \bar{B}$.

QSDs

Assumption A

There exists a probability measure ν on E such that

(A1) there exists $t_0, c_1 > 0$ such that for all $x \in E$,

$$\mathbf{P}_x^{\dagger}(Y_{t_0} \in \cdot | t_0 < \kappa) \geq c_1 \nu(\cdot);$$

(A2) there exists $c_2 > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbf{P}_\nu^{\dagger}(t < \kappa) \geq c_2 \mathbf{P}_x^{\dagger}(t < \kappa).$$

Thank you!