Branching Processes Reading Group
Spatial Pal-Bell Equation and Moment Asymptotic

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MaTHRad PostDoc Retreat Workshop

Two important functions describing branching processes $(X_t)_{t\geq 0}$

For a measure-valued process $(X_t)_{t\geq}$, we define $X_t[g] := \int g dX_t$.
If $X_t := \sum \delta_{x_i(t)}, X_t[g] = \sum_i g(x_i(t))$.

	$\psi_t[g](r,v)$	$v_t[g](r,v)$
Definition	$\mathbb{E}_{\delta_{r,v}}[X_t[g]]$	$\mathbb{E}_{\delta_{r,v}}[e^{-X_t[g]}]$
Describes	First Moment	Moment generating function
Semi-group	$\psi_{t+s}[g] = \psi_t(\psi_s[g])$	$v_{t+s}[g] = v_t(v_s[g])$

Table: Two important functions

Two important functions describing branching processes $(X_t)_{t\geq 0}$

First moment from moment generating function:

$$\psi_t[g](r,v) = -\frac{d}{d\theta} v_t[\theta g](r,v)\bigg|_{\theta=0} \tag{1}$$

Similarly, one can derive all moments of $X_t[g]$ given an evolution equation of $v_t[g](r,v)$.

Evolution of $v_t[g]$

To derive the evolution of $v_t[g]$, we condition on the first event (fission / scatter) for our neutron transport process until time t.

Event	Probability	Contribution
No event / Exit	$e^{-\int_0^{t\wedge\kappa^D_{r,v}}\sigma(r+v\ell,v)d\ell}$	$e^{-g(r+vt,v)}\mathbf{1}_{t<\kappa^D_{r,v}} + \mathbf{1}_{t>\kappa^D_{r,v}}$
Scatter at s Fission at s	$\mathbf{1}_{s<\kappa}\sigma(r+vs,v)$ $\times e^{-\int_0^s \sigma(r+v\ell,v)d\ell}$	$\frac{\sigma_s}{\sigma}\int v_{t-s}(r+vs,v')\pi_s(r+vs,v,v')dv'$ $+\frac{\sigma_f}{\sigma}\mathcal{E}_{r+vs,v}\left[\prod_{i=1}^N v_{t-s}(r+vs,v_i)\right]$

Table: Probability Tables

Evolution of $v_t[g]$

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Event	Probability	Contribution
No event / Exit	$e^{-\int_0^{t\wedge\kappa^D_{r,v}}\sigma(r+v\ell,v)d\ell}$	$e^{-g(r+vt,v)}1_{t<\kappa^D_{r,v}} + 1_{t>\kappa^D_{r,v}}$
Scatter at s Fission at s	$1_{s<\kappa}\sigma(r+vs,v)$ $\times e^{-\int_0^s\sigma(r+v\ell,v)d\ell}$	$U_s\Big[Sv_{t-s}$ $+G\left[v_{t-s}\right]\Big]$

Table: Probability Tables

Combining terms and defining the operators appropriately (in particular subsuming the potential terms $e^{-\int_0^s \cdots}$ into the operators), we have

$$v_t[g] = \hat{U}_t[e^{-g}] + \int_0^t U_s\left[Sv_{t-s}[g] + G[v_{t-s}[g]]\right]ds. \tag{2}$$

Spatial Correlation

- If $g = 1_A$, $v_t[g]$ helps us understand all the moments, and thus the law of $X_t(A)$.
- How about spatial correlation $\mathbb{E}[X_t(A)X_t(B)]$?
- Characterised by correlation of non-local branching, i.e, $\mathcal{V}[f,g](r,v) = \mathbb{E}[\sum_{i\neq j} f(r,v_i)g(r,v_j)]$

Spatial Correlation

Note that

$$2\mathbb{E}[X_t[f]X_t[g]] = \mathbb{E}[X_t[f+g]^2] - \mathbb{E}[X_t[f]^2] - \mathbb{E}[X_t[g]^2], \tag{3}$$

so it suffices to consider expressions for $w_t[g](r,v) := \mathbb{E}_{\delta_{r,v}}[X_t[g]^2]$ in general. Again we condition on first fission / scattering event.

Event	Probability	Contribution
No event / Exit	$e^{-\int_0^{t\wedge\kappa^D_{r,v}} \sigma(r+v\ell,v)d\ell}$	$\underline{g}^2(r+vt,v)1_{t<\kappa^D_{r,v}}$
Scatter at s	$1_{s<\kappa}\sigma(r+vs,v)$	$\frac{\sigma_s}{\sigma}\int \underline{w_{t-s}}(r+vs,v')\pi_s(r+vs,v,v')dv'$
Fission at s	$\times e^{-\int_0^s \sigma(r+v\ell,v)d\ell}$	$+\underline{\frac{\sigma_f}{\sigma}\mathcal{E}_{r+vs,v}\left[\mathbb{E}\left[\left(\sum_{i=1}^N X^i_{t-s}[g]\right)^2\right]\right]}$

Table: Probability Tables

Spatial Correlation

$$\mathcal{E}_{r+vs,v}\left[\mathbb{E}\left[\left(\sum_{i=1}^{N} X_{t-s}^{i}[g]\right)^{2}\right]\right] = \mathcal{E}_{r+vs,v}\left[\mathbb{E}\left[\sum_{i\neq j} X_{t-s}^{i}[g]X_{t-s}^{j}[g] + \sum_{i=1}^{N} X_{t-s}^{i}[g]^{2}\right]\right]$$
$$= \sum_{i\neq j} \psi_{t-s}[g](r+vs, v_i)\psi_{t-s}[g](r+vs, v_j)$$
$$+ \sum_{i=1}^{N} w_{t-s}[g](r+vs, v_i) \quad (4)$$

Spatial Correlation

Combining the terms above, we have that $w_t[g](r,v)$ must solve

$$
\begin{aligned}
w_t[g](r,v) =& U_t[g^2](r,v) + \int_0^t U_s\left[(S+F)w_{t-s}[g]\right](r,v)ds + \int_0^t U_s\left[\sigma_f \mathcal{V}[\psi_{t-s}[g]]\right](r,v)ds \\
=& U_t[g^2](r,v) + \int_0^t U_s\Big[\sigma_f \mathcal{V}[\psi_{t-s}[g]] + (S+F)w_{t-s}[g]\Big](r,v)ds.
\end{aligned} \tag{5}
$$

One could interpret the term $\sigma_f \mathcal{V}[\psi_{t-s}[g]]$ as independent masses / contribution immigrated into the system at time s. Therefore,

$$
w_t[g](r,v) = \psi_t[g^2](r,v) + \int_0^t \psi_s\Big[\sigma_f \mathcal{V}[\psi_{t-s}[g]]\Big](r,v)ds. \tag{6}
$$

Spatial Correlation

Interpretation: The variance of the system at time t, given by,

$$w_t[g](r, v) - \psi_t[g^2](r, v),$$

is the sum ($\int$) of the mean evolution (ψ_s) of the local branching correlation ($\sigma_f \mathcal{V}[\psi_{t-s}[g]]$) along the process.

Asymptotic of 2nd Moments

If $\lambda_* > 0$, for $w_t[g](r,v) = \psi_t[g^2](r,v) + \int_0^t \psi_s\left[\sigma_f \mathcal{V}[\psi_{t-s}[g]]\right](r,v)ds$,

$$\lim_{t\to\infty} e^{-2\lambda_* t} w_t[g](r,v) = \langle\tilde{\varphi}, g\rangle^2 \int_0^\infty e^{-2\lambda_* s}\psi_s[\sigma_f \mathcal{V}[\varphi]](r,v)ds \qquad (7)$$

Heuristics: By Perron-Frobenius results, $\psi_t[g^2] \sim e^{\lambda_* t}$, so first term goes to zero. Furthermore, $\psi_{t-s}[g] = e^{\lambda_*(t-s)}\langle\tilde{\varphi}, g\rangle\varphi + o(e^{\lambda_* t})$, $\mathcal{V}$ is symmetric bilinear form, so $\mathcal{V}[\psi_{t-s}[g]] = e^{2\lambda_*(t-s)}\langle\tilde{\varphi}, g\rangle^2\mathcal{V}[\varphi] + o(e^{2\lambda_* t})$. Therefore

$$e^{-2\lambda_* t}\int_0^t \psi_s\left[\sigma_f \mathcal{V}[\psi_{t-s}[g]]\right](r,v)ds = \int_0^t e^{2\lambda_*(-s)}\langle\tilde{\varphi}, g\rangle^2\psi_s[\mathcal{V}[\varphi]]ds + o(1)$$

Asymptotic of 2nd Moments

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If $\lambda_* < 0$, we write $\int_0^t \psi_s\Big[\sigma_f \mathcal{V}[\psi_{t-s}[g]]\Big](r,v)ds = \int_0^t \psi_{t-s}\Big[\sigma_f \mathcal{V}[\psi_s[g]]\Big](r,v)ds$, then

$$
\begin{aligned}
e^{-\lambda_* t} & \int_0^t \psi_{t-s}\Big[\sigma_f \mathcal{V}[\psi_s[g]]\Big](r,v)ds \\
&= e^{-\lambda_* t}\int_0^t e^{\lambda_*(t-s)}\varphi(r,v)\langle\tilde{\varphi}, \sigma_f\mathcal{V}[\psi_s[g]]\rangle + o(e^{\lambda_*(t-s)})ds \\
&= \int_0^t e^{\lambda_*(-s)}\varphi(r,v)\langle\tilde{\varphi}, \sigma_f\mathcal{V}[\psi_s[g]]\rangle ds + o(1) \qquad (8)
\end{aligned}
$$

Finally, $e^{-\lambda_* t}\psi_t[g^2](r,v) \to \varphi\langle\tilde{\varphi}, g^2\rangle$, so

$$
e^{-\lambda_* t} w_t[g](r,v) \to \varphi\langle\tilde{\varphi}, g^2\rangle + \int_0^t e^{-\lambda_* s}\varphi(r,v)\langle\tilde{\varphi}, \sigma_f\mathcal{V}[\psi_s[g]]\rangle ds.
$$

Asymptotic of 2nd Moments

If $\lambda_* = 0$, first term goes to 0 after rescaling by $1/t$, and

$$\begin{aligned}
\frac{1}{t}\int_0^t \psi_{t-s}\Big[\sigma_f \mathcal{V}[\psi_s[g]]\Big](r,v)ds \\
&= \frac{1}{t}\int_0^t \left\{\varphi(r,v)\langle\tilde{\varphi},\sigma_f\mathcal{V}[\varphi\langle\tilde{\varphi},g\rangle]\rangle + o(1)\right\} ds \\
&= \frac{1}{t}\int_0^t \varphi(r,v)\langle\tilde{\varphi},g\rangle^2\langle\tilde{\varphi},\sigma_f\mathcal{V}[\varphi]\rangle ds + o(1) \\
&= \varphi(r,v)\langle\tilde{\varphi},g\rangle^2\langle\tilde{\varphi},\sigma_f\mathcal{V}[\varphi]\rangle. \qquad (9)
\end{aligned}$$

- What about k-th moment?
- $$\psi_t^{(k)}[g](r,v) = (-1)^k \frac{d^k}{d\theta^k} v_t[\theta g](r,v)\Big|_{\theta=0} \tag{10}$$
- The evolution equation of the k-th moment can be written in terms of that for lower moments through applying product rule k-times
- Moral of the story: Growth of k-th moment is $e^{\lambda_* kt}$ for supercritical, $e^{-\lambda_* t}$ for subcritical, t^{-k} for critical processes.
- More exotic results for the occupational measure: $e^{\lambda_* kt}$ for supercritical, $O(1)$ for subcritical, $O(t^{-2k-1})$ for critical

Optional Extra Discussion - Applications in Nuclear Engineering

Pal-Bell equations

Terminology:

- I precursors, neutron source of strength $S_d(s)$

Description: The Pal-Bell equation is a system of PDEs on the PGF $G(z,t|s) = \mathbb{E}[z^{N(t|s)}]$, $G_{di}(z,t|s)$, $G_S(z,t|s)$.

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- $N_S(t|s)$: descendent neutrons at t from immigrated neutrons at s

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Perron-Frobenius:

- There exists a $\mathbb{R}$-valued Z_∞ and $\rho > 0$ such that $N_S(t|s) \to e^{\rho t} Z_\infty$

Operation Requirement: For some threshold n^*, and $\epsilon = 10^{-8}$, find appropriate $S_d(s)$ such that

$$\inf_{t \leq t_{mat}} \mathbb{P}[N_S(t|s) \leq n^*] \geq 1 - \epsilon.$$

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- The *maturity time* t_{mat} defined when for all $t \geq t_{mat}$,

$$\frac{\sqrt{Var[N_S(t|s)]}}{\mathbb{E}[N_S(t|s)]} < 0.001$$

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Pal-Bell equations

$$-\frac{\partial G(z,t|s)}{\partial s} = \lambda_c(s) - (\lambda_c(s) + \lambda_f(s))G(z,t|s) + \lambda_f(s) f(G(z,t|s)) \prod_{i=1}^{I} f_i(G_{di}(z,t|s)) \quad (7)$$

$$-\frac{\partial G_{di}(z,t|s)}{\partial s} = -\lambda_i G_{di}(z,t|s) + \lambda_i G(z,t|s), \quad i = 1, 2, \dots I \quad (8)$$

$$-\frac{\partial G_S(z,t|s)}{\partial s} = S_d(s)[f_q(G(z,t|s)) - 1]G_S(z,t|s) \quad (9)$$

Pal-Bell equation [WE17]

Pal-Bell equations

Operation Requirement: In [WE17], one can show that

$$
\begin{aligned}
\mathbb{P}[N_S(t|s) \le n^*] &= \sum_{k=1}^{n^*} \frac{1}{n!} \frac{\partial^n}{\partial z^n} G_S(z,t|s)\bigg|_{z=0} \\
&= \frac{1}{2\pi i} \oint G_S(z,t|s) \sum_{k=1}^{n^*} \frac{1}{z^{k+1}} dz \\
&= \frac{1}{2\pi i} \oint G_S(z,t|s) \sum_{k=1}^{n^*} \frac{z}{(1-z)z^{n^*+1}} dz,
\end{aligned} \tag{11}
$$

which can be calculated with numerical methods (saddle point methods).

OPEN PROBLEMS

Spatial Pal-Bell equation

There is a similar Pal-Bell equation for the spatial case, with the PGF $G(z, t, R|\vec{r}_0, \vec{v}_0, s)$.

Problems solved:

- Characterising spatial variations (achieved only through diffusion approximation)

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- Characterising spatial variations (achieved only through diffusion approximation)
- Mean and Variance calculation
- Establishing Perron-Frobenius limiting results

Branching Particle System theory

General theory of BPS with Immigration

- Construct general multi-type spatial branching process with immigration (DONE)

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- Calculate spatial clustering by computing $\mathbb{E}[\langle f, \mu\rangle\langle g, \mu\rangle]$

Monte Carlo methods

What about Monte Carlo methods? (TT+MK+VR)

- Implementation of simulations in SCONE

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- Characterise spatial variations

Ultimate Goal

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