Skeleton Decomposition

"Collapse of the skeleton to the spine"

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Branching Markov processes reading group (Chapter 11.3-11.5)

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Notation

General

$\mu[f]$	integral of f with respect to μ
φ	right eigenfunction from the Perron-Frobenius decomposition
Superscript φ	related to the spine
λ_*	leading eigenvalue
x_i	location of particle i
ξ_t	the Markov process at time t

Branching

γ	branching rate
$m[\cdot]$	mean operator for offspring
$\mathbf{Z}$	branching offspring point process for BMP
$\mathscr{P}$	branching point process probabilities

Spine Decomposition

How to create a spine? We need to mark particles.

1. For an initial configuration of particles μ, the i-th particle is marked "spine" with probability $\frac{\varphi(x_i)}{\mu[\varphi]}$ (For BBM, this is uniform selection.)
2. Unmarked particles evolve as before.
3. The motion process for the spine is determined by the semigroup

$$\mathrm{P}_t^{\varphi}[f](x) := \frac{1}{\varphi(x)} \mathbf{E}_x[e^{-\lambda_* t} e^{\int_0^t \frac{\gamma(\xi_s)}{\varphi(\xi_s)}(m[\varphi](\xi_s) - \varphi(\xi_s))ds} \varphi(\xi_t) f(\xi_t)] \quad (1)$$

4. The branching rate for the marked particle

$$\gamma^{\varphi}(x) := \gamma(x) \frac{m[\varphi(x)]}{\varphi(x)} \quad (2)$$

and it scatters the number of particles that satisfies

$$\frac{d\mathscr{P}_x^{\phi}}{d\mathscr{P}_x} = \frac{\mathbf{Z}[\varphi]}{m[\varphi](x)} \quad (3)$$

Spine Example

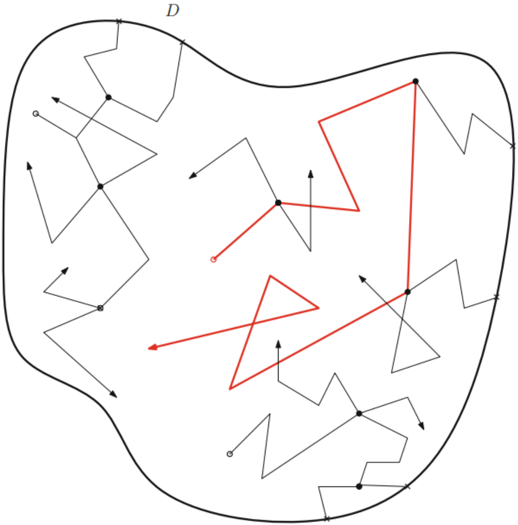

Figure: Neutrons at time 0 are marked with ∘.

Criticality

A spine provides us with a single genealogical trajectory that survives forever.
Now, we will look at the skeleton of our process to analyse what happens if we start with k trajectories that survive until time T.

Notation:

$\mathscr{E}_x$	expectation operator of the point process
$\mathbb{P}$	the law of the BMP
$\mathcal{S}_t$	filtration of BMP
$\emptyset$	label of the initial ancestral particle
ζ	time of extinction

$G[f](x) := \gamma(x)\mathscr{E}_x(\prod_{i=1}^{N} f(x_i) - f(x))$ is the branching mechanism

$\{\uparrow, \downarrow\}$ marks

Let us decompose the process into genealogies that survive until time $T < \infty$, marked $\uparrow$, and those who die out before T, marked $\uparrow$.

Definition: If a particle i at time t has descendants alive at time T, define its mark $c_i^T(t) = \uparrow$. Otherwise, $c_t^T(t) = \downarrow$.

Goal: To decompose the BMP into a thin tree of $\uparrow$-marked individuals dressed with immigrating trees of $\downarrow$-marked individuals.

$w_t(x) := \mathbb{P}_{\delta_x}(\zeta < T) \leftarrow$ extinction probability before time T

Assumptions:

- Extinction by time T is uniformly bounded away from 0.
- Extinction by time T is not a certainty.

$\{\uparrow, \downarrow\}$-marked BMP and the original BMP

Consider a configuration of the BMP at time t, S_t. For a set of particles $\{1, \ldots, N_t\}$ alive at time t, we have

$$\left.\frac{d\mathbb{P}_{\mu}^{\updownarrow, T}}{d\mathbb{P}_{\mu}}\right|_{S_t} = \prod_{i=1}^{N_t} (\mathbf{1}_{(c_i^T(t)=\uparrow)} + \mathbf{1}_{(c_i^T(t)=\downarrow)}) = 1 \tag{4}$$

and

$$\left.\frac{d\mathbf{P}_{\delta_x}^{\updownarrow, T}}{d\mathbf{P}_{\delta_x}}\right|_{S_t} = \mathbf{E}_{\delta_x}\left(\prod_{i=1}^{N_t} (\mathbf{1}_{(c_i^T(t)=\uparrow)} + \mathbf{1}_{(c_i^T(t)=\downarrow)}) | S_t\right) = \ldots = 1 \tag{5}$$

$\downarrow$-Trees

Definition:

$$\mathbb{P}^{\downarrow,T}_{\delta_x}(A) := \mathbb{P}^{\updownarrow,T}_{\delta_x}(A|c^T_\emptyset(0) = \downarrow) = \frac{\mathbb{P}^{\updownarrow,T}_{\delta_x}(A; c^t_i(t) = \downarrow \text{ for each } i = 1, \dots, N_t)}{\mathbb{P}^{\updownarrow,T}_{\delta_x}(c^T_\emptyset(0) = \downarrow)} \tag{6}$$

Lemma 11.2

The law of the Markov process describing the movement of the $\downarrow$-marked particles satisfies

$$\left.\frac{d\mathbf{P}^{\downarrow,T}_x}{d\mathbf{P}_x}\right|_{\sigma(\xi_s, s \leq t)} = \frac{w_{T-t}(\xi_{t\wedge k})}{w_t(s)} \exp\left(\int_0^{t\wedge k} \frac{G[w_{T-s}](\xi_s)}{w_{T-s}(\xi_s)} ds\right) \tag{7}$$

For a branching event at time $s \leq T$, the branching mechanism is given by

$$G^{\downarrow,T}_s[f](x) = \frac{1}{w_{T-s}(x)}[G[fw_{T-s}] - fG[w_{T-s}]](x) \tag{8}$$

Dressed $\uparrow$-Trees

$p_T(x) := \mathbb{P}_{\delta_x}(\zeta > T) = 1 - w_T(x) \leftarrow$ survival until time T probability

Definition:

$$\mathbb{P}^{\updownarrow,T}_{\delta_x}(A|c^T_\emptyset(0) = \uparrow) = \frac{\mathbb{P}^{\updownarrow,T}_{\delta_x}(A; c^t_i(t) = \uparrow \text{ for at least one } i = 1, \ldots, N_t)}{\mathbb{P}^{\updownarrow,T}_{\delta_x}(c^T_\emptyset(0) = \uparrow)} \quad (9)$$

Lemma 11.3

The law of the Markov process describing the movement of the $\uparrow$-marked particles satisfies

$$\left.\frac{d\mathbf{P}^{\uparrow,T}_x}{d\mathbf{P}_x}\right|_{\sigma(\xi_s, s \le t)} = \frac{p_{T-t}(\xi_t)}{p_T(x)} \exp\left(-\int_0^t \frac{G[w_{T-s}](\xi_s)}{p_{T-s}(\xi_s)} ds\right) \quad (10)$$

For a branching event at time $s \le T$, the branching mechanism is given by

$$G^{\uparrow,T}_s[f](x) = \frac{1}{p_{T-s}(x)} (G[p_{T-s}f + w_{T-s}] - (1-f)G[w_{T-s}]) \quad (11)$$

Comparison of ↑-Trees and ↓-Trees

↓-Trees

$$\left.\frac{d\mathbf{P}_x^{\downarrow,T}}{d\mathbf{P}_x}\right|_{\sigma(\xi_s, s\leq t)} = \frac{w_{T-t}(\xi_{t\wedge k})}{w_t(s)} \exp\left(\int_0^{t\wedge k} \frac{G[w_{T-s}](\xi_s)}{w_{T-s}(\xi_s)} ds\right) \tag{12}$$

$$G_s^{\downarrow,T}[f](x) = \frac{1}{w_{T-s}(x)}[G[fw_{T-s}] - fG[w_{T-s}]](x) \tag{13}$$

↑-Trees

$$\left.\frac{d\mathbf{P}_x^{\uparrow,T}}{d\mathbf{P}_x}\right|_{\sigma(\xi_s, s\leq t)} = \frac{p_{T-t}(\xi_t)}{p_T(x)} \exp\left(-\int_0^{t} \frac{G[w_{T-s}](\xi_s)}{p_{T-s}(\xi_s)} ds\right) \tag{14}$$

$$G_s^{\uparrow,T}[f](x) = \frac{1}{p_{T-s}(x)}(G[p_{T-s}f + w_{T-s}] - (1-f)G[w_{T-s}]) \tag{15}$$

The Dressed part of ↑-Trees

The dressing consists of non-locally immigrated particles at the branch points of $X^{\uparrow,T}$ at time s, continuing to evolve as an independent copy of $X^{\downarrow,T-s}$.
Now the joint branching/immigration mechanism is given by

$$G_s^{\updownarrow,T}[f,g](x) := \frac{\gamma(x)}{p_{T-s}(x)} \mathscr{E}_x \Bigg[\sum_{\substack{I\subseteq\{1,\dots,N\} \\ |I|\geq 1}} \prod_{i\in I} p_{T-s}(x_i) f(x_i) \prod_{j\in\{1,\dots,N\}\setminus I} w_{T-s}(x_j) g(x_j) \Bigg] - \gamma^{\updownarrow,T-s}(x) f(x) \quad (16)$$

where $\gamma^{\updownarrow,T-s}(x) = \gamma(x) - \frac{G[w_{T-s}](x)}{p_{T-s}(x)}$.
Intuition: The original process X conditioned on $\zeta > T$ is equal in law to the dressed time-inhomogeneous BMP.

Skeletal Decomposition

Theorem 11.2

$(X, \mathbb{P}^{\updownarrow, T}_{\mu})$ is equal in law to

$$\sum_{i=1}^{n} \left(\Theta_i^T X_t^{\updownarrow, T}(i) + (1 - \Theta_i^T) X_t^{\downarrow, T}(i) \right) \tag{17}$$

where Θ_i^T is an independent Bernoulli random variable with probability of success given by $p_T(x_i)$ and $X^{\downarrow, T}$ and $X^{\updownarrow, T}$ are independent copies of $(X, \mathbb{P}^{\downarrow, T}_{\delta_{x_i}})$ and $(X, \mathbb{P}^{\updownarrow, T}_{\delta_{x_i}}(\cdot, c_{\emptyset}^T(0) = \uparrow))$, respectively.

Eternal survival

Let us consider the critical setting.
We are interested in what happens as $T \to \infty$.
Assume

- G2: There exists an eigenvalue λ_*, a right eigenfunction φ and a left eigenmeasure $\tilde{\varphi}$ such that $\langle \psi_t[\varphi], \mu \rangle = e^{\lambda_* t} \langle \varphi, \mu \rangle$ and $\langle \psi_t[f], \tilde{\varphi} \rangle = e^{\lambda_* t} \langle f, \tilde{\varphi} \rangle$
- G6: Offspring numbers are bounded
- G7: Irreducible branching condition
- G11: $\lim_{T \to \infty} \inf_{x \in E} w_t(x) = 1$ (guaranteed death)

$\downarrow$-Tree Asymptotics

The law of the Markov process describing the movement of $\downarrow$-marked particles $\mathbf{P}^{\downarrow,T}$ converges weakly to $\mathbf{P}$.
$\mathbf{P}_x^{\downarrow,T}$ also converges weakly to the law of the process with associated semigroup $\mathtt{P}^{\varphi}$ (semigroup of the spine).

$$\lim_{T\to\infty} G_s^{\downarrow,T}[f](x) = G[f](x) \tag{18}$$

Intuition: The law of $\downarrow$-marked particles settles down to $(X, \mathbb{P})$.

Relationship to Spines

$N^{\uparrow}$ is the number of individuals at time t=0 marked by $\uparrow$.

$$\lim_{T\to\infty} \mathbb{P}_{\mu}^{\updownarrow,T}(N^{\uparrow}=1|N^{\uparrow}\geq 1) = \lim_{T\to\infty} \frac{\sum_{j=1}^{n} p_T(x_j)\prod_{l\neq j} w_t(x_l)}{1-\prod_{i=1}^{n} w_T(x_i)} = 1 \quad (19)$$

Intuition: The skeleton concentrates on a single individual dressed tree rooted at t=0, which carries all the $\uparrow$-marked individuals in its tree of descendants.

$$\lim_{T\to\infty} \gamma^{\updownarrow,T}(x) = \gamma(x)\frac{m[\varphi](x)}{\varphi(x)} \quad (20)$$

Remark: This is the same as the branching rate for spine.

Collapse of the Skeleton to the Spine

$$\lim_{T\to\infty} G^{\updownarrow,T}[f,g](x) = \ldots = \tag{21}$$

$$\gamma^{\varphi}(x)\mathscr{E}_x\left[\frac{\mathbf{Z}[\varphi]}{m[\varphi](x)}\sum_{i=1}^{N}\frac{\varphi(x_i)}{\mathbf{Z}[\varphi]}f(x_i)\prod_{j\neq i}g(x_j)\right] - \gamma^{\varphi}(x)f(x) \tag{22}$$

Intuition: The rate at which more than one $\uparrow$-marked individual appears at any branching event drops to 0. The law of the motion of the single genealogy of the $\uparrow$-marked particles and the branching mechanism for its dressing settle down to that of the spine.

Branching Brownian Motion

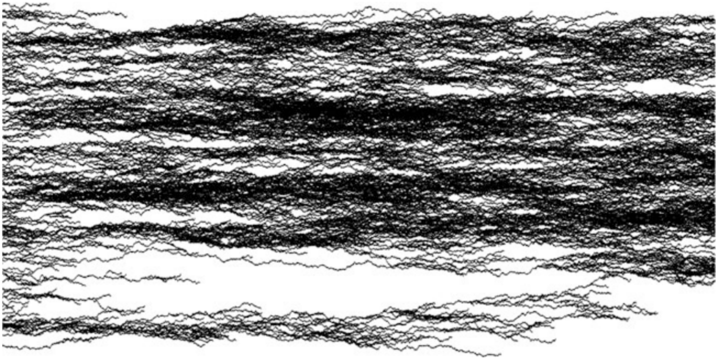

Figure: 150 initial individuals with either 0 or 2 local offspring..

Questions?

Thank you!
Dziękuję!
Merci!
Grazie!