SLLN and CLT in the supercritical setting

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The setting

We take the general setting as given in Section 8 of the book.

- A non-local branching process $X := (X_t, t \geq 0)$ is an atomic measure-valued process whose atoms evolve on $E \cup \{\dagger\}$, where E is a locally compact Hausdorff space and $\dagger$ is a cemetery (absorbing) state.
- Particles at $x \in E$ evolve according to a Markov process (ξ, P_x).
- When at $y \in E$, particles branch at rate $\gamma(y)$.
- If a particle branches at $y \in E$, then it is replaced by $y_1, ..., y_N$, where the number and law of these particles is given by $\mathcal{P}_y$.

Important examples

Motivating example - Neutron branching processes:

- $E = \text{Space} \times \text{Velocity}$.
- (ξ, P_x) - linear motion with absorption at the boundary.
- γ and $\mathcal{P}$ - Absorption, scattering, and fission events.

Useful example - Multitype branching processes:

- The state space is finite, $E = \{x_1, ..., x_d\}$.
- The Markov process has no movement.

Fundamental assumptions - Irreducibility, Moment control, and Supercriticality

To obtain a SLLN and CLT for the branching process, we require the following assumptions on X.

(G2) There exists a real number λ_*, a bounded function $\varphi \in B^+(E)$ and a probability measure $\tilde{\varphi}$ on E, such that $\tilde{\varphi}[\varphi] = 1$ and, for any $x \in E$, $t \geq 0$ and $f \in B(E)$,

$$\psi_t[\varphi](x) = \mathbb{E}_{\delta_x}\left[X_t[\varphi]\right] = \mathrm{e}^{\lambda t}\varphi(x), \quad \text{and} \quad \tilde{\varphi}[\psi_t[f]] = \mathrm{e}^{\lambda t}\tilde{\varphi}[f].$$

Moreover,

$$\sup_{x \in E, f \in B_1(E)} \left| \mathrm{e}^{-\lambda t}\varphi(x)^{-1}\psi_t[f](x) - \tilde{\varphi}[f] \right| \to 0, \quad \text{as } t \to \infty.$$

(G3) For a given $k \geq 1$

$$\sup_{x \in E} \mathcal{E}_x[N^k] < \infty.$$

(GS) $\lambda_* > 0$.

The SLLN - a martingale convergence result

For X with initial state $x \in E$, define the martingale

$$W_t = \mathrm{e}^{-\lambda_* t} \frac{X_t[\varphi]}{\varphi(x)}, t \geq 0.$$

Assuming (G3), this martingale has $k \geq 1$ moments for all $t \geq 0$.

Theorem (12.1 in book)

Under assumptions (G2) and (G3) for $k = 2$, the martingale W satisfies the following:

1. *If $\lambda_* > 0$, then W is $L^2(\mathbb{P})$ convergent (and so has non-trivial limit).*
2. *If $\lambda_* < 0$, then the limit of W, $W_\infty = 0$ almost surely.*
3. *If $\lambda_* = 0$, and additionally*

   $$\text{(G8) For all } t \text{ sufficiently large, } \sup_{x \in E} \mathbb{P}_{\delta_x}(t < \zeta) < 1,$$

   then the limit of W, $W_\infty = 0$ almost surely.

SLLN - Kesten and Stigum type result

(G6) There exists $0 \leq n_{\max} < \infty$, such that $\sup_{x\in E} \mathcal{P}_x(N \leq n_{\max}) = 1$.

Theorem (12.2 in book)

Under assumptions (G2), (G6), and (G8), we have that the events $\{W_\infty = 0\}$ and $\{\zeta < \infty\}$ almost surely agree under $\mathbb{P}$.

Theorem 12.2 implies that, under assumptions (G2), (G6), and (G8), in the case of $\lambda_* > 0$, we have $X_t[\varphi] \sim \mathrm{e}^{\lambda_* t}$, $\mathcal{P}(\cdot|\zeta = \infty)$-almost surely.

Theorem (Athreya and Karlin)

Assume (G2) and (GS). For the multitype branching process,

$$\sup_{x\in E} \mathbb{P}_{\delta_x}(W_\infty > 0) > 0,$$

if and only if

$$\sup_{x\in E} \mathbb{E}_{\delta_x}[N \log(N)] < \infty.$$

SLLN - Main results

We can rewrite the martingale limit as

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{X_t[\varphi]}{\varphi(x)} = W_\infty \tilde{\varphi}[\varphi].$$

Can we extend this result to a more general function f in place of φ?

Yes!!

Theorem (12.4 in book)

Assume that (G2), (G3) for $k = 2$ (second moments), and (GS) hold. Then, for any $f \in B^+(E)$, $\delta > 0$, and $x \in E$,

$$\lim_{n\to\infty} \mathrm{e}^{-\lambda_* n\delta} \frac{X_{n\delta}[f]}{\varphi(x)} = W_\infty \tilde{\varphi}[f], \quad \mathbb{P}_{\delta_x} - a.s.$$

SLLN - Main results

Theorem (12.4 in book)

Assume that (G2), (G3) for $k = 2$ (second moments), and (GS) hold. Moreover, assume that, for all open Ω compactly embedded subsets of E,

$$\liminf_{t \to 0} P_t^{\Omega}[1](x) \geq \mathbf{1}_{\Omega}(x),$$

where $(P_t^{\Omega}[1], t \geq 0)$ is the movement semigroup killed on exiting Ω, for $(X, \mathbb{P})$. Then, for any $f \in B^+(E)$ such that, $f \setminus \varphi \in B^+(E)$, and $x \in E$

$$\lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \frac{X_t[f]}{\varphi(x)} = W_\infty \tilde{\varphi}[f], \quad \mathbb{P}_{\delta_x} - a.s.$$

SLLN - Proof sketch lattice times

In the lattice time setting, the proof is split into two steps:

- The first step is to show that

$$\lim_{n\to\infty} \left| \mathrm{e}^{-\lambda_* n\delta} X_{n\delta}[f] - \mathbb{E}[\mathrm{e}^{-\lambda_* n\delta} X_{n\delta}[f] | \mathcal{F}_{n\delta/2}] \right| = 0, \quad \mathbb{P}_{\delta_x} - a.s.$$

- The second step is to show that

$$\lim_{n\to\infty} \left| \mathbb{E}[\mathrm{e}^{-\lambda_* n\delta} X_{n\delta}[f] | \mathcal{F}_{n\delta/2}] - W_{n\delta} \tilde{\varphi}[f] \varphi(x) \right| = 0, \quad \mathbb{P}_{\delta_x} - a.s.$$

  Finally, we can apply the martingale limit to obtain the theorem.

SLLN - Proof sketch lattice times

To show the first step, we have by the branching property that

$$
\begin{aligned}
&\mathrm{e}^{-\lambda_* n\delta} X_{n\delta}[f] - \mathbb{E}[\mathrm{e}^{-\lambda_* n}\delta X_{n\delta}[f] | \mathcal{F}_{n\delta/2}] \\
&= \mathrm{e}^{-\lambda_* n\delta/2} \sum_{i=1}^{N_{n\delta/2}} \mathrm{e}^{-\lambda_* n\delta/2} \left(X^{(i)}_{n\delta/2} - \mathbb{E}[\delta X^{(i)}_{n\delta/2}[f] | \mathcal{F}_{n\delta/2}] \right),
\end{aligned}
$$

where conditionally on $\mathcal{F}_{n\delta/2}$, the $X^{(i)}$ are indepedent copies of X initiated from x_i, the *ith* particle alive at time $n\delta/2$. Conditionally on $\mathcal{F}_{n\delta/2}$, the right-hand side is a sum of mean-zero independent random variables. Therefore, its second moment is given by

$$
\mathbb{E}\left[\mathrm{e}^{-\lambda_* n\delta} \sum_{i=1}^{N_{n\delta/2}} \mathrm{Var}(\mathrm{e}^{-\lambda_* n\delta/2} X^{(i)}_{n\delta/2} | \mathcal{F}_{n\delta/2}) \right] \leq C\mathrm{e}^{-\lambda_* n\delta/2}
$$

as $n \to \infty$, for some constant C.

SLLN - Proof sketch lattice times

The second step is shown using a similar technique. Again by the branching property, we have that

$$
\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\lambda_* n\delta} X_{n\delta}[f] | \mathcal{F}_{n\delta/2}] - W_{n\delta}\tilde{\varphi}[f]\varphi(x) \\
&= \mathrm{e}^{-\lambda_* n\delta/2} \sum_{i=1}^{N_{n\delta/2}} \mathrm{e}^{-\lambda_* n\delta/2} \left(\mathbb{E}[X^{(i)}_{n\delta/2}[f] | \mathcal{F}_{n\delta/2}] - \tilde{\varphi}[f]\varphi(x_i)\varphi(x) \right).
\end{aligned}
$$

By (G2), the summands tend to 0 as $n \to \infty$. Note, this is why having uniform convergence over E in (G2) is extremely useful.

SLLN - Why irreducibility (G2) is important

Let Y be the following multitype branching process. Let $E = \{r, b\}$, $\gamma(r) = \gamma(b) = 1$, and whenever a type r (resp. b) branches it is replaced by two particles of type r (resp. b).

Theorem (Pólya and Eggenberger)

Assume that at time 0, Y has $\alpha \geq 1$ particles of type r and $\beta \geq 1$ particles of type b, then

$$\frac{(X_t[\mathbf{1}_r], X_t[\mathbf{1}_b])}{X_t[1]} \to (Z, 1 - Z),$$

where $Z \sim \mathrm{Beta}(\alpha, \beta)$.

SLLN - Why irreducibility is important

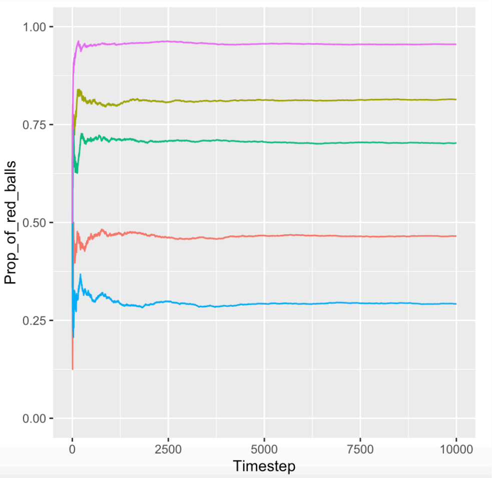

Figure 1: Proportion of red particles in Y with initial condition $(1, 1)$ up to the first 10000 branches.

SLLN - Why irreducibility is important

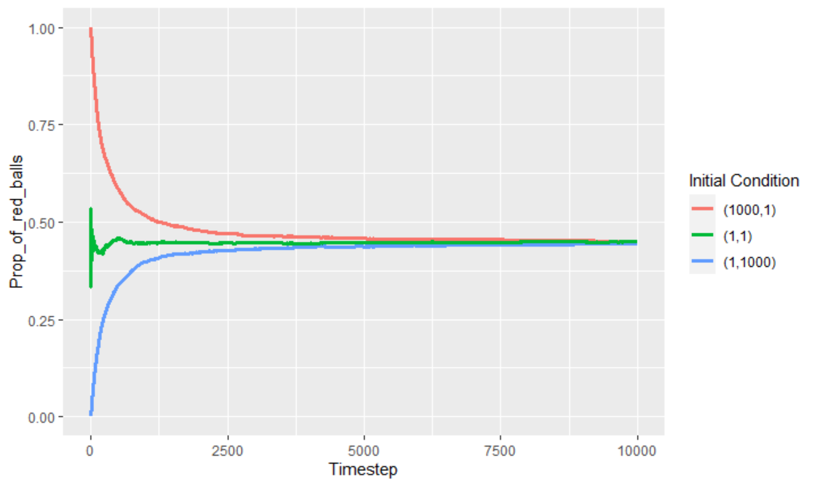

Figure 2: Proportion of red particles in a multitype branching process with varying initial conditions. Each red particle is replaced by 2 red and 3 blue, each blue particle is replaced by 2 red and 2 blue.

The CLT - Additional assumptions required

As in the SLLN, we require (G2), (G6), and (GS). We also need the following further assumption.

(G2b) There exists a constant $\lambda_2 < \lambda_*$, a function $\varphi_2 \in B(E)$ and a probability measure $\tilde{\varphi}_2$, such that, $\tilde{\varphi}_2[\varphi_2] = 1$ and, for any $x \in E$, $t \geq 0$ and $f \in B(E)$,

$$\psi_t[\varphi_2](x) = \mathrm{e}^{\lambda_2 t}\varphi_2(x), \quad \text{and} \quad \tilde{\varphi}_2[\psi_t[f]] = \mathrm{e}^{\lambda_2 t}\tilde{\varphi}_2[f].$$

Moreover,

$$\sup_{x\in E, f\in \ker_1(\tilde{\varphi})} \left|\mathrm{e}^{-\lambda_2 t}\psi_t[f](x) - \tilde{\varphi}_2[f]\varphi_2(x)\right| \to 0, \quad \text{as } t \to \infty,$$

where

$$\ker(\tilde{\varphi}) := \{f \in B(E) : \tilde{\varphi}[f] = 0\}, \quad \ker_1(\tilde{\varphi}) := \{f \in \ker(\tilde{\varphi}) : \|f\| \leq 1\}.$$

The CLT - Three regimes

The behaviour of the second order fluctuations depends on whether λ_2 is larger, equal to, or smaller than $\lambda_*/2$.

We call these the large, critical, and small regimes. The 2nd moment evolution equation gives us some insight as to why these three regimes appear. Recall that, for $f \in \ker(\tilde{\varphi})$ and $x \in E$,

$$\psi_t^{(2)}[f](x) = \psi_t\left[f^2\right](x) + \int_0^t \psi_s\left[\gamma\eta_{t-s}^{(1)}[f]\right](x)\mathrm{d}s, \quad t \geq 0, \tag{1}$$

where,

$$\eta_{t-s}^{(1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\ldots,k_N]_2^2}\binom{2}{k_1,\ldots,k_N}\prod_{j=1}^N \psi_{t-s}^{(k_j)}[f](x_j)\right].$$

The CLT - Large regime $\lambda_2 > \lambda_*/2$

Assume that (G2), (G2b), (G6), and (GS) hold. Furthermore, assume that $\lambda_2 > \lambda_*/2$. Then, for any $f \in \ker(\tilde{\varphi})$ and $x \in E$, as $t \to \infty$

$$\mathrm{e}^{-\lambda_2 t} X_t[f] \xrightarrow{\mathrm{p}} \tilde{\varphi}_2[f] \widetilde{W}_\infty,$$

where $\widetilde{W}_\infty$ is the almost sure limit of the martingale $e^{-\lambda_2 t} X_t[\varphi_2]$.

Furthermore, under additional tightness assumptions, this result holds almost surely.

The CLT - Small regime $\lambda_2 < \lambda_*/2$

Assume that (G2), (G2b), (G6), and (GS) hold. Furthermore, assume that $\lambda_2 > \lambda_*/2$. Then, under additional tightness assumptions, for any $f \in \ker(\tilde{\varphi})$ and $x \in E$, as $n \to \infty$

$$\mathrm{e}^{-\lambda_*(n+t)/2} X_{n+t}[f] \xrightarrow{\mathrm{d}} \varphi(x)^{1/2} W_\infty^{1/2} Z_s(t) \quad in \quad D[0,\infty),$$

where Z_s is a mean-zero Gaussian process independent of W_∞. Furthermore, if X is a multitype branching process, then Z_s is an Ornstein-Uhlenbeck process.

Without additional tightness assumptions, this result still holds in distribution in a non-functional setting.

The CLT - Critical regime $\lambda_2 = \lambda_*/2$

Assume that (G2), (G2b), (G6), and (GS) hold. Furthermore, assume that $\lambda_2 = \lambda_*/2$. Then, under additional tightness assumptions, for any $f \in \ker(\tilde{\varphi})$ and $x \in E$, as $n \to \infty$

$$n^{-1/2} \mathrm{e}^{-\lambda_* nt/2} X_{nt}[f] \xrightarrow{\mathrm{d}} C\varphi(x)^{1/2} W_\infty^{1/2} Z_c(t) \quad in \quad D[0,\infty),$$

where Z_c is a standard Brownian motion independent of W_∞, and

$$C = \left(\tilde{\varphi} \left[\gamma \mathcal{E}_\cdot \left[\sum_{\substack{k_1,k_2=1 \\ k_1 \neq k_2}}^{N} \varphi(x_{k_1})\varphi(x_{k_2}) \right] \right] \right)^{1/2} \tilde{\varphi}_2[f].$$

Without additional tightness assumptions, this result still holds in distribution in a non-functional setting.

The CLT - Example of Small and Large regime

Figure 3: Both lines correspond to the fluctuations in the first type of a 2-type multitype branching process. When a particle branches in the process corresponding to the blue line, the particle is replaced by 2 of its own type and 9 of the opposite. These values are reversed for the red line.

The CLT - Sketch of the proof

In each regime, the proof is split into three steps:

- The first step is to show the asymptotic behaviour of the kth moment evolution equation when $f \in \ker(\tilde{\varphi})$. This is done using a similar argument to that used for the general f.
- The second step is to show the convergence along lattice times (Large regime) and the convergence of the finite dimensional distributions (Small and Critical regime).
- The final step is to show tightness of the processes to obtain functional convergence.